Annales Mathematicae et Informaticae
***39** (2012) pp. 173–191*

Proceedings of the Conference on Stochastic Models and their Applications
Faculty of Informatics, University of Debrecen, Debrecen, Hungary, August 22–24, 2011

Renewal theorems in the case of attraction to the stable law with characteristic exponent smaller than unity*

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Dedicated to Mátyás Arató on his eightieth birthday

1. Introduction

Let X be a non-negative integer-valued random variable, $p_n = \mathbf{P}(X = n)$. Put $S_n = \sum_1^n X_j,\ n \geq 1$, where X_j are i.i.d. random variables which have the same distribution as X. In what follows we assume that $S_0 = 0$. Let $u_n = \sum_{k=0}^{\infty} \mathbf{P}(S_k = n)$ be the renewal probability at the instant n. Put $f(z) = \sum_{k=0}^{\infty} p_k z^k$. If $g(z)$ is an analytical function in some neighbourhood of zero, we denote the coefficient at z^n in Taylor series for $g(z)$ by $C_n(g(z))$.

In 1963 Garsia and Lamperti [1] proved that under the condition

$$\mathbf{P}(X > n) \sim L(n)n^{-\alpha}, \tag{1.1}$$

where $L(x)$ is a slowly-varying function, the asymptotic formula

$$u_n \sim \frac{\sin \pi\alpha}{\pi} L^{-1}(n)n^{\alpha-1}, \tag{1.2}$$

is valid, provided $1/2 < \alpha < 1$. The relation $a_n \sim b_n$ here and below indicates that $\lim\limits_{n\to\infty} a_n/b_n = 1$.

In 1968 Williamson [3] extended Garsia-Lamperti's result to the case that X belongs to the domain of attraction of a non-degenerate d - dimensional stable law with characteristic exponent α, $\ d/2 < \alpha < \min(d; 2)$.

To prove (1.2) Garsia and Lamperti used the purely analytical method based on analysis of behavior of the generating function $f(z)$ on the unit circle. On the

*This work was supported by RFBR 09-01-00048-a.

contrary, Williamson's approach is probabilistic with the local limit theorem by Rvacheva [4] as the starting point.

As to case $0 < \alpha \leq 1/2$, formula (1.2), generally speaking, is not true if we restrict our selves to condition (1.1). Corresponding counter-example is given in [3]. The point is that in the case $0 < \alpha \leq 1/2$ the existence of lacunas in the sequence p_n influences on the behavior of u_n. Therefore, additional constraints are necessary to provide the validity of (1.2). One such constraint was suggested by De Bruijn and Erdos [2] before [1] appeared, namely,

$$p_{n-1}p_{n+1} > p_n^2, \tag{1.3}$$

i.e. the sequence $\ln p_n$ is convex. Williamson [3] noticed that (1.2) remains true if the sequence p_n does not increase beginning with some number n. This condition is weaker than (1.3).

In the present work we use the condition

$$p_n \sim \frac{l(n)}{n^{1+\alpha}}, \ 0 < \alpha < 1, \tag{1.4}$$

where the function $l(x)$ is slowly varying. Notice that condition (1.1) with $L(n) = \alpha^{-1}l(n)$ follows from (1.4) (see Lemma 2.1 below). Condition (1.4) is discussed in our previous paper [5], namely, it is shown therein that if above-mentioned Williamson's condition is fulfilled, then (1.4) hold.

Theorem 1.1. *If condition (1.4) holds, then*

$$u_n \sim c(\alpha)\frac{\mathbf{P}(X = n)}{\mathbf{P}^2(X \geq n)} \sim \frac{\alpha^2 c(\alpha)}{l(n)n^{1-\alpha}}, \tag{1.5}$$

where $c(\alpha) = \sin \pi\alpha/\pi\alpha$.

The extreme case $p_n \sim n^{-1}l(n)$ is studied in [5]. It turns out that under this condition $u_n \sim \mathbf{P}(X = n)/\mathbf{P}^2(X \geq n)$. Since $c(\alpha) \to 1$ as $\alpha \to 0$, it implies that representation

$$u_n \sim c(\alpha)\frac{\mathbf{P}(X = n)}{\mathbf{P}^2(X \geq n)},$$

which is given in Theorem 1.1 is stable as $\alpha \to 0$. However, we can not say this about the relation $u_n \sim \alpha^2 c(\alpha)/l(n)n^{1-\alpha}$.

In proving Theorem 1.1 we apply the same approach as in [5]. However, to realize it was found more difficult in this case.

Remark. In [6] the renewal theorem is proved under condition that (1.1) holds and

$$p_n < c\mathbf{P}(X > n)n^{-1}$$

using Williamson's method. The proof is based on the following statement:
Assume that $F(0) = 0$ *and (2.1) holds. Then for all* $n \geq 1, z$ *large enough and* $x \geq z$

$$\mathbf{P}\{S_n \geq x, M_n \leq z\} \leq \{cz/x\}^{x/z},$$

where $M_n = \max\{X_1, X_2, ..., X_n\}$ *and* $S_n = \sum_1^n X_i$ (see Lemma 2 in [6]).

The author of [6] asserts that this lemma is an immediate consequence of the inequality

$$\mathbf{P}(S_n \geq x) \leq \sum_{i=1}^{n} \mathbf{P}(X_i > y_i) + (eA_t^+/xy^{t-1})^{x/y},$$

where $S_n = \sum_{j=1}^{n} X_j$, X_j are independent random variables, $y > \max_i y_i$, $A_t^+ = \sum_{j=1}^{n} \{X_j^t; X_j > 0\}$, $0 < t < 1$ (see Corollary 1.5 in [7]).

If X_j are i.i.d. equal to X by distribution, then

$$\mathbf{P}(S_n \geq x) \leq n\mathbf{P}(X > y) + \left(\frac{en\mathbf{E}\{X^t; X > 0\}}{xy^{t-1}}\right)^{x/y}.$$

If $X \leq y$, then

$$\mathbf{E}\{X^t; X > 0\} \leq y^t.$$

Consequently, in this case

$$\mathbf{P}(S_n \geq x) \leq \left(\frac{eny}{x}\right)^{x/y}.$$

This inequality differs from the inequality stated in [6] by the presence of n in the right-hand side. Thus, Lemma 2 of [6] does not follow from Corollary 1.5 of [7]), and, therefore, the former can not be considered as being proved.

Let $h_n = \sum_{k=0}^{\infty} n^{-1}\mathbf{P}(S_k = n)$.

Theorem 1.2. *If condition (1.4) holds, then*

$$h_n \sim \frac{\alpha}{n}. \tag{1.6}$$

Notice that h_n is the derivative of the measure $\nu(A) := \sum_{k\in A} h_k$ with respect to the counting measure. The measure $\nu(A)$ is a particular case of so called harmonic renewal measure. Recall that that the measure $\nu(\cdot) = \sum_1^{\infty} n^{-1}F_n(\cdot)$, where F_n is n-th convolution of any distribution F on $\mathbf{R}^+$ is said to be harmonic renewal measure associated with F. In our case the distribution F is concentrated on the lattice of non-negative integers.The harmonic renewal function is defined by the equality $H(x) = \nu([0, x))$.

The next statement concerning the asymptotic behavior of $H(n)$ as $n \to \infty$ follows from Theorem 1.2.

Corollary 1.3. *If condition (1.4) holds, then*

$$H(n) \sim \alpha \ln n. \tag{1.7}$$

The asymptotic behavior of $H(x)$ for $x \to \infty$ is studied in [9, 10, 11, 12]. The case that F attracts to a stable law is considered in [9], namely , it is proved therein that under the condition $1 - F(x) \sim x^{-\alpha} L(x)$

$$\lim_{x\to\infty} (H(x) - \alpha \ln x + \ln L(x)) = \alpha \mathbf{C} - \ln \Gamma(1-\alpha),$$

where $\mathbf{C}$ is the Euler constant, $\Gamma(\cdot)$ is the gamma function. Of course, the last assertion is sharper than (1.7). Formula (1.7) is presented by reason of simplicity of proving.

2. Auxiliary results

Lemma 2.1. *For any $\alpha > 0$*

$$\sum_{k=n}^{\infty} \frac{l(k)}{k^{\alpha+1}} \sim \int_n^{\infty} \frac{l(y)}{y^{\alpha+1}} dy. \tag{2.1}$$

Proof. Put $p(x) = l(x)/x^{\alpha+1}$. Obviously,

$$\inf_{n\le y\le n+1} \frac{p(y)}{p(n)} \le \frac{1}{p(n)} \int_n^{n+1} p(y)dy \le \sup_{n\le y\le n+1} \frac{p(y)}{p(n)}. \tag{2.2}$$

It is easily seen that for every $n \le y \le n+1$

$$\left(\frac{n}{n+1}\right)^{\alpha+1} \inf_{n\le y\le n+1} \frac{l(y)}{l(n)} \le \frac{p(y)}{p(n)} \le \sup_{n\le y\le n+1} \frac{l(y)}{l(n)}. \tag{2.3}$$

In what follows we need Kamarata's representation

$$l(x) = a(x) \exp\left\{ \int_1^x \frac{\epsilon(u)}{u} du \right\}, \; x \ge 1, \tag{2.4}$$

where $\lim_{n\to\infty} \varepsilon(u) = 0,\; \lim_{x\to\infty} a(x) = a,\; 0 < a < \infty$. Hence,

$$\frac{l(y)}{l(n)} = \frac{a(y)}{a(n)} \exp\left\{ \int_n^y \frac{\epsilon(u)}{u} du \right\}.$$

Obviously,

$$\lim_{n\to\infty} \sup_{n\le y\le n+1} \left| \int_n^y \frac{\varepsilon(u)}{u} du \right| = 0.$$

It follows from last two relations that

$$\lim_{n\to\infty} \sup_{n\le y\le n+1} \left| \frac{l(y)}{l(n)} - 1 \right| = 0. \tag{2.5}$$

Combining (2.2), (2.3) and (2.5), we have

$$\lim_{n\to\infty} \frac{1}{p(n)} \int_n^{n+1} p(y)dy = 1. \tag{2.6}$$

It is easily seen that

$$\inf_{k\ge n} \frac{1}{p(k)} \int_k^{k+1} p(y)dy \le \frac{\int_n^\infty p(y)dy}{\sum_{k=n}^\infty p(k)} \le \sup_{k\ge n} \frac{1}{p(k)} \int_k^{k+1} p(y)dy. \tag{2.7}$$

The conclusion of the Lemma follows from (2.6) and (2.7). □

Lemma 2.2. *For any $\alpha > 0$*

$$\int_x^\infty \frac{l(y)}{y^{\alpha+1}} dy \sim \frac{l(x)}{\alpha x^\alpha}. \tag{2.8}$$

Proof. By using (2.4), we have

$$\int_x^\infty \frac{l(y)}{y^{\alpha+1}} dy \sim \int_x^\infty \frac{l_0(y)}{y^{\alpha+1}} dy, \tag{2.9}$$

where

$$l_0(y) = \exp\left\{ \int_1^y \frac{\varepsilon(u)}{u} du \right\}. \tag{2.10}$$

Integrating by parts, we conclude that

$$\begin{aligned} \int_x^\infty \frac{l_0(y)}{y^{\alpha+1}} dy &= \frac{l_0(x)}{\alpha x^\alpha} + \frac{1}{\alpha} \int_x^\infty \frac{\varepsilon(u) l_0(y)}{y^{\alpha+1}} dy \\ &= \frac{l_0(x)}{\alpha x^\alpha}(1+o(1)) = \frac{l(x)}{\alpha x^\alpha}(1+o(1)). \end{aligned} \tag{2.11}$$

The desired result follows from (2.9) and (2.11). □

Note that (2.8) can be deduced from the asymptotic formula

$$\int_{\alpha}^{\infty} f(t)l(xt)dt \sim l(x)\int_{\alpha}^{\infty} f(t)dt,$$

where $\alpha > 0$, and $f(t)t^{\eta}$, $\eta > 0$, is integrable (see [8], Theorem 2.6), but not immediately. For this purpose one needs to make the change of variables $y = xt$ in the integral $\int_x^{\infty} y^{-\alpha-1}l(y)dy$. On the other hand, the method which is used in proving Lemma 2.2 allows to obtain very easily the statement the above mentioned Theorem 2.6 of [8].

Corollary 2.3. *Under condition (1.4)*

$$\mathbf{P}(X \geq n) \sim \frac{l(n)}{\alpha n^{\alpha}}. \tag{2.12}$$

Proof. Evidently,

$$\inf_{k\geq n} \frac{l(k)}{k^{\alpha+1}p_k} \leq \frac{\sum\limits_{k=n}^{\infty} l(k)k^{-\alpha-1}}{\sum\limits_{k=n}^{\infty} p_k} \leq \sup_{k\geq n} \frac{l(k)}{k^{\alpha+1}p_k}.$$

Hence, by (2.7)

$$\mathbf{P}(X \geq n) = \sum_{k\geq n} p_k \sim \sum_{k\geq n} \frac{l(k)}{k^{\alpha+1}} \sim \frac{l(n)}{\alpha n^{\alpha}},$$

which was to be proved. □

Lemma 2.4. *For any $\alpha < 1$*

$$\sum_{k=1}^{n} \frac{l(k)}{k^{\alpha}} \sim \frac{l(n)}{1-\alpha} n^{1-\alpha}. \tag{2.13}$$

Proof. Let $l_0(x)$ be defined by (2.10). Since $l_0(x) \sim l(x)$,

$$\sum_{k=1}^{n} \frac{l_0(k)}{k^{\alpha}} \sim \sum_{k=1}^{n} \frac{l(k)}{k^{\alpha}}. \tag{2.14}$$

Indeed,

$$1-\varepsilon < \frac{\sum\limits_{k=n(\varepsilon)}^{n} k^{-\alpha}l_0(k)}{\sum\limits_{k=n(\varepsilon)}^{n} k^{-\alpha}l(k)} < 1+\varepsilon$$

if $n(\varepsilon)$ is such that for $x > n(\varepsilon)$

$$1 - \varepsilon < \frac{l_0(x)}{l(x)} < 1 + \varepsilon.$$

It is easily seen that

$$\lim_{n\to\infty} \sum_{k=n(\epsilon)}^{n} k^{-\alpha} l(k) = \infty.$$

Therefore for sufficiently large n

$$1 - 2\varepsilon < \frac{\sum\limits_{k=n(\varepsilon)}^{n} k^{-\alpha} l_0(k)}{\sum\limits_{k=n(\varepsilon)}^{n} k^{-\alpha} l(k)} < 1 + 2\varepsilon.$$

Since ε can be made as small as we wish, hence the validity of (2.14) follows. By applying the Abel transform, we get

$$\sum_{k=1}^{n} \frac{l_0(k)}{k^\alpha} = l_0(n) \sum_{k=1}^{n} k^{-\alpha} + \sum_{k=1}^{n-1} (l_0(k) - l_0(k+1)) \sum_{j=1}^{k} j^{-\alpha}. \tag{2.15}$$

It is easily seen that

$$l_0(k) - {}_0(k+1) = l_0(k)\left(1 - \exp\left\{ \int_k^{k+1} \frac{\varepsilon(u)}{u} du \right\}\right).$$

Hence

$$|l_0(k) - {}_0(k+1)| < l_0(k) \left| \int_k^{k+1} \frac{\varepsilon(u)}{u} du \right| = o(l_0(k) k^{-1}). \tag{2.16}$$

Further,

$$\sum_{k=1}^{n} k^{-\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}. \tag{2.17}$$

It follows from (2.16) and (2.17)

$$\sum_{k=1}^{n-1} (l_0(k) - l_0(k+1)) \sum_{j=1}^{k} j^{-\alpha} = o\left(\sum_{k=1}^{n} \frac{l_0(k)}{k^\alpha} \right). \tag{2.18}$$

Combining (2.15)–(2.17), we conclude that

$$\sum_{k=1}^{n} l_0(k) k^{-\alpha} \sim \frac{l_0(n)}{1-\alpha} n^{1-\alpha}. \tag{2.19}$$

From (2.14) and (2.19) the result follows. □

Corollary 2.5. *Under conditions of Theorem 1.1*

$$\sum_{k=1}^{n} \mathbf{P}(X \geq k) \sim \frac{l(n)}{\alpha(1-\alpha)} n^{1-\alpha}. \tag{2.20}$$

Proof. According to Corollary 2.3 for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for $n > n(\varepsilon)$

$$1 - \varepsilon < \mathbf{P}(X \geq n) \Big/ \frac{l(n)}{\alpha n^{\alpha}} < 1 + \varepsilon.$$

Hence,

$$1 - \varepsilon < \sum_{n(\varepsilon) < k \leq n} \mathbf{P}(X \geq k) / \alpha^{-1} \sum_{n(\varepsilon) < k \leq n} \frac{l(k)}{k^{\alpha}} < 1 + \varepsilon.$$

On the other hand, since

$$\lim_{n \to \infty} \sum_{n(\varepsilon) < k \leq n} \frac{l(k)}{k^{\alpha}} = \infty$$

for every $\varepsilon > 0$,

$$\sum_{n(\varepsilon) < k \leq n} \frac{l(k)}{k^{\alpha}} \sim \sum_{k=1}^{n} \frac{l(k)}{k^{\alpha}}, \quad \sum_{n(\varepsilon) < k \leq n} \mathbf{P}(X \geq k) \sim \sum_{k=1}^{n} \mathbf{P}(X \geq k).$$

Therefore, for sufficiently large n

$$1 - 2\varepsilon < \alpha \sum_{k=1}^{n} \mathbf{P}(X \geq k) / \sum_{k=1}^{n} \frac{l(k)}{k^{\alpha}} < 1 + 2\varepsilon.$$

Hence, since ε is arbitrary, it follows that

$$\sum_{k=1}^{n} \mathbf{P}(X \geq k) \sim \alpha^{-1} \sum_{k=1}^{n} \frac{l(k)}{k^{\alpha}}.$$

To complete the proof it remains to apply Lemma 2.4. □

Lemma 2.6. *Under conditions of Theorem 1.1*

$$1 - f(z) \sim (1-z)^{\alpha} L\left(\frac{1}{1-z}\right), \tag{2.21}$$

where

$$L(x) = \frac{\Gamma(1-\alpha)}{\alpha} l(x).$$

Proof. First of all,

$$\sum_{k=0}^{n} \mathbf{P}(X > k) z^k = \frac{1 - f(z)}{1 - z}.$$

It is easily seen that

$$\mathbf{P}(X > k) \sim \mathbf{P}(X \geq k).$$

Now, using Corollary 2.5 and the Abelian theorem (see, e.g. [13], Ch. XIII, section 5, Th. 5), we have

$$\begin{aligned}\frac{1 - f(z)}{1 - z} &\sim \frac{\Gamma(2 - \alpha)}{\alpha(1 - \alpha)}(1 - z)^{\alpha - 1} L(1 - z) \\ &= \alpha^{-1}\Gamma(1 - \alpha)(1 - z)^{\alpha - 1} l\Big(\frac{1}{1 - z}\Big) = (1 - z)^{\alpha - 1} L\Big(\frac{1}{1 - z}\Big),\end{aligned}$$

which is equivalent to the assertion of the Lemma. □

Lemma 2.7. *Under conditions of Theorem 1.1*

$$\sum_{k=0}^{n} u_k \sim \frac{n^\alpha}{\Gamma(\alpha + 1) L(n)}, \tag{2.22}$$

where $L(x)$ is defined in Lemma 2.6.

Proof. Obviously,

$$u_k = C_k\Big(\frac{1}{1 - f(z)}\Big).$$

Applying Lemma 2.6 and the Tauberian theorem (see ref. in the proof of Lemma 2.6), we obtain the desired result. □

The next assertion is borrowed from [5].

Lemma 2.8. *The identity*

$$n u_n = \sum_{k=0}^{n-1} (n - k) p_{n-k} u_k^{(2)} \tag{2.23}$$

holds, where $u_n = \sum\limits_{k=0}^{\infty} \mathbf{P}(S_k = n)$, $u_n^{(2)} = \sum\limits_{k=0}^{n} u_{n-k} u_k$.

Lemma 2.9. *Under condition of Theorem 1.1 there exists the sequence θ_n such that $\lim\limits_{n \to \infty} \theta_n = 1$ and*

$$u_n^{(2)} \leq \frac{2^{1-\alpha}\theta_n n^\alpha}{\Gamma(\alpha + 1) L(n)} \max_{n/2 \leq k \leq n} u_k. \tag{2.24}$$

Proof. It is easily seen that

$$u_n^{(2)} \le 2 \sum_{0\le k\le n/2} u_k u_{n-k} \le 2 \max_{n/2\le k\le n} u_k \sum_{0\le k\le n/2} u_k.$$

To complete the proof it is sufficient to apply Lemma 2.7. □

Lemma 2.10. *Under conditions of Theorem 1.1*

$$\sum_{k=1}^{n} u_k^{(2)} \sim \frac{n^{2\alpha}}{\Gamma(2\alpha+1)L^2(n)}, \tag{2.25}$$

where $L(x)$ *is defined in Lemma 2.6.*

Proof. It is easily seen that

$$u_k^{(2)} = C_k\left(\frac{1}{(1-f(z))^2}\right).$$

According to Lemma 2.6

$$(1-f(z))^{-2} \sim (1-z)^{-2\alpha} L^{-2}\left(\frac{1}{1-z}\right).$$

Applying the Tauberian theorem (see ref. in the proof of Lemma 2.6), we get the desired result. □

Lemma 2.11. *Under conditions of Theorem 1.1 for every fixed* $0 < a < b < 1$

$$\sum_{na\le k\le nb} l^{-2}(k)k^{2\alpha-1}(n-k)^{-\alpha} \sim l^{-2}(n)n^{\alpha}\int_a^b u^{2\alpha-1}(1-u)^{-\alpha}du. \tag{2.26}$$

Proof. First of all, notice that

$$\ln\frac{l_0(n)}{l_0(k)} = \int_k^n \frac{\varepsilon(u)}{u}du. \tag{2.27}$$

Consequently,

$$\lim_{n\to\infty} \sup_{na\le k\le nb}\left|\frac{l_0(n)}{l_0(k)} - 1\right| = 0. \tag{2.28}$$

This implies that

$$\sum_{na\le k\le nb} l_0^{-2}(k)k^{2\alpha-1}(n-k)^{-\alpha} \sim l_0^{-2}(n)\sum_{na\le k\le nb} k^{2\alpha-1}(n-k)^{-\alpha}.$$

Hence it follows that

$$\sum_{na\le k\le nb} l^{-2}(k)k^{2\alpha-1}(n-k)^{-\alpha} \sim l^{-2}(n) \sum_{na\le k\le nb} k^{2\alpha-1}(n-k)^{-\alpha}.$$

Further,

$$\begin{aligned}\sum_{na\le k\le nb} k^{2\alpha-1}(n-k)^{-\alpha} &= n^{\alpha-1} \sum_{na\le k\le nb} \left(\frac{k}{n}\right)^{2\alpha-1}\left(1-\frac{k}{n}\right)^{-\alpha}\\ &\sim n^{\alpha}\int_a^b u^{2\alpha-1}(1-u)^{-\alpha}du.\end{aligned}$$

The result follows from last two relations. □

3. The proof of Theorem 1.1

Let us write down formula (2.23) in the form

$$\begin{aligned} nu_n &= \Bigg(\sum_{0\le k<\sqrt{n}} + \sum_{\sqrt{n}\le k\le(1-\eta)n} + \sum_{(1-\eta)n<k\le n}\Bigg)(n-k)p_{n-k}u_k^{(2)} \\ &\equiv \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \end{aligned} \tag{3.1}$$

where $0<\eta<1$. For any $\varepsilon>0$, sufficiently large n, and $k<\sqrt{n}$

$$p_{n-k} < (1+\varepsilon)\frac{l(n-k)}{(n-\sqrt{n})^{\alpha+1}}. \tag{3.2}$$

If $n-\sqrt{n}\le k\le n$, then

$$\frac{l_0(n)}{l_o(k)} = \exp\left\{\int_k^n \frac{\varepsilon(u)}{u}du\right\} = 1+o(\ln n - \ln(n-\sqrt{n})) = 1+o\left(\frac{1}{\sqrt{n}}\right).$$

Consequently,

$$\max_{n-\sqrt{n}\le k\le n} l_0(k) \sim l_0(n). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\sum\nolimits_1 = O\left(\frac{l(n)}{n^{\alpha}}\sum_{k=1}^{[\sqrt{n}]} u_k^{(2)}\right).$$

By Lemma 2.10

$$\sum_{k=1}^{[\sqrt{n}]} u_k^{(2)} = O\left(\frac{n^\alpha}{l^2(\sqrt{n})}\right). \tag{3.4}$$

Thus,

$$\sum\nolimits_1 = O\left(\frac{1}{l(\sqrt{n})}\right). \tag{3.5}$$

Let us turn to estimating $\sum_2$. It is easily seen that

$$\sum\nolimits_2 \sim \sum_{\sqrt{n}\le k\le (1-\eta)n} u_k^{(2)} \frac{l_0(n-k)}{(n-k)^\alpha} \equiv \sum\nolimits_4 . \tag{3.6}$$

Applying Abel's transformation, we have

$$\begin{aligned}\sum\nolimits_4 \sim \frac{l_0(n-\sqrt{n})}{(n-\sqrt{n})^\alpha} &\sum_{\sqrt{n}\le k\le (1-\eta)n} u_k^{(2)} \\ - \sum_{\sqrt{n}\le k\le (1-\eta)n} &\left(\frac{l_0(n-k-1)}{(n-k-1)^\alpha} - \frac{l_0(n-k)}{(n-k)^\alpha}\right) \sum_{j=[\sqrt{n}]}^{k} u_j^{(2)}.\end{aligned} \tag{3.7}$$

By Lemma 2.10

$$\sum_{\sqrt{n}\le k\le (1-\eta)n} u_k^{(2)} = \sum_{k\le (1-\eta)n} u_k^{(2)} - \sum_{k<\sqrt{n}} u_k^{(2)} \sim \frac{(1-\eta)^{2\alpha} n^{2\alpha}}{\Gamma(2\alpha+1)L^2(n)}. \tag{3.8}$$

Further,

$$\frac{l_0(k)}{k^\alpha} - \frac{l_0(k+1)}{(k+1)^\alpha} = l_0(k)\left(\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha}\right) + \frac{l_0(k)-l_0(k+1)}{(k+1)^\alpha}. \tag{3.9}$$

Obviously,

$$\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} \sim \frac{\alpha}{k^{\alpha+1}}. \tag{3.10}$$

On the other hand,

$$\begin{aligned} l_0(k+1) - l_0(k) &= l_0(k)\left(\frac{l_0(k+1)}{l_0(k)} - 1\right) \\ &= l_0(k)\left(\exp\left\{\int_k^{k+1} \frac{\varepsilon(u)}{u} du\right\} - 1\right) = o\left(\frac{l_0(k)}{k}\right).\end{aligned} \tag{3.11}$$

It follows from (3.9)–(3.11) that

$$\frac{l_0(k)}{k^\alpha} - \frac{l_0(k+1)}{(k+1)^\alpha} \sim \frac{\alpha l_0(k)}{k^{\alpha+1}}. \tag{3.12}$$

Combining (3.6)–(3.8) and (3.12), we obtain

$$\begin{aligned}
\sum\nolimits_2 &\sim \frac{(1-\eta)^{2\alpha} l_0(n) n^\alpha}{\Gamma(2\alpha+1)L^2(n)} - \alpha \sum_{\sqrt{n}\le k\le(1-\eta)n} \frac{l_0(n-k)}{(n-k)^{\alpha+1}} \sum_{j=[\sqrt{n}]}^{k} u_j^{(2)} \\
&= \frac{(1-\eta)^{2\alpha}\alpha n^\alpha}{\Gamma(1-\alpha)\Gamma(2\alpha+1)a(n)L(n)} \\
&- \alpha \sum_{\sqrt{n}\le k\le(1-\eta)n} \frac{l_0(n-k)}{(n-k)^{\alpha+1}} \sum_{j=0}^{k} u_j^{(2)} + \alpha \sum_{j=0}^{\sqrt{[n]}-1} u_j^{(2)} \sum_{\sqrt{n}\le k\le(1-\eta)n} \frac{l_0(n-k)}{(n-k)^{\alpha+1}} \\
&= \frac{(1-\eta)^{2\alpha}\alpha n^\alpha}{\Gamma(1-\alpha)\Gamma(2\alpha+1)a(n)L(n)} - \alpha \sum\nolimits_5 + \alpha \sum\nolimits_6 .
\end{aligned} \tag{3.13}$$

Here $a(\cdot)$ is a factor in Karamata's representation (2.4) for $l(x)$. In view of (3.4)

$$\sum\nolimits_6 = O\left(\frac{l_0(n)}{l_0^2(\sqrt{n})}\right). \tag{3.14}$$

We now proceed to estimating $\sum_5$. By Lemma 2.10

$$\sum\nolimits_5 \sim c(\alpha) \sum_{\sqrt{n}\le k\le(1-\eta)n} L^{-2}(k)k^{2\alpha} \frac{l_0(n-k)}{(n-k)^{\alpha+1}} \equiv c(\alpha)\sum\nolimits_7, \tag{3.15}$$

where $c(\alpha) = 1/\Gamma(2\alpha+1)$. Applying the Abel transformation, we have

$$\begin{aligned}
\sum\nolimits_7 &\sim L^{-2}(n)(1-\eta)^{2\alpha} n^{2\alpha} \sum_{\sqrt{n}\le k\le(1-\eta)n} \frac{l_0(n-k)}{(n-k)^{\alpha+1}} \\
&- \sum_{\sqrt{n}\le k\le(1-\eta)n} (L^{-2}(k+1)(k+1)^{2\alpha} - L^{-2}(k)k^{2\alpha}) \sum_{j=[\sqrt{n}]}^{k} \frac{l_0(n-j)}{(n-j)^{\alpha+1}}.
\end{aligned} \tag{3.16}$$

In the same way as (3.12) we deduce that

$$L^{-2}(k+1)(k+1)^{2\alpha} - L^{-2}(k)k^{2\alpha} \sim 2\alpha L^{-2}(k)k^{2\alpha-1}.$$

Hence, denoting the second summand in (3.16) by $\sum_8$, we obtain

$$\sum\nolimits_8 \sim 2\alpha \sum_{\sqrt{n}\le k\le(1-\eta)n} L^{-2}(k)k^{2\alpha-1} \sum_{j=[\sqrt{n}]}^{k} \frac{l_0(n-j)}{(n-j)^{\alpha+1}}$$

$$\sim 2\alpha l_0(n) \sum_{\sqrt{n} \le k \le (1-\eta)n} L^{-2}(k) k^{2\alpha-1} \sum_{j=[\sqrt{n}]}^{k} (n-j)^{-\alpha-1}. \tag{3.17}$$

It is not difficult to check that for $\sqrt{n} \le k \le (1-\eta)n$

$$\alpha \sum_{j=\lfloor\sqrt{n}\rfloor} (n-j)^{-\alpha-1} = (n-k)^{-\alpha} - n^{-\alpha} + o(n^{-\alpha}).$$

Consequently,

$$\sum\nolimits_8 + 2n^{-\alpha} \sum_{\sqrt{n} \le k \le (1-\eta)n} L^{-2}(k) k^{2\alpha-1} \sim 2l_0(n) \sum_{\sqrt{n} \le k \le (1-\eta)n} L^{-2}(k) k^{2\alpha-1} (n-k)^{-\alpha}. \tag{3.18}$$

We need the identity

$$\begin{aligned} \sum_{\sqrt{n} \le k \le (1-\eta)n} &= \Bigg(\sum_{\sqrt{n} \le k < \varepsilon n} + \sum_{\varepsilon n \le k \le (1-\eta)n} \Bigg) L^{-2}(k) k^{2\alpha-1} (n-k)^{-\alpha} \\ &\equiv \sum\nolimits_9 + \sum\nolimits_{10}. \end{aligned} \tag{3.19}$$

It is easily seen that

$$\sum\nolimits_9 < (1-\varepsilon)^{-\alpha} n^{-\alpha} \sum_{\sqrt{n} \le k \le \varepsilon n} L^{-2}(k) k^{2\alpha-1}.$$

By using Lemma 2.4, we obtain that

$$\sum_{\sqrt{n} \le k \le \varepsilon n} L^{-2}(k) k^{2\alpha-1} \sim \frac{(\varepsilon n)^{2\alpha}}{2\alpha L^2(n)}.$$

Therefore, for sufficiently large n

$$\sum\nolimits_9 < (1-\varepsilon)^{-\alpha} \frac{\varepsilon^{2\alpha} n^{\alpha}}{2\alpha L^2(n)}. \tag{3.20}$$

On the other hand, by Lemma 2.11

$$\sum\nolimits_{10} \sim L^{-2}(n) n^{\alpha} \int_{\varepsilon}^{1-\eta} u^{2\alpha-1} (1-u)^{-\alpha} du. \tag{3.21}$$

It follows from (3.18) – (3.21) that

$$\sum\nolimits_8 + \frac{(1-\eta)^{2\alpha} n^{\alpha}}{\alpha L^2(n)} \sim \frac{2\alpha^2 n^{\alpha}}{\Gamma^2(1-\alpha) l(n)} \int_0^{1-\eta} u^{2\alpha-1} (1-u)^{-\alpha} du. \tag{3.22}$$

Combining (3.15), (3.16), (3.18) and (3.22) we obtain

$$\sum\nolimits_5 \sim \frac{(1-\eta)^{2\alpha}\alpha n^{\alpha}}{\Gamma(1-\alpha)\Gamma(2\alpha+1)L(n)} - \frac{2\alpha^2 n^{\alpha}}{\Gamma^2(1-\alpha)\Gamma(2\alpha+1)l(n)} I(\eta), \tag{3.23}$$

where $I(\eta) = \int\limits_0^{1-\eta} u^{2\alpha-1}(1-a)^{-\alpha}du$. Finally, it follows from (3.13), (3.14) and (3.23) that

$$\sum\nolimits_2 \sim \frac{2\alpha^3 n^{\alpha}}{\Gamma^2(1-\alpha)\Gamma(2\alpha+1)l(n)} I(\eta). \tag{3.24}$$

We now turn to estimating $\sum_3$. Evidently,

$$\sum\nolimits_3 < \max_{(1-\eta)n<k\le n} u_k^{(2)} \sum_{(1-\eta)n<k\le n} (n-k)p_{n-k}.$$

By Lemma 2.4

$$\sum_{(1-\eta)n<k\le n} (n-k)p_{n-k} \sim \sum_1^{[\eta n]} \frac{l(j)}{j^{\alpha}} \sim \frac{l(n)}{1-\alpha}(\eta n)^{1-\alpha}.$$

On the other hand, in view of (2.24)

$$\max_{(1-\eta)n<k\le n} u_k^{(2)} < \frac{2^{1-\alpha}n^{\alpha}}{\Gamma(\alpha+1)} \max_{(1-\eta)n<k\le n} \frac{\theta_k}{L(k)} \max_{(1-\eta)n/2\le j\le n} u_j.$$

As a result we obtain that

$$\sum\nolimits_3 = n\psi(n)(2\eta)^{1-\alpha} \max_{\delta n\le j\le n} u_j, \tag{3.25}$$

where

$$\psi(n) = \frac{\alpha b_n}{\Gamma(\alpha+1)\Gamma(1-\alpha)(1-\alpha)}, \quad 0 < \limsup_{n\to\infty} b_n \le 1, \quad \delta = \frac{1-\eta}{2}.$$

Notice that

$$\frac{\alpha}{\Gamma(\alpha+1)\Gamma(1-\alpha)} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin\pi\alpha}{\pi}$$

(see [14], formula 8.334, 3). Consequently,

$$\psi(n) = \frac{\sin\pi\alpha}{(1-\alpha)\pi} b_n. \tag{3.26}$$

It follows from (3.1), (3.5), (3.24) and (3.25) that

$$u_n = \varphi(n) + (2\eta)^{1-\alpha}\psi(n) \max_{\delta n\le j\le n} u_j, \tag{3.27}$$

where

$$\varphi(n) = \frac{2\alpha^3 a_n n^{\alpha-1} I(\eta)}{\Gamma^2(1-\alpha)\Gamma(2\alpha+1)l(n)}, \ a_n \sim 1.$$

Let us fix $0 < \varepsilon < 1/2$. Let η be such that $(2\eta)^{1-\alpha} < \varepsilon$. Chose N so that $\psi(n) < 1$ for $n > N$. Let n_1 be the value of k for which $\max\limits_{\delta n \le k \le n} u_k$ is attained. In particular, it may be that $n_1 = n$. In this case $u_n < \varphi(n)/(1-\varepsilon)$. If $N < n_1 < n$, then

$$u_{n_1} < \varphi(n_1) + \varepsilon \max_{\delta n_1 \le j \le n_1} u_j$$

and consequently

$$u_n < \varphi(n) + \varepsilon\varphi(n_1) + \varepsilon^2 \max_{\delta n_1 \le j \le n_1} u_j. \tag{3.28}$$

If $\max\limits_{\delta n_1 \le j \le n_1} u_j = u_{n_1}$, then $u_{n_1} < \varphi(n_1)/(1-\varepsilon)$. Substituting this bound in (3.28), we have

$$u_n < \varphi(n) + \varepsilon\varphi(n_1) + \frac{\varepsilon^2}{1-\varepsilon}\varphi(n_1).$$

If $\max\limits_{\delta n_1 \le j \le n_1} u_j$ is attained for $N < j < n_1$, then, similarly, the following inequality is deduced

$$u_n < \varphi(n) + \varepsilon\varphi(n_1) + \varepsilon^2\varphi(n_2) + \frac{\varepsilon^3}{1-\varepsilon} \max_{\delta n_2 \le j \le n_2} u_j$$

and so forth.

There exist two possibilities: either for some $n_k > N$

$$\max_{\delta n_k \le j \le n_k} u_j = u_{n_k},$$

or for some $k = k_0$ the inequality $n_k < N$ is fulfilled. Consider the first case. First of all, notice that $n_k \ge \delta^k n$. Using Karamata's representation (2.4) for $l(n)$, we obtain

$$\frac{\varphi(n_j)}{\varphi(n)} = \frac{a_n a(n)}{a_{n_j} a(n_j)} \left(\frac{n}{n_j}\right)^{1-\alpha} \exp\left\{ -\int_{n_j}^{n} \frac{\varepsilon(u)}{u} \right\}.$$

Evidently,

$$\left| \int_{n_j}^{n} \frac{\varepsilon(u)}{u} du \right| < \sup_{n_j \le u \le n} |\varepsilon(u)| \ln \frac{n}{n_j} < -j\gamma \ln \delta, \ \gamma = \sup_{u > N} |\varepsilon(u)|.$$

Consequently, there exists ε_0 such that for $\varepsilon < \varepsilon_0$

$$\varepsilon^j \varphi(n_j) < \varepsilon^j \varphi(n) \exp\left\{ j\gamma \ln 2 \right\} < \varepsilon^{j/2}.$$

As a result we get that for $\varepsilon < \varepsilon_0$

$$u_n < \sum_{j=0}^{k-1} \varepsilon^j \varphi(n_j) + \frac{\varepsilon^k}{1-\varepsilon}\varphi(n_k) < \left(\sum_{j=0}^{k-1} \varepsilon^{j/2} + \frac{\varepsilon^{k/2}}{1-\varepsilon}\right)\varphi(n) < \frac{\varphi(n)}{1-\varepsilon^{1/2}}. \quad (3.29)$$

In the second case the recursion stops for $k = k_0 = \min\{k : n_k < N\}$. As a result we arrive at the bound

$$u_n < \frac{\varphi(n)}{1-\varepsilon^{1/2}} + \frac{\varepsilon^{k_0-1}}{1-\varepsilon} \max_{k\geq 0} u_k. \quad (3.30)$$

Since $n_k \geq \delta^k n$, $k_0 \geq \log_\delta \frac{N}{n}$. It implies that $\varepsilon^{k_0} \leq \exp\{-2^{-1}\ln\varepsilon \log_\delta n\}$ for $n > N^2$. Consequently, for sufficiently small ε

$$\varepsilon^{k_0} = o(n^{-2}) = o(\varphi(n)). \quad (3.31)$$

It follows from (3.30) and (3.31) that $u_n < 2\varphi(n)$ for $n > N^2$ if ε sufficiently small. Returning to (3.27) we conclude that for sufficiently large n

$$0 < l(n)n^{1-\alpha}u_n - a_n c_1(\alpha)I(\eta) < 2\varepsilon n^{1-\alpha} l(n) \max_{\delta n \leq k \leq n} \varphi(k),$$

where $c_1(\alpha) = 2\alpha^3/\Gamma^2(1-\alpha)\Gamma(2\alpha+1)$. It is easily seen that

$$\limsup_{n\to\infty} n^{1-\alpha} l(n) \max_{\delta n \leq k \leq n} \varphi(k) \leq \delta^{\alpha-1} c_1(\alpha) I(\eta).$$

It follows from two latter relations that

$$\lim_{n\to\infty} l(n)n^{1-\alpha}u_n = c_1(\alpha)I(0). \quad (3.32)$$

It remains to calculate $c_1(\alpha)I(0)$. Obviously,

$$I(0) = B(2\alpha, 1-\alpha) = \frac{\Gamma(2\alpha)\Gamma(1-\alpha)}{\Gamma(1+\alpha)}.$$

Consequently,

$$c_1(\alpha)I(0) = \frac{2\alpha^3\Gamma(2\alpha)}{\Gamma(1-\alpha)\Gamma(2\alpha+1)\Gamma(1+\alpha)} = \frac{\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)} = \frac{\alpha\sin\pi\alpha}{\pi}. \quad (3.33)$$

It follows from (3.32) and (3.33) that

$$\lim_{n\to\infty} l(n)n^{1-\alpha}u_n = \frac{\alpha\sin\pi\alpha}{\pi}$$

On the other hand, by (2.12)

$$\frac{\mathbf{P}(X=n)}{\mathbf{P}^2(X\geq n)} \sim \frac{\alpha^2}{l(n)n^{1-\alpha}}.$$

Hence,

$$\frac{\sin\pi\alpha}{\pi\alpha}\frac{\mathbf{P}(X=n)}{\mathbf{P}^2(X\geq n)} \sim \frac{\alpha\sin\pi\alpha}{\pi l(n)n^{1-\alpha}} \sim u_n,$$

which was to be proved.

4. The proof of Theorem 1.2

According to definition

$$h_n = C_n(-\ln(1-f(z))).$$

Hence,

$$nh_n = C_n\left(\frac{f'(z)}{1-f(z)}\right).$$

Consequently,

$$h_n = \frac{1}{n}\sum_{k=0}^{n}(k+1)p_{k+1}u_{n-k}. \tag{4.1}$$

Applying Theorem 1.1, we have

$$\sum_{\varepsilon n\le k\le(1-\varepsilon)n}(k+1)p_{k+1}u_{n-k} \sim \frac{\alpha\sin\pi\alpha}{\pi}\sum_{\varepsilon n\le k\le(1-\varepsilon)n}(k+1)^{-\alpha}(n-k)^{\alpha-1}$$

$$\sim \frac{\alpha\sin\pi\alpha}{\pi}\int_{\varepsilon}^{1-\varepsilon}u^{-\alpha}(1-u)^{\alpha-1}du \equiv \frac{\alpha\sin\pi\alpha}{\pi}I(\varepsilon). \tag{4.2}$$

On the other hand, applying Lemmas 2.4 and 2.7, we have

$$\limsup_{n\to\infty}\sum_{0\le k<\varepsilon n}(k+1)p_{k+1}u_{n-k} < \frac{\alpha}{\pi(1-\alpha)}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-\alpha} \tag{4.3}$$

and

$$\limsup_{n\to\infty}\sum_{(1-\varepsilon)n<k\le n}(k+1)p_{k+1}u_{n-k} < \frac{1}{\pi}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{\alpha}. \tag{4.4}$$

It follows from (4.2)–(4.4) that

$$\lim_{n\to\infty}\sum_{k=0}^{n}(k+1)p_{k+1}u_{n-k} = \alpha\frac{\sin\pi\alpha}{\pi}I(0). \tag{4.5}$$

Obviously,

$$I(0) = B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}. \tag{4.6}$$

Combining (4.1), (4.5), (4.6), we obtain that

$$h_n \sim \frac{\alpha}{n},$$

which was to be proved.

Acknowledgments. I thank the referee for helpful remarks.

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