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On rate of convergence in distribution of asymptotically normal statistics based on samples of random size

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

In the present paper we prove a general theorem which gives the rates of convergence in distribution of asymptotically normal statistics based on samples of random size. The proof of the theorem uses the rates of convergences in distribution for the random size and for the statistics based on samples of nonrandom size.

Keywords: sample of random size; asymptotically normal statistic; transfer theorem; rate of convergence; mixture of distributions; Laplace distribution; Student's distribution

1. Introduction

Asymptotic properties of distributions of sums of random number of random variables are subject of many papers (see e.g. Gnedenko&Fahim, 1969; Gnedenko, 1989; Kruglov&Korolev, 1990; Gnedenko&Korolev, 1996; Bening&Korolev, 2002; von Chossy&Rappl, 1983). This kind of sums are widely used in insurance, economics, biology, etc. (see Gnedenko, 1989; Gnedenko, 1998; Bening&Korolev, 2002). However, in mathematical statistics and its applications, there are common statistics that are not sums of observations. Examples are the rank statistics, U-statistics, linear combinations of order statistics, etc. In this case the statistics are often situations when the sample size is not predetermined and can be regarded as random. For example, in reliability testing the number of failed devices at a particular time is a random variable.

Generally, in most cases related to the analysis and processing of experimental data, we can assume that the number of random factors, influencing the observed values, is itself random and varies from observation to observation. Therefore, instead of different variants of the central limit theorem, proving the normality of the limiting distribution of classical statistics, in such situations we should rely on their analogues for samples of random size. This makes it natural to study the asymptotic behavior of distributions of statistics of general form, based on samples of random size. For example, Gnedenko (1989) examines the asymptotic properties of the distributions of sample quantiles constructed from samples of random size.

In this paper we estimate the rate of convergence of distribution functions of asymptotically normal statistics based on samples of random size. The estimations depend on the rates of convergences of distributions of the random size of sample and the statistic based on sample of nonrandom size. Such statements are usually called transfer theorems. In the present paper we prove transfer theorems concerning estimates of convergence rate.

In this paper we use the following notation and symbols: $\mathbb{R}$ as real numbers, $\mathbb{N}$ as positive integers, $\Phi(x)$, $\varphi(x)$ as standard normal distribution function and density.

In Section 2 we give a sketch of the proof of a general transfer theorem, Sections 3, 4 and 5 contain the main theorems, their proofs and examples.

Consider random variables $N_1, N_2, \ldots$ and $X_1, X_2, \ldots$ defined on a common measurable space $(\Omega, \mathcal{A}, \mathsf{P})$. The random variables $X_1, X_2, \ldots X_n$ denote observations, n is a nonrandom size of sample, the random variable N_n denotes a random size of sample and depends on a natural parameter $n \in \mathbb{N}$. Suppose that the random variables N_n take on positive integers for any $n \geq 1$, that is $N_n \in \mathbb{N}$, and do not depend on $X_1, X_2, \ldots$. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed observations having a distribution function $F(x)$.

Let $T_n = T_n(X_1, \ldots, X_n)$ be some statistic, that is a real measurable function on observations $X_1, \ldots, X_n$. The statistic T_n is called asymptotically normal with parameters $(\mu, 1/\sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$, if

$$\mathsf{P}(\sigma\sqrt{n}(T_n - \mu) < x) \longrightarrow \Phi(x), \quad n \to \infty, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\Phi(x)$ is the standard normal distribution function.

The asymptotically normal statistics are abundant. Recall some examples of asymptotically normal statistics: the sample mean (assuming nonzero variances), the maximum likelihood estimators (under weak regularity conditions), the central order statistics and many others.

For any $n \geq 1$ define the random variable T_{N_n} by

$$T_{N_n}(\omega) \equiv T_{N_n(\omega)}(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)), \quad \omega \in \Omega. \tag{1.2}$$

Therefore, T_{N_n} is a statistic constructed from the statistic T_n and from the sample of random size N_n.

In Gnedenko&Fahim (1969) and Gnedenko (1989), the first and second transfer theorems are proved for the case of sums of independent random variables and sample quantiles.

Theorem 1.1 (Gnedenko, 1989)**.** *Let $X_1, X_2, \dots$ be independent and identically distributed random variables and $N_n \in \mathbb{N}$ denotes a sequence of random variables which are independent of $X_1, X_2, \dots$. If there exist real numbers $b_n > 0$, $a_n \in \mathbb{R}$ such that*

$$\text{1.} \qquad \mathsf{P}\Big(\frac{1}{b_n}\sum_{i=1}^{n}(X_i - a_n) < x\Big) \longrightarrow \Psi(x), \quad n \to \infty$$

and

$$\text{2.} \qquad \mathsf{P}\Big(\frac{N_n}{n} < x\Big) \longrightarrow H(x), \quad H(0+) = 0, \quad n \to \infty,$$

where $\Psi(x)$ and $H(x)$ are distribution functions, then, as $n \to \infty$,

$$\mathsf{P}\Big(\frac{1}{b_n}\sum_{i=1}^{N_n}(X_i - a_n) < x\Big) \longrightarrow G(x), \quad n \to \infty,$$

where the distribution function $G(x)$ is defined by its characteristic function

$$g(t) = \int_0^\infty (\psi(t))^z \,\mathrm{d}H(z)$$

and $\psi(t)$ is the characteristic function of $\Psi(x)$.

The proof of the theorem can be read in Gnedenko (1998).

Theorem 1.2 (Gnedenko, 1989)**.** *Let $X_1, X_2, \dots$ be independent and identically distributed random variables and $N_n \in \mathbb{N}$ is a sequence of random variables which are independent of $X_1, X_2, \dots$, and let $X_{\gamma:n}$ be the sample quantile of order $\gamma \in (0,1)$ constructed from sample $X_1, \dots, X_n$. If there exist real numbers $b_n > 0$, $a_n \in \mathbb{R}$ such that*

$$\text{1.} \qquad \mathsf{P}\Big(\frac{1}{b_n}(X_{\gamma:n} - a_n) < x\Big) \longrightarrow \Phi(x), \quad n \to \infty$$

and

$$\text{2.} \qquad \mathsf{P}\Big(\frac{N_n}{n} < x\Big) \longrightarrow H(x), \quad H(0+) = 0, \quad n \to \infty,$$

where $H(x)$ is a distribution function, then, as $n \to \infty$,

$$\mathsf{P}\Big(\frac{1}{b_n}(X_{\gamma:N_n} - a_n) < x\Big) \longrightarrow G(x), \quad n \to \infty$$

where the distribution function $G(x)$ is a mixture of normal distribution with the mixing distribution H

$$G(x) = \int\limits_0^\infty \Phi(x\sqrt{y})\,dH(y).$$

In Bening&Korolev (2005), the following general transfer theorem is proved for asymptotically normal statistics (1.1).

Theorem 1.3. *Let $\{d_n\}$ be an increasing and unbounded sequence of positive integers. Suppose that $N_n \to \infty$ in probability as $n \to \infty$. Let $T_n(X_1, \dots, X_n)$ be an asymptotically normal statistics, that is*

$$\mathsf{P}(\sigma\sqrt{n}(T_n - \mu) < x) \longrightarrow \Phi(x), \quad n \to \infty.$$

Then a necessary and sufficient condition for a distribution function $G(x)$ to satisfy

$$\mathsf{P}(\sigma\sqrt{d_n}(T_{N_n} - \mu) < x) \longrightarrow G(x), \quad n \to \infty,$$

is that there exists a distribution function $H(x)$ with $H(0+) = 0$ satisfying

$$\mathsf{P}(N_n < d_n x) \longrightarrow H(x), \quad n \to \infty, \quad x > 0,$$

and $G(x)$ has a form

$$G(x) = \int\limits_0^\infty \Phi(x\sqrt{y})\,dH(y), \quad x \in \mathbb{R},$$

that is the distribution $G(x)$ is a mixture of the normal law with the mixing distribution H.

Now, we give a brief sketch of proof of Theorem 1.3 to make references later.

2. Sketch of proof of Theorem 1.3

The proof of Theorem 1.3 is closely related to the proof of Theorems 6.6.1 and 6.7.3 for random sums in Kruglov&Korolev (1990).

By the formula of total probability, we have

$$\mathsf{P}\Big(\sigma\sqrt{d_n}(T_{N_n} - \mu) < x\Big) - G(x)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \mathsf{P}(N_n = k)\mathsf{P}\Big(\sigma\sqrt{k}(T_k - \mu) < \sqrt{k/d_n}x\Big) - G(x) \\
&= \sum_{k=1}^{\infty} \mathsf{P}(N_n = k)\Big(\Phi\big(\sqrt{k/d_n}x\big) - G(x)\Big) \\
&\quad + \sum_{k=1}^{\infty} \mathsf{P}(N_n = k)\Big(\mathsf{P}\Big(\sigma\sqrt{k}(T_k - \mu) < \sqrt{k/d_n}x\Big) - \Phi\big(\sqrt{k/d_n}x\big)\Big) \\
&\equiv J_{1n} + J_{2n}. \qquad (2.1)
\end{aligned}
$$

From definition of $G(x)$ the expression for J_{1n} can be written in the form

$$
\begin{aligned}
J_{1n} &= \int_0^{\infty} \Phi(x\sqrt{y})\, \mathrm{d}\mathsf{P}(N_n < d_n y) - \int_0^{\infty} \Phi(x\sqrt{y})\, \mathrm{d}H(y) \\
&= \int_0^{\infty} \Phi(x\sqrt{y})\, \mathrm{d}\big(\mathsf{P}(N_n < d_n y) - H(y)\big).
\end{aligned}
$$

Using the formula of integration by parts for Lebesgue integral (see e.g. Theorem 2.6.11 in Shiryaev, 1995) yields

$$
J_{1n} = -\int_0^{\infty} \big(\mathsf{P}(N_n < d_n y) - H(y)\big)\, \mathrm{d}\Phi(x\sqrt{y}). \qquad (2.2)
$$

By the condition of the present theorem,

$$
\mathsf{P}(N_n < d_n y) - H(y) \longrightarrow 0, \quad n \to \infty
$$

for any fixed $y \in \mathbb{R}$, therefore, by the dominated convergence theorem (see e.g. Theorem 2.6.3 in Shiryaev, 1995), we have

$$
J_{1n} \longrightarrow 0, \quad n \to \infty.
$$

Consider J_{2n}. For simplicity, instead of the condition for the statistic T_n to be asymptotically normal (see (1.1)), we suggest a stronger condition which describes the rate of convergence of distributions of T_n to the normal law. Suppose that the following condition is satisfied.

Condition 1. *There exist real numbers $\alpha > 0$ and $C_1 > 0$ such that*

$$
\sup_x \Big|\mathsf{P}\Big(\sigma\sqrt{n}(T_n - \mu) < x\Big) - \Phi(x)\Big| \le \frac{C_1}{n^{\alpha}}, \quad n \in \mathbb{N}.
$$

From the condition we obtain estimates for J_{2n}. We have

$$
|J_{2n}| = \Big|\sum_{k=1}^{\infty} \mathsf{P}(N_n = k)\Big(\mathsf{P}\Big(\sigma\sqrt{k}(T_k - \mu) < \sqrt{k/d_n}x\Big) - \Phi\big(\sqrt{k/d_n}x\big)\Big)\Big|
$$

$$\leq C_1 \sum_{k=1}^{\infty} \mathsf{P}(N_n = k) \frac{1}{k^\alpha} = C_1 \mathsf{E}(N_n)^{-\alpha} = \frac{C_1}{d_n^\alpha} \mathsf{E}(N_n/d_n)^{-\alpha}. \tag{2.3}$$

Since, by the condition of theorem, the random variables N_n/d_n have a weak limit, then the expectation $\mathsf{E}(N_n/d_n)^{-\alpha}$ is typically bounded. Because $d_n \to \infty$, from the last inequality it follows that

$$J_{2n} \longrightarrow 0, \quad n \to \infty.$$

3. The main results

Suppose that the limiting behavior of distribution functions of the normalized random size is described by the following condition.

Condition 2. *There exist real numbers $\beta > 0$, $C_2 > 0$ and a distribution $H(x)$ with $H(0+) = 0$ such that*

$$\sup_{x \geq 0} \Big| \mathsf{P}\Big(\frac{N_n}{n} < x\Big) - H(x) \Big| \leq \frac{C_2}{n^\beta}, \quad n \in \mathbb{N}.$$

Theorem 3.1. *If for the statistic $T_n(X_1, \ldots, X_n)$ condition 1 is satisfied, for the random sample size N_n condition 2 is satisfied, then the following inequality holds*

$$\sup_x \Big| \mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n} - \mu) < x\Big) - G(x) \Big| \leq C_1 \mathsf{E} N_n^{-\alpha} + \frac{C_2}{2n^\beta},$$

where the distribution $G(x)$ has the form

$$G(x) = \int_0^\infty \Phi(x\sqrt{y}) \,\mathrm{d}H(y), \quad x \in \mathbb{R}.$$

Corollary 3.2. *The statement of the theorem remains valid if the normal law is replaced by any limiting distribution.*

Corollary 3.3. *If the moments $\mathsf{E}(N_n/n)^{-\alpha}$ are bounded uniformly in n, that is*

$$\mathsf{E}\Big(\frac{N_n}{n}\Big)^{-\alpha} \leq C_3, \quad C_3 > 0, \quad n \in \mathbb{N},$$

then the right side of the inequality in the statement of the theorem has the form

$$\frac{C_1 C_3}{n^\alpha} + \frac{C_2}{2n^\beta} = \mathcal{O}\big(n^{-\min(\alpha,\beta)}\big).$$

Corollary 3.4. *By Hölder's inequality for* $0 < \alpha \leq 1$*, the following estimate holds*

$$\mathsf{E}N_n^{-\alpha} \leq \Big(\mathsf{E}\frac{1}{N_n}\Big)^{\alpha},$$

which is useful from practical viewpoint. In this case, the right side of the inequality has the form

$$C_1\Big(\mathsf{E}\frac{1}{N_n}\Big)^{\alpha} + \frac{C_2}{2n^{\beta}}.$$

Corollary 3.5. *Note that, condition 2 means that the random variables* N_n/n *converge weakly to* V *which has the distribution* $H(x)$*. From the definition of weak convergence with function* $x^{-\alpha}, \quad x \geq 1$*, for* $N_n \geq n, n \in \mathbb{N}$*, it follows that*

$$\mathsf{E}\Big(\frac{N_n}{n}\Big)^{-\alpha} \longrightarrow \mathsf{E}\frac{1}{V^{\alpha}}, \quad n \to \infty,$$

that is the moments $\mathsf{E}(N_n/n)^{-\alpha}$ *are bounded in n and, therefore, the estimate from Corollary 3.3 holds.*

The case $N_n \geq n$ appears when the random variable N_n takes on values $n, 2n, \ldots, kn$ with equal probabilities $1/k$ for any fixed $k \in \mathbb{N}$. In this case, the random variables N_n/n do not depend on n and, therefore, converge weakly to V which takes values $1, 2, \ldots, k$ with equal probability $1/k$.

Corollary 3.6. *From the proof of the theorem it follows that skipping of conditions 1 and 2 yields the following statement*

$$\begin{aligned}
&\sup_x \Big|\mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n} - \mu) < x\Big) - G(x)\Big| \\
&\leq \sum_{k=1}^{\infty} \mathsf{P}\big(N_n = k\big) \sup_x \Big|\mathsf{P}\Big(\sigma\sqrt{k}(T_k - \mu) < x\Big) - \Phi(x)\Big| \\
&\quad + \frac{1}{2}\sup_{x \geq 0}\Big|\mathsf{P}\Big(\frac{N_n}{n} < x\Big) - H(x)\Big|.
\end{aligned}$$

Following the proof of Theorem 3.1 (see Section 2 and 4), we can formulate more general result.

Theorem 3.7. *Let a random element* $\mathbf{X}_n$ *in some measurable space and random variable* N_n *be defined on a common measurable space and independent for any* $n \in \mathbb{N}$*. Suppose that a real-valued statistic* $T_n = T_n(\mathbf{X}_n)$ *and the random variable* N_n *satisfy the following conditions.*

1. There exist real numbers $\alpha > 0$*,* $\sigma > 0$*,* $\mu \in \mathbb{R}$*,* $C_1 > 0$ *and a sequence* $0 < d_n \uparrow +\infty, n \to \infty$*, such that*

$$\sup_x \Big|\mathsf{P}\Big(\sigma\sqrt{d_n}(T_n - \mu) < x\Big) - \Phi(x)\Big| \leq \frac{C_1}{n^{\alpha}}, \quad n \in \mathbb{N}.$$

2. There exist a number $C_2 > 0$, a sequence $0 < \delta_n \downarrow 0$, $n \to \infty$ and a distribution function $H(x)$ with $H(0+) = 0$ such that

$$\sup_{x\geq 0}\Big|\mathsf{P}\Big(\frac{N_n}{d_n} < x\Big) - H(x)\Big| \leq C_2\delta_n, \quad n \in \mathbb{N}.$$

Then the following inequality holds

$$\sup_{x}\Big|\mathsf{P}\Big(\sigma\sqrt{d_n}(T_{N_n} - \mu) < x\Big) - G(x)\Big| \leq C_1\mathsf{E}N_n^{-\alpha} + \frac{C_2}{2}\delta_n,$$

where the distribution function $G(x)$ has the form

$$G(x) = \int\limits_0^\infty \Phi(x\sqrt{y})\,\mathrm{d}H(y), \quad x \in \mathbb{R}.$$

4. Proof of Theorem 3.1

Suppose $x \geq 0$. Using formulas (2.1)–(2.3) with $d_n = n$ yields

$$\sup_{x\geq 0}\Big|\mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n} - \mu) < x\Big) - G(x)\Big| \leq I_{1n} + I_{2n}, \tag{4.1}$$

where

$$I_{1n} = \sup_{x\geq 0}\int\limits_0^\infty \big|\mathsf{P}(N_n < ny) - H(y)\big|\,\mathrm{d}\Phi(x\sqrt{y}), \tag{4.2}$$

$$I_{2n} = \sum_{k=1}^\infty \mathsf{P}(N_n = k)\sup_{x\geq 0}\Big|\mathsf{P}\Big(\sigma\sqrt{k}(T_k - \mu) < \sqrt{k/n}x\Big) - \Phi\big(\sqrt{k/n}x\big)\Big|. \tag{4.3}$$

To estimate the variable I_{1n} we use equality (4.2) and condition 2,

$$I_{1n} \leq \frac{C_2}{n^\beta}\sup_{x\geq 0}\int\limits_0^\infty \mathrm{d}\Phi(x\sqrt{y}) = \frac{C_2}{2n^\beta}. \tag{4.4}$$

The series in I_{2n} (see (4.3)) is estimated by using condition 1.

$$I_{2n} \leq C_1\sum_{k=1}^\infty \frac{1}{k^\alpha}\mathsf{P}(N_n = k) = C_1\mathsf{E}N_n^{-\alpha}. \tag{4.5}$$

Note that the estimate (4.5) is valid for $x < 0$. For I_{1n} and negative x, we have (see (2.1) and (2.2))

$$I_{1n} = \sup_{x<0}\Big|\int\limits_0^\infty \big(\mathsf{P}(N_n < ny) - H(y)\big)\,\mathrm{d}\Phi(x\sqrt{y})\Big|$$

$$= \sup_{x<0} \Big| \int_0^\infty \big(\mathsf{P}(N_n < ny) - H(y)\big) \, \mathrm{d}\Phi(|x|\sqrt{y}) \Big|$$

$$\le \sup_{x\ge 0} \int_0^\infty \big|\mathsf{P}(N_n < ny) - H(y)\big| \, \mathrm{d}\Phi(x\sqrt{y}),$$

and we can use (4.4) again. The statement of the theorem follows from (4.1), (4.4) and (4.5). The theorem is proved.

5. Examples

We consider two examples of use of Theorem 3.1 when the limiting distribution function $G(x)$ is known.

5.1. Student's distribution

Bening&Korolev (2005) shows that if the random sample size N_n has the negative binomial distribution with parameters $p = 1/n$ and $r > 0$, that is (in particular, for $r = 1$, it is the geometric distribution)

$$\mathsf{P}(N_n = k) = \frac{(k+r-2)\cdots r}{(k-1)!} \frac{1}{n^r} \Big(1 - \frac{1}{n}\Big)^{k-1}, \quad k \in \mathbb{N},$$

then, for an asymptotically normal statistic T_n the following limiting relationship holds (see Corollary 2.1 in Bening&Korolev, 2005)

$$\mathsf{P}(\sigma\sqrt{n}(T_{N_n} - \mu) < x) \longrightarrow G_{2r}(x\sqrt{r}), \quad n \to \infty, \tag{5.1}$$

where $G_{2r}(x)$ is Student's distribution with parameter $\gamma = 2r$, having density

$$p_\gamma(x) = \frac{\Gamma((\gamma+1)/2)}{\sqrt{\pi\gamma}\Gamma(\gamma/2)} \Big(1 + \frac{x^2}{\gamma}\Big)^{-(\gamma+1)/2}, \quad x \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the gamma function, and $\gamma > 0$ is a shape parameter (if the parameter γ is a positive integer, then it is called the number of degrees of freedom). In our situation the parameter may be arbitrary small, and we have typical heavy-tailed distribution. If $\gamma = 2$, that is $r = 1$, then the distribution function $G_2(x)$ can be found explicitly

$$G_2(x) = \frac{1}{2}\Big(1 + \frac{x}{\sqrt{2+x^2}}\Big), \quad x \in \mathbb{R}.$$

For $r = 1/2$, we obtain the Cauchy distribution.

Bening et al. (2004) gives an estimate of rate of convergence for random sample size, for $0 < r < 1$,

$$\sup_{x\ge 0} \Big| \mathsf{P}\Big(\frac{N_n}{\mathsf{E}N_n} < x\Big) - H_r(x) \Big| \le \frac{C_r}{n^{r/(r+1)}}, \quad C_r > 0, \quad n \in \mathbb{N}, \tag{5.2}$$

where

$$H_r(x) = \frac{r^r}{\Gamma(r)} \int\limits_0^x e^{-ry} y^{r-1}\,\mathrm{d}y, \quad x \geq 0,$$

for $r = 1$, the right side of the inequality can be replaced by $1/(n-1)$. So, $H_r(x)$ is a distribution with parameter $r \in (0, 1]$, and

$$\mathsf{E} N_n = r(n-1) + 1. \tag{5.3}$$

From

$$(1+x)^\gamma = \sum_{k=0}^{\infty} \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!} x^k, \quad |x| < 1, \quad \gamma \in \mathbb{R},$$

we have

$$\mathsf{E} N_n^{-1} = \frac{1}{(n-1)(1-r)} \Big(\frac{1}{n^{r-1}} - 1\Big) = \mathcal{O}(n^{-r}), \quad 0 < r < 1, \quad n \in \mathbb{N}. \tag{5.4}$$

If the Berry-Esseen estimate is valid for the rate of convergence of distribution of T_n, that is

$$\sup_x \Big| \mathsf{P}\Big(\sigma\sqrt{n}(T_n - \mu) < x\Big) - \Phi(x) \Big| = \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big), \quad n \in \mathbb{N}, \tag{5.5}$$

then from Theorem 3.1 with $\alpha = 1/2$, $\beta = r/(r+1)$, from relations (5.1)–(5.4) and Corollary 3.4, we have the following estimate

$$\begin{aligned} &\sup_x \Big| \mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n} - \mu) < x\Big) - G_{2r}(x\sqrt{r}) \Big| \\ &= \mathcal{O}\Big(\frac{1}{n^{r/2}}\Big) + \mathcal{O}\Big(\frac{1}{n^{r/(r+1)}}\Big) = \mathcal{O}\Big(\frac{1}{n^{r/2}}\Big), \quad r \in (0,1), \ n \in \mathbb{N}. \end{aligned} \tag{5.6}$$

5.2. Laplace distribution

Consider Laplace distribution with distribution function $\Lambda_\gamma(x)$ and density

$$\lambda_\gamma(x) \ = \ \frac{1}{\gamma\sqrt{2}} \exp\Big\{-\frac{\sqrt{2}|x|}{\gamma}\Big\}, \quad \gamma > 0, \quad x \in \mathbb{R}.$$

Bening&Korolev (2008) gives a sequence of random variables $N_n(m)$ which depends on the parameter $m \in \mathbb{N}$. Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with some continuous distribution function. Let m be a positive integer and

$$N(m) = \min\{i \geq 1 : \max_{1 \leq j \leq m} Y_j < \max_{m+1 \leq k \leq m+i} Y_k\}.$$

It is well-known that such random variables have the discrete Pareto distribution

$$\mathsf{P}(N(m) \geq k) = \frac{m}{m+k-1}, \quad k \geq 1. \tag{5.7}$$

Now, let $N^{(1)}(m), N^{(2)}(m), \dots$ be independent random variables with the same distribution (5.7). Define the random variable

$$N_n(m) = \max_{1 \le j \le n} N^{(j)}(m),$$

then Bening&Korolev (2008) shows that

$$\lim_{n\to\infty} \mathsf{P}\Big(\frac{N_n(m)}{n} < x\Big) = e^{-m/x}, \quad x > 0, \tag{5.8}$$

and, for an asymptotically normal statistic T_n, the following relationship holds

$$\mathsf{P}(\sigma\sqrt{n}(T_{N_n(m)} - \mu) < x) \longrightarrow \Lambda_{1/m}(x), \quad n \to \infty,$$

where $\Lambda_{1/m}(x)$ is the Laplace distribution function with parameter $\gamma = 1/m$.

Lyamin (2010) gives the estimate for the rate of convergence for (5.8),

$$\sup_{x\ge 0}\Big|\mathsf{P}\Big(\frac{N_n(m)}{n} < x\Big) - e^{-m/x}\Big| \le \frac{C_m}{n}, \quad C_m > 0, \quad n \in \mathbb{N}. \tag{5.9}$$

If the Berry-Esseen estimate is valid for the rate of convergence of distribution for the statistic (see (5.5)), then from Corollary 3.4 for $\alpha = 1/2$, $\beta = 1$ and from inequality (5.9), we have

$$\sup_x\Big|\mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n(m)} - \mu) < x\Big) - \Lambda_{1/m}(x)\Big| = \mathcal{O}\Big((\mathsf{E}N_n^{-1}(m))^{1/2}\Big) + \mathcal{O}\big(n^{-1}\big). \tag{5.10}$$

Consider the variable $\mathsf{E}N_n^{-1}(m)$. From definition of $N_n(m)$ and inequality (5.7), we have

$$\mathsf{P}(N_n(m) = k) = \Big(\frac{k}{m+k}\Big)^n - \Big(\frac{k-1}{m+k-1}\Big)^n = mn\int\limits_{k-1}^{k} \frac{x^{n-1}}{(m+x)^{n+1}}\,\mathrm{d}x,$$

therefore,

$$\begin{aligned}\mathsf{E}N_n^{-1}(m) &= \sum_{k=1}^{\infty} \frac{1}{k}\mathsf{P}(N_n(m) = k) = mn\sum_{k=1}^{\infty}\frac{1}{k}\int\limits_{k-1}^{k}\frac{x^{n-1}}{(m+x)^{n+1}}\,\mathrm{d}x \\ &\le mn\sum_{k=1}^{\infty}\int\limits_{k-1}^{k}\frac{x^{n-2}}{(m+x)^{n+1}}\,\mathrm{d}x mn\int\limits_{0}^{\infty}\frac{x^{n-2}}{(m+x)^{n+1}}\,\mathrm{d}x.\end{aligned}$$

To calculate the last integral we use the following formula (see formula 856.12 in Dwight, 1961)

$$\int\limits_0^{\infty}\frac{x^{m-1}}{(a+bx)^{m+n}}\,\mathrm{d}x = \frac{\Gamma(m)\Gamma(n)}{a^n b^m \Gamma(m+n)} \quad a, b, m, n > 0.$$

We have

$$\mathsf{E}N_n^{-1}(m) \leq mn\frac{\Gamma(n-1)\Gamma(2)}{m^2\Gamma(n+1)} = \frac{1}{m(n-1)} = O(n^{-1}).$$

Now, by this formula and (5.10), we obtain

$$\sup_x \Big|\mathsf{P}\Big(\sigma\sqrt{n}(T_{N_n(m)} - \mu) < x\Big) - \Lambda_{1/m}(x)\Big| = \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big).$$

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