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Strong limit theorems for random fields

Allan Gut

Department of Mathematics, Uppsala University, Uppsala, Sweden
e-mail: allan.gut@math.uu.se

Dedicated to Mátyás Arató on his eightieth birthday

Abstract

The aim of the present paper is to review some joint work with Ulrich Stadtmüller concerning random field analogs of the classical strong laws.

In the first half we start, as background information, by quoting the law of large numbers and the law of the iterated logarithm for random sequences as well as for random fields, and the law of the single logarithm for sequences. We close with a one-dimensional LSL pertaining to windows, whose edges expand in an "almost linear fashion", viz., the length of the nth window equals, for example, $n/\log n$ or $n/\log\log n$. A sketch of the proof will also be given.

The second part contains some extensions of the LSL to random fields, after which we turn to convergence rates in the law of large numbers. Departing from the now legendary Baum–Katz theorem in 1965, we review a number of results in the multiindex setting. Throughout main emphasis is on the case of "non-equal expansion rates", viz., the case when the edges along the different directions expand at different rates. Some results when the power weights are replaced by almost exponential weights are also given.

We close with some remarks on martingales and the strong law.

Keywords: i.i.d. random variables, law of large numbers, law of the iterated logarithm, law of the single logarithm, random field, multiindex.

MSC: Primary 60F05, 60F15, 60G70, 60G60; Secondary 60G40.

1. Introduction

Let $X, X_1, X_2, \dots$ be independent, identically distributed (i.i.d.) random variables with partial sums S_n, $n \geq 1$, and set $S_0 = 0$. The two most famous strong laws are the Kolmogorov strong law and the Hartman–Wintner Law of the iterated logarithm:

Theorem 1.1 (The Kolmogorov strong law — LLN)**.** *Suppose that X, X_1, X_2, ... are i.i.d. random variables with partial sums S_n, $n \geq 1$.*

(a) *If $E|X| < \infty$ and $E\,X = \mu$, then*

$$\frac{S_n}{n} \stackrel{a.s.}{\to} \mu \quad as \quad n \to \infty.$$

(b) *If $\frac{S_n}{n} \stackrel{a.s.}{\to} c$ for some constant c, as $n \to \infty$, then*

$$E|X| < \infty \quad and \quad c = E\,X.$$

(c) *If $E|X| = \infty$, then*

$$\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty.$$

Remark 1.2. Strictly speaking, we presuppose in (b) that the limit can only be a constant. That this is indeed the case follows from the Kolmogorov zero–one law. Considering this, (c) is somewhat more general than (b). For proofs and details, see e.g. Gut (2007), Chapter 6.

Theorem 1.3 (The Hartman–Wintner law of the iterated logarithm — LIL)**.** *Suppose that X, X_1, X_2, ... are i.i.d. random variables with mean 0 and finite variance σ^2, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Then*

$$\limsup_{n\to\infty} \, (\liminf_{n\to\infty}) \frac{S_n}{\sqrt{2\sigma^2 n \log\log n}} = +1 \quad (-1) \quad a.s. \tag{1.1}$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n \log\log n}} < \infty\Big) > 0,$$

then $E\,X^2 < \infty$, $E\,X = 0$, and (1.1) holds.

The sufficiency is due to Hartman and Wintner (1941). The necessity is due to Strassen (1966). For this and more, see e.g. Gut (2007), Chapter 8.

Remark 1.4. The Kolmogorov zero–one law tells us that the limsup is finite with probability zero or one, and, if finite, the limit equals a constant almost surely. Thus, assuming in the converse that the probability is positive is in reality assuming that it is equal to 1. This remark also applies to (e.g.) Theorem 1.8.

The Kolmogorov strong law, which relates to the first moment, was generalized by Marcinkiewicz and Zygmund (1937) into a result relating to moments of order between 0 and 2; cf. also Gut (2007), Section 6.7:

Theorem 1.5 (The Marcinkiewicz–Zygmund strong law)**.** *Let $0 < r < 2$. Suppose that X, X_1, X_2, ... are i.i.d. random variables. If $E|X|^r < \infty$ and $E\,X = 0$ when $1 \leq r < 2$, then*

$$\frac{S_n}{n^{1/r}} \stackrel{a.s.}{\to} 0 \quad as \quad n \to \infty \quad \Longleftrightarrow \quad E|X|^r < \infty \quad and, if \quad 1 \leq r < 2, \; E\,X = 0.$$

The results so far pertain to partial sums, summing from X_1 and onwards. There exist, however, analogs pertaining to *delayed sums* or *windows* or *lag sums*, that have not yet reached the same level of attention, most likely because they are more recent.

In order to describe these results we define the concept of a *window*, say. Namely for any given sequence $X_1, X_2, \ldots$ we set

$$T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j, \quad n \geq 0,\, k \geq 1.$$

The analogs of the strong law large numbers and the law of the iterated logarithm are due to Chow (1973) and Lai (1974), respectively.

Theorem 1.6 (Chow's strong law for delayed sums)**.** *Let* $0 < \alpha < 1$*, suppose that* $X, X_1, X_2, \ldots$ *are i.i.d. random variables, and set* $T_{n,n+n^\alpha} = \sum_{k=n+1}^{n+n^\alpha} X_k$*,* $n \geq 1$*. Then*

$$\frac{T_{n,n+n^\alpha}}{n^\alpha} \overset{a.s.}{\to} 0 \quad \Longleftrightarrow \quad E|X|^{1/\alpha} < \infty \quad \text{and} \quad E\,X = 0.$$

This result has been extended in Bingham and Goldie (1988) by replacing the window width n^α by a self-neglecting function $\phi(n)$ which includes regularly varying functions $\phi(\cdot)$ of order $\alpha \in (0, 1)$.

Remark 1.7. As pointed out in Chow (1973), the strong law remains valid for $\alpha = 1$, since

$$\frac{T_{n,2n}}{n} = 2 \cdot \frac{S_{2n}}{2n} - \frac{S_n}{n} \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

whenever the mean is finite and equals zero.

In analogy with the LIL, where an *iterated logarithm* appears in the normalisation, the following result, due to Lai (1974), is called the *law of the single logarithm* (LSL).

Theorem 1.8 (Lai's law of the single logarithm — LSL)**.** *Let* $0 < \alpha < 1$*. Suppose that* $X, X_1, X_2, \ldots$ *are i.i.d. random variables with mean* 0 *and variance* σ^2*, and set* $T_{n,n+n^\alpha} = \sum_{k=n+1}^{n+n^\alpha} X_k$*,* $n \geq 1$*. If*

$$E\,|X|^{2/\alpha}\big(\log^+|X|\big)^{-1/\alpha} < \infty,$$

then,

$$\limsup_{n\to\infty}\,(\liminf_{n\to\infty})\frac{T_{n,n+n^\alpha}}{\sqrt{2n^\alpha \log n}} = \sigma\sqrt{1-\alpha} \quad (-\sigma\sqrt{1-\alpha}) \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{n,n+n^\alpha}|}{\sqrt{n^\alpha \log n}} < \infty\Big) > 0,$$

then

$$E\,|X|^{2/\alpha}\big(\log^+|X|\big)^{-1/\alpha} < \infty \quad \text{and} \quad E\,X = 0.$$

We remark, in passing, that results of this kind may be useful for the evaluation of weighted sums of i.i.d. random variables for certain classes of weights, for example in connection with certain summability methods; see e.g., Bingham (1984), Bingham and Goldie (1983), Bingham and Maejima (1985), Chow (1973).

The aim of this paper is, in the first half, to present a survey of random field analogs, although with main focus on the LSL. We shall therefore content ourselves by simply providing appropriate references for the law of large numbers and the law of the iterated logarithm. However, our first result is an LSL for sequences, where the windows expand in an "almost linear fashion", viz., the length of the nth window equals, for example, $n/\log n$ or $n/\log\log n$. A skeleton of the proof will be given in Subsection 2.1, and a sketch in Subsection 2.2.

In the second part we first present some extensions of the LSL to random fields, that is, we consider a collection of i.i.d. random variables indexed by $\mathbf{Z}_+^d$, the positive integer d-dimensional lattice, and prove analogous results in that setting. Main emphasis is on the case when the expansion rates in the components are different.

Finally we turn to convergence rates in the law of large numbers. Departing from the legendary Baum–Katz (1965) theorem, more precisely, the Hsu–Robbins–Erdős–Spitzer–Baum–Katz theorem, relating the finiteness of sums such as $\sum_{n=1}^{\infty} n^{\text{power}} P(|S_n| > n^{\text{power}}\varepsilon)$ to moment conditions, we review a number of results in the multiindex setting. Once again, the non-equal expansion rates are the main point. Some results when the power weights are replaced by almost exponential weights are also presented.

A final section contains some remarks on martingale proofs of the law of large numbers and their relation to the classical proofs.

We close this introduction with some pieces of notation and conventions:

- For all results concerning the limsup of a sequence there exist "obvious" analogs for the liminf.
- In the following we shall, at times, for mutual convenience, abuse the notation "iff" to be interpreted as in, for example, Theorems 1.3 and 1.8 in LIL- and LSL-type results.
- C with or without indices denote(s) numerical constants of no importance that may differ between appearances.
- Any random variable without index denotes a generic random variable with respect to the sequence or field of i.i.d. random variables under investigation.
- $\log^+ x = \max\{\log x, 1\}$ for $x > 0$. We shall, however, occasionally be sloppy about the additional $+$-sign within computations.
- For simplicity, we shall permit ourselves, when convenient, to treat quantities such as n^α or $n/\log n$, and so on, as integers.
- Empty products, such as $\prod_{i=1}^{0} = 1$.

2. Between the LIL and LSL

There exist two boundary cases with respect to Theorem 1.8; the cases $\alpha = 0$ and $\alpha = 1$.

The case $\alpha = 0$ contains the trivial one; when the window reduces to a single random variable. More interesting are the windows $T_{n,n+\log n}$, $n \geq 1$, for which the so-called Erdős–Rényi law (cf. Erdős and Rényi (1970), Theorem 2, Csörgő and Révész (1981), Theorem 2.4.3) tells us that if $E\,X = 0$, and the moment generating function $\psi_X(t) = E\,\exp\{tX\}$ exists in a neighborhood of 0, then, for any $c > 0$,

$$\lim_{n\to\infty} \max_{0\leq k\leq n-k} \frac{T_{k,k+c\log k}}{c\log k} = \rho(c) \quad \text{a.s.},$$

where

$$\rho(c) = \sup\{x : \inf_t e^{-tx}\psi_X(t) \geq e^{-1/c}\},$$

where, in particular, we observe that the limit depends on the actual distribution of the summands.

For a generalization to more general window widths a_n, such that $a_n/\log n \to \infty$ as $n \to \infty$, but still assuming that the moment generating function exists, we refer, e.g., to Csörgő and Révész (1981), Theorem 3.1.1. Results where the moment condition is somewhat weaker than existence of a moment generating function were discussed in Lanzinger and Stadtmüller (2000).

For the boundary case at the other end, viz., $\alpha = 1$, one has $a_n = n$ and $T_{n,2n} \stackrel{d}{=} S_n$ and the correct norming is as in the LIL.

An interesting remaining case is when the window size is larger than any power less than one, and at the same time not quite linear. In order to present that one we need the concept of slow variation.

Definition 2.1. Let $a > 0$. A positive measurable function L on $[a,\infty)$ *varies slowly* at infinity, denoted $L \in \mathcal{SV}$, iff

$$\frac{L(tx)}{L(t)} \to 1 \quad \text{as} \quad t\to\infty \quad \text{for all} \quad x > 0.$$

The typical example one should have in mind is $L(x) = \log x$ (or possibly $L(x) = \log\log x$). Every positive function with a finite limit as $x \to \infty$ is slowly varying. An excellent source is Bingham, Goldie and Teugels (1987). Some basic facts can be found in Gut (2007), Section A.7.

With this definition in mind, our windows thus are of the form

$$T_{n,n+n/L(n)}, \tag{2.1}$$

where

$$L \in \mathcal{SV},\ L(\cdot) \nearrow \infty,\ L \text{ is differentiable, and } \frac{xL'(x)}{L(x)} \searrow \quad \text{as} \quad x\to\infty. \tag{2.2}$$

Here is now the corresponding LSL from Gut et al. (2010).

Theorem 2.2. *Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with mean* 0 *and finite variance σ^2. Set, for $n \geq 2$,*

$$d_n = \log \frac{n}{a_n} + \log\log n = \log L(n) + \log\log n,$$

and

$$f(n) = \min\{a_n \cdot d_n,\, n\},$$

where $f(\cdot)$ is an increasing interpolating function, i.e., $f(x) = f_{[x]}$ for $x > 0$. Then, with $f^{-1}(\cdot)$ being the corresponding (suitably defined) inverse function,

$$\limsup_{n\to\infty} \frac{T_{n,n+a_n}}{\sqrt{2a_n d_n}} = \sigma \quad a.s. \quad \Longleftrightarrow \quad E\big(f^{-1}(X^2)\big) < \infty.$$

Remark 2.3. The "natural" necessary moment assumption is the given one with $f(n) = a_n d_n$. However, for very slowly increasing functions, such as $L(x) = \log\log\log\log x$, we have $f(n) = n$, that is the moment condition is equivalent to finite variance in such cases.

In order to get a flavor of the result, we begin by providing some examples. In the following two subsections we shall encounter a skeleton of the proof as well as a sketch of the same.

First, the two "obvious ones".

Example 2.4. If for some $p > 0$

$$E\,X^2 \frac{(\log^+ |X|)^p}{\log^+ \log^+ |X|} < \infty,$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/(\log n)^p}}{\sqrt{2(p+1)\frac{n}{(\log n)^p} \log\log n}} = \sigma \quad \text{a.s.}$$

Example 2.5. If $\sigma^2 = \operatorname{Var} X < \infty$, then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\log\log n}}{\sqrt{2n}} = \sigma \quad \text{a.s.}$$

And here are two more elaborate ones.

Example 2.6. Let, for $n \geq 9$, $a_n = n(\log\log n)^q/(\log n)^p$, p, $q > 0$. Then

$$d_n = \log\Big(\frac{n(\log\log n)^q}{n/(\log n)^p}\Big) + \log\log n \sim (p+1)\log\log n \quad \text{as} \quad n \to \infty,$$

so that, $f(n) = (p+1)n(\log\log n)^{q+1}/(\log n)^p$, and, hence,

$$f^{-1}(n) \sim Cn(\log n)^p/(\log\log n)^{q+1}$$

as $n \to \infty$, and the following result emerges.

If, for some p, $q > 0$,

$$E\,X^2 \frac{(\log^+ |X|)^p}{(\log^+ \log^+ |X|)^{q+1}} < \infty,$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n(\log\log n)^q/(\log n)^p}}{\sqrt{2(p+1)\frac{n}{(\log n)^p}(\log\log n)^{q+1}}} = \sigma \quad \text{a.s.}$$

Example 2.7. Let $a_n = n/\exp\{\sqrt{\log n}\}$, $n \geq 1$, that is,

$$d_n = \log\exp\{\sqrt{\log n}\} + \log\log n = \sqrt{\log n} + \log\log n \sim \sqrt{\log n} \quad \text{as} \quad n \to \infty,$$

which yields $f(n) \sim n\sqrt{\log n}/\exp\{\sqrt{\log n}\}$ as $n \to \infty$, so that

$$f^{-1}(n) \sim n\exp\{\sqrt{\log n + 1/2}\}/\sqrt{\log n} \quad \text{as} \quad n \to \infty,$$

which tells us that if

$$E\,X^2 \frac{\exp\{\sqrt{2\log^+ |X|}\}}{\sqrt{\log^+ |X|}} < \infty,$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\exp\{\sqrt{\log n}\}}}{\sqrt{2\frac{n}{\exp\{\sqrt{\log n}\}}\sqrt{\log n}}} = \sigma \quad \text{a.s.}$$

We refer to Gut et al. (2010) for details and further examples.

The proof of Theorem 2.2 has some common ingredients with that of the LIL, in the sense that one needs two truncations. One to match the Kolmogorov exponential bounds and one to match the moment requirement. Typically (and somewhat frustratingly) it is the thin central part that causes the main trouble in the proof. A weaker result is obtained if only the first truncation is made. The cost is that too much (although not much too much) integrability will be required. A proof in this weaker setting is hinted at in Remark 2.10. For more we refer to Gut et al. (2010), Section 6.

2.1. Skeleton of the proof of Theorem 2.2

As indicated a few lines ago, one begins by truncating at two levels—b_n and c_n, where the former is chosen to match the exponential inequalities, and the latter to match the moment assumption, after which one defines the truncated summands,

$$\begin{aligned} X_n' &= X_n I\{|X_n| \leq b_n\}, \\ X_n'' &= X_n I\{b_n < |X_n| < c_n\}, \\ X_n''' &= X_n I\{|X_n| \geq c_n\}, \end{aligned} \tag{2.3}$$

and, along with them, their expected values, partial sums, and windows: $E\,X'_n$, $E\,X''_n$, $E\,X'''_n$, S'_n, S''_n, S'''_n, and $T'_{n,n+n/L(n)}$, $T''_{n,n+n/L(n)}$, $T'''_{n,n+n/L(n)}$, respectively, where, in the following any object with a prime or a multiple prime refers to the respective truncated component.

Since truncation generally destroys centering one then shows that the truncated means are "small" and that $\mathrm{Var}(T'_{n,n+n/L(n)}) \approx n\sigma^2$.

With these quantities one now proceeds as follows:

The upper estimate:

1. Dispose of $T'''_{n_k,n_k+n_k/L_{n_k}}$;
2. Dispose of $T''_{n_k,n_k+n_k/L_{n_k}}$ (frequently the hard(est) part);
3. Upper exponential bounds for a suitable subsequence $T'_{n_k,n_k+n_k/L_{n_k}}$;
4. Borel–Cantelli 1 $\Longrightarrow$ $T'_{n_k,n_k+n_k/L_{n_k}}$ is OK;
5. $1+2+4$ $\Longrightarrow$ $\limsup T_{n_k,n_k+n_k/L_{n_k}} \leq \cdots$;
6. Filling gaps;
7. $5+6$ $\Longrightarrow$ $\limsup T_{n,n+n/L(n)} \leq \cdots$;

The lower estimate:

8. Lower exponential for a suitable subsequence $T'_{n_k,n_k+n_k/L_{n_k}}$;
9. Subsequence is sparse $\Longrightarrow$ independence;
10. Borel–Cantelli 2 $\Longrightarrow$ $T'_{n_k,n_k+n_k/L_{n_k}}$ is OK;
11. $1+2+10$ $\Longrightarrow$ $\limsup T_{n_k,n_k+n_k/L_{n_k}} \geq \cdots$;
12. $\limsup T_{n,n+n/L(n)} \geq \limsup T_{n_k,n_k+n_k/L_{n_k}} \geq \cdots$;
13. $7+12$ $\Longrightarrow$ $\limsup T_{n,n+n/L(n)} = \cdots$;
14. $\Box$

Remark 2.8. This is the procedure in Gut et al. (2010). However, for some results one can even dispose of $T'''_{n,n+n/L(n)}$ and $T''_{n,n+n/L(n)}$ in Steps 1 and 2, respectively.

When it comes to choosing the appropriate subsequence it turns out that the choice should satisfy the relation

$$d_{n_k} \sim \log k \quad \text{as} \quad k \to \infty, \tag{2.4}$$

and for this to happen, the following lemma, which is due to Fredrik Jonsson, Uppsala, is crucial.

Lemma 2.9. *Suppose that $L \in \mathcal{SV}$ satisfies (2.2). Then*

$$\frac{\log(L(t)\ \log t)}{\log \varphi(t)} \to 1 \quad as \quad t \to \infty.$$

Before presenting the proof we note that the lemma is more or less trivially true for slowly varying functions made up by logarithms or iterated ones.

Proof. Setting $\varphi^*(t) = L(t)\log t$ we have $\varphi(t) \le \varphi^*(t)$ since $L(\cdot) \nearrow$. For the opposite inequality an appeal to (2.2) shows that

$$\begin{aligned}\varphi^*(t) &= \int_1^t \Big(L'(u)\log u + \frac{L(u)}{u}\Big)du = \int_1^t \frac{L'(u)\,u\,L(u)}{L(u)\,u}\Big(\int_1^u \frac{1}{v}\,dv\Big)du + \varphi(t)\\ &\le \int_1^t \frac{L(u)}{u}\Big(\int_1^u \frac{L'(v)}{L(v)}dv\Big)du + \varphi(t) \le \varphi(t)(1+\log(L(t))),\end{aligned}$$

from which we conclude that

$$1 \ge \frac{\log\varphi(t)}{\log\varphi^*(t)} \ge 1 - \frac{\log(1+\log L(t))}{\log(L(t)\log t)} \to 1 \quad \text{as} \quad t\to\infty. \qquad \square$$

2.2. Sketch of the proof of Theorem 2.2

We introduce the parameters $\delta > 0$ and $\varepsilon > 0$ and truncate at

$$b_n = \frac{\sigma\delta}{\varepsilon}\sqrt{\frac{a_n}{d_n}} \quad \text{and} \quad c_n = \delta\sqrt{f(n)},$$

recalling that

$$a_n = n/L(n), \quad d_n = \log L(n) + \log\log n, \quad f(n) = \min\{a_n d_n, n\},$$

and set, in accordance with (2.3),

$$\begin{aligned} X_n' &= X_n I\{|X_n| \le b_n\},\\ X_n'' &= X_n I\{b_n < |X_n| < \delta\sqrt{f(n)}\},\\ X_n''' &= X_n I\{|X_n| \ge \delta\sqrt{f(n)}\},\end{aligned}$$

after which we check the appropriate smallness of the truncated means.

Next we choose a subsequence such that $d_{n_k} \sim \log k$.

In order to dispose of $T'''_{n_k,n_k+a_{n_k}}$ we observe that if $|T'''_{n_k,n_k+a_{n_k}}|$ surpasses the $\eta\sqrt{a_{n_k}d_{n_k}}$ then, necessarily, at least one of the corresponding X''':s is nonzero, which leads to

$$\sum_{k=1}^{\infty} P(|T'''_{n_k,n_k+a_{n_k}}| > \eta\sqrt{a_{n_k}d_{n_k}}) \le \sum_{k=1}^{\infty} a_{n_k} P(|X| > \frac{\eta}{2}\sqrt{f(n_k)}) < \infty, \qquad (2.5)$$

where the finiteness is a consequence of the moment assumption.

As for the second step, this is a technically pretty involved matter for which we refer to Gut et al. (2010).

For the analysis of $T'_{n_k,n_k+a_{n_k}}$ we use the Kolmogorov upper exponential bounds (see e.g., Gut (2007), Lemma 8.2.1) and obtain (after having taken care of the centering inflicted by the truncation),

$$\begin{aligned} P(|T'_{n,n+a_n}| > \varepsilon\sqrt{2a_n d_n}) &\le P(|T'_{n,n+a_n} - ET'_{n,n+a_n}| > \varepsilon(1-\delta)\sqrt{2a_n d_n}) \\ &\le 2\exp\Big\{-\frac{\varepsilon^2(1-\delta)^3}{\sigma^2}\cdot d_n\Big\} \quad \text{for } n \text{ large,} \end{aligned}$$

which, together with the previous estimates, shows that

$$\sum_{k=1}^{\infty} P(|T_{n_k,n_k+a_{n_k}}| > (\varepsilon+2\eta)\sqrt{2a_{n_k}d_{n_k}}) < \infty,$$

provided $\varepsilon > \sigma/(1-\delta)^{3/2}$, and thus, due to the arbitrariness of η and δ, and the first Borel–Cantelli lemma, that

$$\limsup_{k\to\infty} \frac{T_{n_k,a_{n_k}}}{\sqrt{2a_{n_k}d_{n_k}}} \le \sigma \quad \text{a.s.} \tag{2.6}$$

The next step (Step 6 in the above list) amounts to proving the same for the entire sequence, and this is achieved by showing that

$$\sum_k P\Big(\max_{n_k\le n\le n_{k+1}} \frac{S_{n+a_n}-S_n}{\sqrt{2a_n\,d_n}} > \sigma\Big) < \infty, \tag{2.7}$$

implying that

$$P(\max_{n_k\le n\le n_{k+1}} \frac{S_{n+a_n}-S_n}{\sqrt{2a_n\,d_n}} > \sigma \text{ i.o}) = 0,$$

which, together with (2.6), then will tell us that

$$\limsup_{n\to\infty} \frac{T_{n,n+a_n}}{\sqrt{2a_n d_n}} \le \sigma \quad \text{a.s.}$$

In order to prove (2.7) we first observe that, for any $\eta > 0$,

$$\begin{aligned} &P\Big(\max_{n_k\le n\le n_{k+1}} \frac{S_{n+a_n}-S_n}{\sqrt{2a_n\,d_n}} > (1+6\,\eta)\sigma\Big) \\ &\le P(\max_{n_k\le n\le n_{k+1}} (S_{n+a_n} - S_{n_k+a_{n_k}}) > 2\eta\,\sigma\sqrt{2a_{n_k}\,d_{n_k}}) \\ &\quad + P(\max_{n_k\le n\le n_{k+1}} (-S_n + S_{n_k}) > 2\eta\,\sigma\sqrt{2a_{n_k}\,d_{n_k}}) \\ &\quad + P(\max_{n_k\le n\le n_{k+1}} (S_{n_k+a_{n_k}} - S_{n_k}) > (1+2\eta)\,\sigma\sqrt{2a_{n_k}d_{n_k}}), \end{aligned}$$

after which (2.7), broadly speaking, follows by applying the Lévy inequality (cf. e.g. Gut (2007), Theorem 3.7.2) to each of the four terms.

This finishes the "proof" of the upper estimate, and it remains to take care of the lower one (Step 8 and onwards in the skeleton list).

After having checked that

$$\operatorname{Var} X'_k \geq \sigma^2 - 2E\,X^2 I\{|X_k| \geq b_k\} \geq \sigma^2(1-\delta),$$

for n large, so that

$$\operatorname{Var}(T'_{n,n+a_n}) \geq a_n\sigma^2(1-\delta) \quad \text{for } n \text{ large},$$

we obtain, exploiting the lower exponential bound (see e.g. Gut (2007), Lemma 8.2.2), that, for any $\gamma > 0$,

$$\begin{aligned}
&P(T'_{n,n+a_n} > \varepsilon\sqrt{2a_n d_n}) \\
&\geq P\big(T'_{n,n+a_n} - ET'_{n,n+a_n} > \frac{\varepsilon(1+\delta)}{\sigma\sqrt{(1-\delta)}}\sqrt{2\operatorname{Var}(T'_{n,n+a_n})d_n}\big) \\
&\geq \exp\big\{-\frac{\varepsilon^2(1+\delta)^2(1+\gamma)}{\sigma^2(1-\delta)}\cdot d_n\big\} \quad \text{for } n \text{ large.}
\end{aligned}$$

Applying this lower bound to our subsequence and combining the outcome with (2.5) and the omitted analog for $T''_{n,n+n/L(n)}$ then yields

$$\limsup_{k\to\infty} \frac{T_{n_k,n_k+a_{n_k}}}{\sqrt{2a_{n_k}d_{n_k}}} \geq \sigma \quad \text{a.s.} \tag{2.8}$$

Finally, since the limsup for the entire sequence certainly is at least as large as that of the subsequence (Step 12 in the skeleton), we conclude that the lower bound (2.8) also holds for the entire sequence.

This completes (the sketch of) the proof (Step 14).

Remark 2.10. We close this section by recalling that a slightly weaker result may be obtained by truncation at $b_n = \sqrt{a_n/d_n}$ only, in which case $T''_{n,n+n/L(n)}$ and $T'''_{n,n+n/L(n)}$ are joined into one "outer" contribution. With the same argument as above, the previous computation then is replaced by

$$\sum_{n=1}^{\infty} P(|X| > \frac{\sigma\delta}{\varepsilon}b_n) < \infty,$$

where finiteness holds iff

$$E\,b^{-1}(|X|) < \infty.$$

If, for example, $L(n) = \log n$, then the moment condition $E\,X^2\frac{\log^+|X|}{\log^+\log^+|X|} < \infty$ in Theorem 2.2 is replaced by the condition $E\,X^2\log^+|X|\log^+\log^+|X| < \infty$; cf. Gut et al. (2010), Section 6.

3. The LLN and the LIL for random fields

We now turn our attention to random fields. But first, in order to formulate our results, we need to define the setup. Toward that end, let $\mathbf{Z}_+^d$, $d \geq 2$, denote the positive integer d-dimensional lattice with coordinate-wise partial ordering $\leq$, viz., for $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, $\mathbf{m} \leq \mathbf{n}$ means that $m_k \leq n_k$, for $k = 1, 2, \ldots, d$. The "size" of a point equals $|\mathbf{n}| = \prod_{k=1}^d n_k$, and $\mathbf{n} \to \infty$ means that $n_k \to \infty$, for all $k = 1, 2, \ldots, d$.

Next, let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$.

For random fields with i.i.d. random variables $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ the analog of Kolmogorov's strong law (see Smythe (1973)) reads as follows:

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|} = \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \overset{a.s.}{\to} 0 \quad \Longleftrightarrow \quad E|X| \, (\log^+ |X|)^{d-1} < \infty \quad \text{and} \quad E\,X = 0. \tag{3.1}$$

For more general index sets, see Smythe (1974).

The analogous Marcinkiewicz–Zygmund law of large numbers was proved in Gut (1978):

$$\frac{1}{|\mathbf{n}|^{1/r}} S_{\mathbf{n}} \overset{a.s.}{\to} 0 \quad \Longleftrightarrow \quad E|X|^r (\log^+ |X|)^{d-1} < \infty \quad \text{and, if} \quad 1 \leq r < 2, \; E\,X = 0. \tag{3.2}$$

The Hartman–Wintner analog is due to Wichura (1973):

$$\begin{gathered} \limsup_{\mathbf{n} \to \infty} \frac{S_{\mathbf{n}}}{\sqrt{2|\mathbf{n}| \log\log |\mathbf{n}|}} = \sigma\sqrt{d} \quad \text{a.s.} \\ \Longleftrightarrow \\ E\,X^2 \frac{(\log^+ |X|)^{d-1}}{\log^+ \log^+ |X|} < \infty \quad \text{and} \quad E\,X = 0, \; E\,X^2 = \sigma^2. \end{gathered} \tag{3.3}$$

A variation on the theme concerns the same problems when one considers the index set $\mathbf{Z}_+^d$ restricted to a *sector*, which, for the case $d = 2$, equals

$$S_\theta^{(2)} = \{(x, y) \in \mathbb{Z}_+^2 : \theta x \leq y \leq \theta^{-1} x, \; 0 < \theta < 1\}. \tag{3.4}$$

In the limiting case $\theta = 1$, the sector degenerates into a diagonal ray, in which case the sums $S_{\mathbf{n}}$, $\mathbf{n} \in S_\theta^{(2)}$, are equivalent to the subsequence S_{n^2}, more generally, S_{n^d}, $n \geq 1$, of the sequence $\{S_n, n \geq 1\}$ when $d = 1$. In that case it is clear that the usual one-dimensional assumptions are sufficient for the LLN and the LIL. One may therefore wonder about the proper conditions for the sector—since extra logarithms are needed "at the other end" (as $\theta \to 0$).

Without going into any details we just mention that it has been shown in Gut (1983) that the law of large numbers as well as the law iterated logarithm hold under the same moment conditions as in the case $d = 1$, and that the limit points in the latter case are the same as in the Hartman–Wintner theorem (Theorem 1.3).

For some additional comments on this we refer to Section 10 toward the end of the paper.

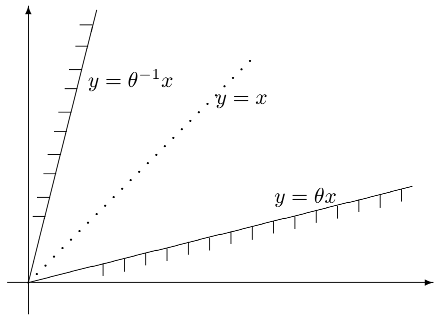

Figure 1: A sector ($d = 2$)

4. The LLN and LSL for windows

Having defined the general setup we also need the extension of the concept delayed sums or windows to this setting. A window here is an object $T_{\mathbf{n},\mathbf{n}+\mathbf{k}}$. For $d = 2$ we this is an incremental rectangle

$$T_{\mathbf{n},\mathbf{n}+\mathbf{k}} = S_{n_1+k_1,n_2+k_2} - S_{n_1+k_1,n_2} - S_{n_1,n_2+k_2} + S_{n_1,n_2} :$$

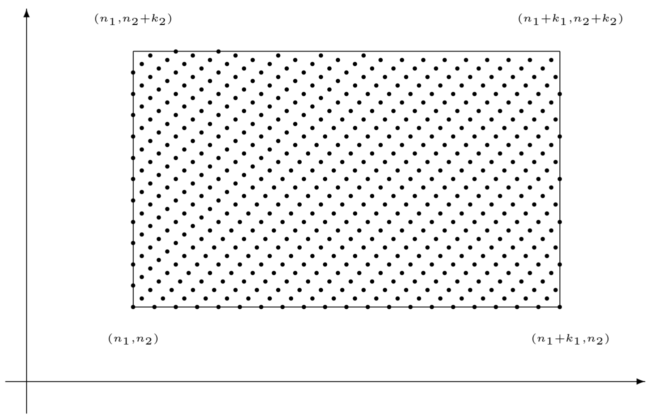

Figure 2: A typical window ($d = 2$)

In higher dimensions it is the analogous d-dimensional cube. A strong law for this setting can be found in Thalmaier (2009), Stadtmüller and Thalmaier (2009).

The extension of Theorem 1.8 to random fields runs as follows.

Theorem 4.1. *Let $0 < \alpha < 1$, and suppose that $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. random variables with mean 0 and finite variance σ^2. If*

$$E\,X^{2/\alpha}(\log^+|X|)^{d-1-1/\alpha} < \infty,$$

then

$$\limsup_{\mathbf{n}\to\infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha} \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{\mathbf{n}\to\infty} \frac{|T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}|}{\sqrt{|\mathbf{n}|^\alpha \log|\mathbf{n}|}} < \infty\Big) > 0,$$

then $E\,X^{2/\alpha}(\log^+|X|)^{d-1-1/\alpha} < \infty$ and $E\,X = 0$.

Some remarks on the proof will be given in Section 6.

4.1. An LSL for subsequences

The proof of the theorem is in the LIL-style, which, i.a., means that one begins by proving the sufficiency as well as the necessity along a suitable subsequence. Sticking to this fact one can, with very minor modifications of the proof of Theorem 4.1, prove the following *LSL for subsequences.* The inspiration for this result comes from the LIL-analog in Gut (1986).

Theorem 4.2. *Let $0 < \alpha < 1$, suppose that $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. random variables with mean 0 and finite variance σ^2, and set*

$$\Lambda = \{\mathbf{n} \in \mathbf{Z}_+^d : \mathbf{n}_i = i^{\beta/(1-\alpha)},\ i \geq 1\}.$$

If

$$E\,X^{2/\alpha}(\log^+|X|)^{d-1-1/\alpha} < \infty,$$

then, for $\beta > 1$,

$$\limsup_{\substack{\mathbf{n}\to\infty \\ \{\mathbf{n}\in\mathbf{\Lambda}^*\}}} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{\frac{1-\alpha}{\beta}} \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{\substack{\mathbf{n}\to\infty \\ \{\mathbf{n}\in\mathbf{\Lambda}^*\}}} \frac{|T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}|}{\sqrt{|\mathbf{n}|^\alpha \log|\mathbf{n}|}} < \infty\Big) > 0,$$

then $E\,X^{2/\alpha}(\log^+|X|)^{d-1-1/\alpha} < \infty$ and $E\,X = 0$.

For further details, see Gut and Stadtmüller (2008a), Section 6.

4.2. Different α:s

During a seminar in Uppsala on the previous material Fredrik Jonsson asked the question: "What happens if the α:s are different?"

In Theorem 4.1 the windows grow at the same rate in each coordinate; the edges of the windows are equal to n_k^α for all $k = 1, 2, \dots, d$. The focus now is to allow for different growth rates in different directions; viz., the edges of the windows will be $n_k^{\alpha_k}$, $k = 1, 2, \dots, d$, where, w.l.o.g., we assume that

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d < 1.$$

Next, we define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$, and set, for ease of notation,

$$\mathbf{n}^{\boldsymbol{\alpha}} = (n_1^{\alpha_1}, n_2^{\alpha_2}, \dots, n_d^{\alpha_d}), \quad \text{and} \quad |\mathbf{n}^{\boldsymbol{\alpha}}| = \prod_{k=1}^{d} n_k^{\alpha_k}.$$

Furthermore, following Stadtmüller and Thalmaier (2009), we let p be equal to the number of α:s that are equal to the smallest one.

As for the strong law, the results in Thalmaier (2009), Stadtmüller and Thalmaier (2009), in fact, also cover the case of unequal α:s. For a Marcinkiewicz–Zygmund analog we refer to Gut and Stadtmüller (2009). For completeness we also mention Gut and Stadtmüller (2010), where some results concerning Cesàro summation are proved.

Here is now the generalization of Theorem 4.1. For a proof and further details we refer to Gut and Stadtmüller (2008b).

Theorem 4.3. *Suppose that* $\{X_{\mathbf{k}},\ \mathbf{k} \in \mathbf{Z}_+^d\}$ *are i.i.d. random variables with mean* 0 *and finite variance* σ^2*. If*

$$E|X|^{2/\alpha_1}(\log^+|X|)^{p-1-1/\alpha_1} < \infty,$$

then

$$\limsup_{\mathbf{n}\to\infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}}{\sqrt{2|\mathbf{n}^{\boldsymbol{\alpha}}|\log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha_1} \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{\mathbf{n}\to\infty} \frac{|T_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}|}{\sqrt{|\mathbf{n}^{\boldsymbol{\alpha}}|\log|\mathbf{n}|}} < \infty\Big) > 0,$$

then $E|X|^{2/\alpha_1}(\log^+|X|)^{p-1-1/\alpha_1} < \infty$ *and* $E\,X = 0$*.*

Remark 4.4. If $\alpha_1 = \alpha_2 = \cdots = \alpha_d = \alpha$, then $p = d$ and $|\mathbf{n}^{\boldsymbol{\alpha}}| = |\mathbf{n}|^\alpha$, and the theorem reduces to Gut and Stadtmüller (2008a), Theorem 2.1 = Theorem 4.1 above.

Remark 4.5. For a result for subsequences analogous to Theorem 4.2; see Gut and Stadtmüller (2008b), Section 6.

We observe that the moment condition as well as the extreme limit points depend on the *smallest* α and its multiplicity. Heuristically this can be explained as follows. The longer the stretch of the window along a specific axis, the more cancellation may occur in that direction. Equivalently, the shorter the stretch, the wilder the fluctuations. This means that in order to "tame" the fluctuations it is (only) necessary to put conditions on the shortest edge(s).

4.3. Different α:s, $\log$, and $\log\log$

One can exaggerate the mixtures even further, namely, by combining edges that expand at different α-rates with edges that expand with different almost linear rates. Some results in this direction concerning the LLN can be found in Gut and Stadtmüller (2011b).

The paper Gut and Stadtmüller (2011a) is devoted to the LSL. First a result from that paper that extends Gut et al. (2010) to random fields for (iterated) logarithmic expansions and mixtures of them. For simplicity and illustrative purposes we stick to the case $d = 2$.

Theorem 4.6. *Let $\{X_{i,j},\, i,j \geq 1\}$ be i.i.d. random variables.*
(i) *If*

$$E\,X^2 \frac{(\log^+|X|)^3}{\log^+\log^+|X|} < \infty \quad \textit{and} \quad E\,X = 0,\ E\,X^2 = \sigma^2,$$

then

$$\limsup_{m,n\to\infty} \frac{T_{(m,n)\,,\,(m+m/\log m, n+n/\log n)}}{\sqrt{4mn\frac{\log\log m+\log\log n}{\log m\log n}}} = \sigma \quad \textit{a.s.}$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{(m,n)\,,\,(m+m/\log m, n+n/\log n)}|}{\sqrt{mn\frac{\log\log m+\log\log n}{\log m\log n}}} < \infty\Big) > 0,$$

then $E\,X^2 \frac{(\log^+|X|)^3}{\log^+\log^+|X|} < \infty$ *and* $E\,X = 0$.
(ii) *If*

$$E\,X^2\log^+|X|\,\log^+\log^+|X| < \infty \quad \textit{and} \quad E\,X = 0,\ E\,X^2 = \sigma^2,$$

then

$$\limsup_{m,n\to\infty} \frac{T_{(m,n)\,,\,(m+m/\log\log m, n+n/\log\log n)}}{\sqrt{2mn\frac{\log\log m+\log\log n}{\log\log m\log\log n}}} = \sigma \quad \textit{a.s.}$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{(m,n)\,,\,(m+m/\log\log m, n+n/\log\log n)}|}{\sqrt{mn\frac{\log\log m+\log\log n}{\log\log m\log\log n}}} < \infty\Big) > 0,$$

then $E\,X^2 \log^+|X|\,\log^+\log^+|X| < \infty$ *and* $E\,X = 0$.

(iii) *If*

$$E\,X^2(\log^+|X|)^2 < \infty \quad \text{and} \quad E\,X = 0,\ E\,X^2 = \sigma^2,$$

then

$$\limsup_{m,n\to\infty} \frac{T_{(m,n)\,,\,(m+m/\log m, n+n/\log\log n)}}{\sqrt{4mn\frac{\log\log m + \log\log n}{\log m \log\log n}}} = \sigma \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{(m,n)\,,\,(m+m/\log m, n+n/\log\log n)}|}{\sqrt{4mn\frac{\log\log m + \log\log n}{\log m \log\log n}}} < \infty\Big) > 0,$$

then $E\,X^2(\log^+|X|)^2 < \infty$ *and* $E\,X = 0$.

We conclude with an example where a logarithmic expansion is mixed with a power.

Theorem 4.7. *Let* $0 < \alpha < 1$, *and let* $\{X_{i,j},\ i,j \geq 1\}$ *be i.i.d. random variables. If*

$$E\,X^{2/\alpha}(\log^+|X|)^{-1/\alpha} < \infty \quad \text{and} \quad E\,X = 0,\ E\,X^2 = \sigma^2,$$

then

$$\limsup_{m,n\to\infty} \frac{T_{(m,n)\,,\,(m+m^\alpha,\, n+n/\log n)}}{\sqrt{2m^\alpha n\frac{(1-\alpha)\log(mn)}{\log n}}} = \sigma \quad a.s.$$

Conversely, if

$$P\Big(\limsup_{m,n\to\infty} \frac{|T_{(m,n)\,,\,(m+m^\alpha,\, n+n/\log n)}|}{\sqrt{m^\alpha n\frac{\log(mn)}{\log n}}} < \infty\Big) > 0,$$

then $E\,X^{2/\alpha}(\log^+|X|)^{-1/\alpha} < \infty$ *and* $E\,X = 0$.

5. Preliminaries

Proposition 5.1. *Let* $r > 0$ *and let* X *be a non-negative random variable. Then*

$$E\,X^r < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} n^{r-1}P(X \geq n) < \infty,$$

More precisely,

$$\sum_{n=1}^{\infty} n^{r-1}P(X \geq n) \leq E\,X^r \leq 1 + \sum_{n=1}^{\infty} n^{r-1}P(X \geq n).$$

As an example, consider the case $r = 1$, and suppose that $X_1, X_2, \dots$ is an i.i.d. sequence. It then follows from the proposition that, for any $\varepsilon > 0$,

$$P(|X_n| > n\varepsilon \text{ i.o.}) = 0 \iff \sum_{n=1}^{\infty} P(|X_n| > n\varepsilon) < \infty \iff E\,|X| < \infty.$$

Suppose instead that we are facing an i.i.d. random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d\}$. What is then the relevant moment condition that ensures that

$$\sum_{n} P(|X_{\mathbf{n}}| > |\mathbf{n}|) < \infty\,? \quad \text{or, equivalently, that} \quad \sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) < \infty\,? \tag{5.1}$$

In order to answer this question it turns out that we need the quantities

$$d(j) = \text{Card}\,\{\mathbf{k} : |\mathbf{k}| = j\} \quad \text{and} \quad M(j) = \text{Card}\,\{\mathbf{k} : |\mathbf{k}| \le j\},$$

which describe the "size" of the index set, and their asymptotics

$$\frac{M(j)}{j(\log j)^{d-1}} \to \frac{1}{(d-1)!} \quad \text{and} \quad d(j) = o(j^{\delta}) \text{ for any } \delta > 0 \quad \text{as} \quad j \to \infty; \tag{5.2}$$

cf. Hardy and Wright (1954), Chapter XVIII and Titchmarsh (1951), relation (12.1.1) (for the case $d = 2$). The quantity $d(j)$ itself has no pleasant asymptotics; $\liminf_{j\to\infty} d(j) = d$, and $\limsup_{j\to\infty} d(j) = +\infty$.

Now, exploiting the fact that all terms in expressions such as the second sum in (5.1) with equisized indices are equal, we conclude that

$$\sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) = \sum_{j=1}^{\infty} \sum_{|\mathbf{n}|=j} d(j) P(|X| > j), \tag{5.3}$$

which, via partial summation yields the first half of following lemma. The second half follows via a change of variable.

Lemma 5.2. *Let $r > 0$, and suppose that X is a random variables. Then*

$$\sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) < \infty \iff E\,M(|X|) < \infty \iff E|X|(\log^+ |X|)^{d-1} < \infty,$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{r-1} P(|X| > |\mathbf{n}|) < \infty \iff E\,M(|X|^r) < \infty \iff E|X|^r(\log^+ |X|)^{d-1} < \infty.$$

Reviewing the steps leading to the lemma one finds that if, instead, we consider the sector (recall (3.4)) one finds that

$$\sum_{\mathbf{n}\in S_\theta^d} P(|X| > |\mathbf{n}|) < \infty \iff E\,M(|X|) < \infty \iff E|X| < \infty. \tag{5.4}$$

Remark 5.3. Note that the first equivalence is the same as in Lemma 5.2, and that the second one is a consequence of the "size" of the index set.

For results such as Theorem 4.3, as well as for some of the results in Section 8 below, we shall need the more general index sets

$$M_{\boldsymbol{\alpha}}(j) = \text{Card}\,\{\mathbf{k} : |\mathbf{k}^{\boldsymbol{\alpha}}| \le j^{\alpha_1}\} = \text{Card}\,\{\mathbf{k} : \prod_{\nu=1}^{d} k_\nu^{\alpha_\nu/\alpha_1} \le j\}. \tag{5.5}$$

Generalizing Lemma 3 in Stadtmüller and Thalmaier (2009) in a straight forward manner yields the following analog of (5.2):

$$M_{\boldsymbol{\alpha}}(j) \sim c_{\boldsymbol{\alpha}}\, j\,(\log j)^{p-1} \quad \text{as} \quad j \to \infty \tag{5.6}$$

where $c_{\boldsymbol{\alpha}} > 0$, which, in turn, via partial summation, tells us that

$$\sum_{\mathbf{n}} P(|X| > |\mathbf{n}^{\boldsymbol{\alpha}}|) \asymp \sum_{j=1}^{\infty} (\log j)^{p-1} P(|X| > j^{\alpha_1}).$$

Using a slight modification of this, together with the fact that the inverse of the function $y = x^\alpha (\log x)^\kappa$ behaves asymptotically like $x = y^{1/\alpha}(\log y)^{-(\kappa/\alpha)}$, yields the next tool (Gut and Stadtmüller (2008a), Lemma 3.2, Gut and Stadtmüller (2008b), Lemma 3.1).

Lemma 5.4. *Let $\kappa \in \mathbb{R}$ and suppose that X is a random variable. Then,*

$$\sum_{\mathbf{n}} P\big(|X| > |\mathbf{n}^{\boldsymbol{\alpha}}|(\log|\mathbf{n}|)^\kappa\big) < \infty \quad \Longleftrightarrow \quad E|X|^{1/\alpha_1}(\log^+|X|)^{p-1-\kappa/\alpha_1} < \infty.$$

In particular, if $\alpha_1 = \alpha_2 = \cdots = \alpha_d = \kappa = 1/2$, then

$$\sum_{\mathbf{n}} P\big(|X| > \sqrt{|\mathbf{n}|\log|\mathbf{n}|}\big) < \infty \quad \Longleftrightarrow \quad E\,X^2(\log^+|X|)^{d-2} < \infty.$$

For illustrative reasons we also quote Gut and Stadtmüller (2008a), Lemma 3.3, as an example of the kind of technical aid that is required at times.

Lemma 5.5. *Let $\kappa \ge 1$, $\theta > 0$, and $\eta \in \mathbb{R}$.*

$$\sum_{i=2}^{\infty} \sum_{\{\mathbf{n}:|\mathbf{n}|=i^\kappa(\log i)^\eta\}} \frac{1}{|\mathbf{n}|^\theta} = \sum_{i=2}^{\infty} \frac{d(i^\kappa(\log i)^\eta)}{i^{\kappa\theta}(\log i)^{\eta\theta}} \begin{cases} < \infty, & \textit{when} \quad \theta > \frac{1}{\kappa}, \\ = \infty, & \textit{when} \quad \theta < \frac{1}{\kappa}. \end{cases}$$

6. Sketch of the proofs of Theorems 4.1 and 4.3

In this section we give som hints on the proofs of Theorems 4.1 and 4.3, in the sense that we shall point to differences and modifications compared to the proof of Theorem 2.2 in Section 2.2.

6.1. On the proof of Theorem 4.1

This time truncation is at

$$b_{\mathbf{n}} = b_{|\mathbf{n}|} = \frac{\sigma\delta}{\varepsilon}\frac{\sqrt{|\mathbf{n}|^{\alpha}}}{\log|\mathbf{n}|} \quad \text{and} \quad c_{\mathbf{n}} = \delta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|},$$

for some (arbitrarily) small $\delta > 0$.

The first step differs slightly from the analog in the proof of Theorem 2.2, in that we now start by dispensing of the full double- and triple primed sequences (recall Remark 2.8).

As for the double primed contribution we argue that in order for the $|T'''_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}|$:s to surpass the level $\eta\sqrt{|\mathbf{n}^{\boldsymbol{\alpha}}|\log|\mathbf{n}|}$ infinitely often, for some $\eta > 0$ small, it is necessary that infinitely many of the X''':s are nonzero, and the latter event has probability zero by the first Borel–Cantelli lemma, since

$$\sum_{\mathbf{n}} P(|X_{\mathbf{n}}| > \eta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}) = \sum_{\mathbf{n}} P(|X| > \eta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}) < \infty,$$

where the finiteness is a consequence of the moment assumption and the second half of Lemma 5.4.

Taking care of $T''_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}$ is a bit easier this time, the argument being that in order for $|T''_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}|$ to surpass the level $\eta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}$ it is necessary that at least $N \geq \eta/\delta$ of the X'':s are nonzero, which, by stretching the truncation bounds to the extremes, some elementary combinatorics, and the moment assumption implies that

$$\begin{aligned}
&P(|T''_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}| > \eta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}) \\
&\leq \binom{|\mathbf{n}|^{\alpha}}{N}\Big(P\big(b_{\mathbf{n}} < |X| \leq \delta\sqrt{(|\mathbf{n}|+|\mathbf{n}|^{\alpha})\log(|\mathbf{n}|+|\mathbf{n}|^{\alpha})}\,\big)\Big)^{N} \\
&\leq C\frac{(\log|\mathbf{n}|)^{N((3/\alpha)+1-d)}}{|\mathbf{n}|^{N(1-\alpha)}},
\end{aligned}$$

and, hence, that

$$\sum_{\mathbf{n}} P(|T''_{\mathbf{n},\mathbf{n}+\mathbf{n}/L(\mathbf{n})}| > \eta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}) < \infty \quad \text{for all} \quad \eta > \frac{\delta}{1-\alpha},$$

whenever $N(1-\alpha) > 1$ (and $N\delta \geq \eta$), after which another application of the first Borel–Cantelli lemma concludes that part of the proof.

As for $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}$, the exponential bounds do the job as before;

$$P\big(T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}} > \varepsilon\sqrt{2|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}\big) \begin{cases} \leq \exp\Big\{-\dfrac{2\varepsilon^2(1-\delta)^2}{2\sigma^2}\log|\mathbf{n}|(1-\delta)\Big\}, \\ \geq \exp\Big\{-\dfrac{2\varepsilon^2(1+\delta)^2}{2\sigma^2(1-\delta)}\log|\mathbf{n}|(1+\gamma)\Big\}. \end{cases}$$

Putting things together proves the theorem for suitably selected subsequences, and thus, in particular also the lower bound for the full field (remember Step 12 in the skeleton list).

It thus remains to verify the upper bound for the entire field.

Now, for the LIL and LSL one investigates the gaps between subsequence points with the aid of the Lévy inequalities, as we have seen in the proof of Theorem 2.2, Step 6. When $d \geq 2$, however, there are no gaps in the usual sense and one must argue somewhat differently.

Let us have a quick look at the situation when $d = 2$. First we must show that the selected subsequence (which we have not explicitly presented) is such that the subset of windows overlap, viz., that they cover all of $\mathbf{Z}_+^2$. Next, we select an arbitrary window

$$T_{((m,n),(m+m^\alpha,n+n^\alpha))}$$

and note that it is always contained in the union of (at most) four of the earlier selected ones:

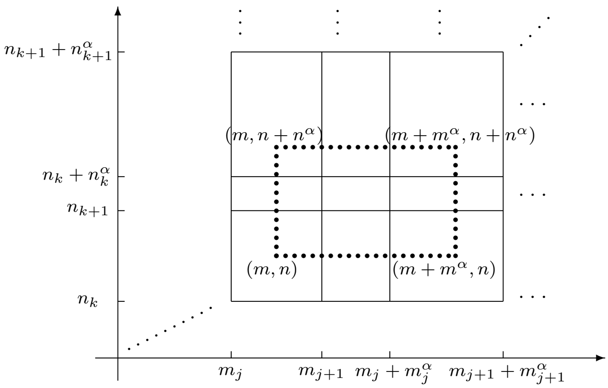

Figure 3: A dotted arbitrary window

One, finally, shows that the discrepancy between the arbitrary window and the selected ones is asymptotically negligible. This is a technical matter which we omit. Except for mentioning that one has to distinguish between the cases when the arbitrary window is located in "the center" of the index set or "close" to one of the coordinate axes (for a similar discussion cf. also Gut (1980), Section 4).

6.2. On the proof of Theorem 4.3

This proof runs along the same lines as the previous one with some additional technical complications, due to the non-equalness of the α:s. In order to illustrate this, consider the triple-primed windows.

Truncation now is at

$$b_{\mathbf{n}} = b_{|\mathbf{n}|} = \frac{\sigma\delta}{\varepsilon}\frac{\sqrt{|\mathbf{n}^{\boldsymbol{\alpha}}|}}{\log|\mathbf{n}|} \quad \text{and} \quad c_{\mathbf{n}} = \delta\sqrt{|\mathbf{n}^{\boldsymbol{\alpha}}|\log|\mathbf{n}|},$$

for $\delta > 0$ small; note $|\mathbf{n}^{\boldsymbol{\alpha}}|$ instead of $|\mathbf{n}|^{\alpha}$.

The argument for $T'''_{\mathbf{n},\mathbf{n}+\mathbf{n}^{\boldsymbol{\alpha}}}$:s is verbatim as before, and leads to the sum

$$\sum_{\mathbf{n}} P(|X_{\mathbf{n}}| > \eta\sqrt{|\mathbf{n}^{\boldsymbol{\alpha}}|\log|\mathbf{n}|}) = \sum_{\mathbf{n}} P(|X| > \eta\sqrt{|\mathbf{n}^{\alpha}|\log|\mathbf{n}|}) < \infty,$$

where the finiteness is a consequence of the moment assumption, which this time is a consequence of the first half of Lemma 5.4.

The remaining part of the proof amounts to analogous changes.

7. The Hsu–Robbins–Erdős–Spitzer–Baum–Katz theorem

One aspect of the seminal paper Hsu and Robbins (1947) is that it started an area of research related to convergence rates in the law of large numbers, which, in turn, culminated in the now classical paper Baum and Katz (1965), in which the equivalence of (7.1), (7.2), and (7.4) below was demonstrated. Namely, in Hsu and Robbins (1947) the authors introduced the concept of *complete convergence*, and proved that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value of the variables provided their variance is finite. The necessity was proved by Erdős (1949, 1950).

Theorem 7.1. *Let $r > 0$, $\alpha > 1/2$, and $\alpha r \geq 1$. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. If*

$$E|X|^r < \infty \quad \textit{and, if } r \geq 1, \quad E\,X = 0, \tag{7.1}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P(|S_n| > n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0; \tag{7.2}$$

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P(\max_{1\leq k\leq n} |S_k| > n^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{7.3}$$

If $\alpha r > 1$ we also have

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P(\sup_{k\geq n} |S_k/k^{\alpha}| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{7.4}$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then so are the others (for appropriate values of r and α), $E|X|^r < \infty$ and, if $r \geq 1$, $E\,X = 0$.

The Hsu–Robbins–Erdős part corresponds to the equivalence of (7.1) and (7.2) for the case $r = 2$ and $p = 1$. Spitzer (1956) verified the same for the case $r = p = 1$, and Katz (1963), followed by Baum and Katz (1965) took care of the equivalence between (7.1), (7.2), and (7.4) as formulated in the theorem. Chow (1973) proved that (7.3) holds iff (7.1) does, somewhat differently.

On the other hand, the equivalence of (7.2) and (7.3) is trivial one way and follows via the Lévy inequalities (more precisely via the standard Lévy inequalities as given in e.g. Gut (2007), Theorem 3.7.1 in conjunction with Proposition 3.6.1 there). The implication (7.4) $\Longrightarrow$ (7.2) is also trivial and the converse follows via a "slicing device" introduced in Baum and Katz (1965).

Remark 7.2. Strictly speaking, if one of the sums is finite for *some* $\varepsilon > 0$, then so are the others, and $E|X|^r < \infty$. However, we need convergence *for all* $\varepsilon > 0$ in order to infer that $E\,X = 0$ for the case $r \geq 1$. The same remark applies below.

Before continuing we pause for a moment and consider, for simplicity, the Hsu–Robbins–Erdős case $r = 2$ and $\alpha = 1$, for which the original proof of the implication (7.1) $\Longrightarrow$ (7.2) was technically very intricate.

The first and obvious attempt in order to find a simple proof of this implication fails, as is frequently the case, because of the divergence of the harmonic series. Namely, if $E\,X = 0$ and $\operatorname{Var} X = \sigma^2 < \infty$, then, by Chebyshev's inequality, we have

$$\sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) \leq \sum_{n=1}^{\infty} \frac{\sigma^2}{n\varepsilon^2} = +\infty \quad \text{for any} \quad \varepsilon > 0.$$

However, a fascinating inequality, due to Kahane (1985) and Hoffmann-Jørgensen (1974), see also Gut (2007), Theorem 3.7.5, turns out to be an extremely efficient remedy.

Namely, the KHJ-inequality tells us that for independent *symmetric* random variables one has

$$P(|S_n| > 3n\varepsilon) \leq P(\max_{1\leq k\leq n} |X_k| > n\varepsilon) + 4\big(P(|S_n| > n\varepsilon)\big)^2, \tag{7.5}$$

which, since $P(\max_{1\leq k\leq n} |X_k| > n\varepsilon) \leq nP(|X| > n\varepsilon)$, yields

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n| > 3n\varepsilon) &\leq \sum_{n=1}^{\infty} nP(|X| > n\varepsilon) + 4\big(P(|S_n| > n\varepsilon)\big)^2 \\ &\leq E(X/\varepsilon)^2 + 4\sum_{n=1}^{\infty}\Big(\frac{\sigma^2}{n\varepsilon^2}\Big)^2 = \frac{E\,X^2}{\varepsilon^2} + 4\frac{\sigma^4}{\varepsilon^4}\frac{\pi^2}{6},\end{aligned}$$

where, in the last inequality, we exploited Proposition 5.1.

Symmetrizing and desymmetrizing follow standard procedures. For a complete proof of the implication in the general case, one can iterate the KHJ-inequality and exploit the Marcinkiewicz–Zygmund (moment) inequalities in order to cover everything (except for the case $r = p$ for which truncation and a WLLN-type of argument is used). For details and a full proof we refer to Gut (2007), Section 6.11, the proof of which is based on Gut (1978), where the random field version, Theorem 8.1 below, was proved.

The beauty of this proof, thanks to KHJ, is the squaring of the Chebyshev estimate, in that $\sum_{n=1}^{\infty} n^{-1}$ (which is divergent) is replaced by $\sum_{n=1}^{\infty} n^{-2}$ (which is convergent).

We close by mentioning that for the limiting case $p = 2$ one is in the realm of the central limit theorem, and since the individual probabilities do not converge to zero in that case, there is of course no way of having their sums converge. However, by replacing, what would then be $\sqrt{n}$ by $\sqrt{n \log n}$ or even by $\sqrt{n \log\log n}$ there exist positive results; cf. Davis (1968a, 1968b), Lai (1974) for more.

8. The H-R-E-S-B-K theorem for random fields

The obvious question at this point is: What about random field versions?

Theorem 8.1. *Let $r > 0$ and $\alpha > 1/2$ with $\alpha r \geq 1$, suppose that $\{X_{\mathbf{k}},\, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. random variables, and set $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r(\log^+|X|)^{d-1} < \infty \quad \textit{and, if } r \geq 1, \quad E\,X = 0, \tag{8.1}$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0\,; \tag{8.2}$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(\max_{\mathbf{k}\leq\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}|^{\alpha}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{8.3}$$

If $\alpha r > 1$ we also have

$$\sum_{j=1}^{\infty} j^{\alpha r-2} P(\sup_{j\leq|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^{\alpha}| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{8.4}$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then $E|X|^r(\log^+|X|)^{d-1} < \infty$ and, if $r \geq 1$, $E\,X = 0$.

This is Theorem 4.1 in Gut (1978). As for the proof we only mention that the KHJ- and the Marcinkiewicz–Zygmund inequalities concern sums and consequently remain valid also for random fields. The proof of (8.1) $\Longrightarrow$ (8.2) therefore follows along the same lines as above (with an application to Lemma 5.2 for the appropriate moment condition).

The same can be said about the equivalence (8.2) $\Longleftrightarrow$ (8.3) (with a $\mathbf{Z}_+^d$-version of the Lévy inequality replacing the standard one). The implication (8.4) $\Longrightarrow$ (8.2) is trivial again, and the converse follows via an elaboration of the slicing device of Baum and Katz (1965). We refer to Gut (1978) for details in the multiindex setting.

As the reader may have guessed by now, the next point on the agenda is the case of unequal α:s. Toward that end we first recall, from Subsection 4.2, that α is replaced by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where, as before,

$$p = \max\{k : \alpha_k = \alpha_1\},$$

although now,

$$\frac{1}{2} \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d \le 1,$$

The reason for the lower bound 1/2 is, as was hinted at before, the central limit theorem. In fact, supposing that $\alpha_1 = 1/2$, then, for any $\varepsilon > 0$, we have

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) \ge \sum_{i=1}^{\infty} i^{(r/2)-2} P(|S_{i,1,1,\ldots,1}| > \sqrt{i}\cdot 1\cdot 1 \cdots 1\cdot 1\cdot \varepsilon) = +\infty.$$

Our first result extends Theorem 8.1. The proof follows the basic lines of that of Theorem 8.1 with obvious changes, such as $|\mathbf{n}^{\boldsymbol{\alpha}}|$ instead of $|\mathbf{n}|^{\alpha}$, and the additional technicalities inflicted by the unequalness of the α:s. We refer to Gut and Stadtmüller (2012) for details.

Theorem 8.2. *Let $r > 0$, suppose that $\alpha_1 > 1/2$, that $\alpha_1 r \ge 1$, let $\{X_{\mathbf{k}},\, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables, and set $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r (\log^+ |X|)^{p-1} < \infty \quad \textit{and, if } r \ge 1, \quad E\,X = 0,$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0\,;$$
$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

If $\alpha_1 r > 1$ we also have

$$\sum_{j=1}^{\infty} j^{\alpha_1 r-2} P(\sup_{j\le|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}^{\boldsymbol{\alpha}}|| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then $E|X|^r(\log^+|X|)^{p-1} < \infty$ and, if $r \ge 1$, $E\,X = 0$.

In order to illustrate, once more, the efficiency of the KHJ-inequality we show how the proof for the first sum works in the special case when $\alpha_1 r = 2$ and the summands are symmetric. Following the procedure from the proof of Theorem 7.1 we obtain

$$\begin{aligned}
\sum_{\mathbf{n}} P(|S_{\mathbf{n}}| > 3^j |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) &\le \sum_{\mathbf{n}} P(|X| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) + 4 \sum_{\mathbf{n}} \Big(P(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)\Big)^2 \\
&\le \sum_{\mathbf{n}} P(|X| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) + \frac{4\sigma^4}{\varepsilon^4} \sum_{\mathbf{n}} \Big(\frac{|\mathbf{n}|\sigma^2}{|\mathbf{n}^{\boldsymbol{\alpha}}|^2\varepsilon^2}\Big)^2 \\
&= \sum_{\mathbf{n}} P(|X| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) + \frac{4\sigma^4}{\varepsilon^4} \prod_{i=1}^{d} \sum_{n_i=1}^{\infty} n_i^{-2(2\alpha_i - 1)}.
\end{aligned}$$

Now, the first sum is finite iff the desired moment condition is fulfilled (Lemma 5.2), and the second one is finite, since the last exponent $2(2\alpha_i - 1) > 1$ for all i.

Full details are given in Gut and Stadtmüller (2012), Section 3.

As mentioned some lines ago, there are no positive results when $\alpha_1 = 1/2$. However, by adding logarithms as in Lai (1974), Gut (1980), maybe ...?

In the following we first let "some" ($= p \leq d$) of the α:s be equal to $1/2$ with additional logarithms or iterated logarithms and some ($= d - p \geq 0$) of them be strictly larger than $1/2$, after which we consider the complete mixture with $q > p$ of the α:s being equal to $1/2$, the p first of them with additional logarithms, the $q-p$ next ones with additional iterated logarithms, and the $d-q$ largest ones being $> 1/2$. For proofs we refer to Gut and Stadtmüller (2012).

Theorem 8.3. *Let $r \geq 2$, suppose that $\alpha_1 = 1/2$ (and thus, in particular, that $\alpha_1 r \geq 1$), let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables, and set $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r(\log^+|X|)^{p-1-r/2} < \infty, \quad E\,X = 0, \quad \textit{and} \quad \operatorname{Var} X = \sigma^2 < \infty,$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P\Big(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\big(\prod_{i=1}^{p} n_i\big) \prod_{i=p+1}^{d} n_i^{\alpha_i}} \cdot \varepsilon\Big) < \infty$$

for $\varepsilon > \sigma\sqrt{r-2}$;

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P\Big(\max_{\mathbf{k}\leq\mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\big(\prod_{i=1}^{p} n_i\big) \prod_{i=p+1}^{d} n_i^{\alpha_i}} \cdot \varepsilon\Big) < \infty$$

for $\varepsilon > \sigma\sqrt{r-2}$. If $\alpha_1 r > 1$, i.e. if $r > 2$, then we also have

$$\sum_{j=1}^{\infty} j^{(r/2)-2} P\Big(\sup_{j\leq|\mathbf{k}|} \Big|S_{\mathbf{k}} \Big/ \sqrt{\prod_{i=1}^{p} k_i \log k_i \prod_{i=p+1}^{d} k_i^{\alpha_i}}\Big| > \varepsilon\Big) < \infty \ \textit{for all}\ \varepsilon > \sigma\sqrt{r-2}.$$

Conversely, suppose that either $r = 2$ and $p \geq 2$, or that $r > 2$. If one of the sums is finite for some $\varepsilon > 0$, then $E|X|^r\,(\log^+|X|)^{p-1-r/2} < \infty$ and $E\,X = 0$.

Theorem 8.4. *Suppose that $\alpha_1 = 1/2$, that $p \geq 2$, let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables, and set $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E\,X^2 \frac{(\log^+|X|)^{p-1}}{\log^+\log^+|X|} < \infty, \quad E\,X = 0, \quad \textit{and} \quad \operatorname{Var} X = \sigma^2,$$

then, for $\varepsilon > \sigma\sqrt{2p}$,

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\Big(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\log\big(\prod_{i=1}^{p} n_i\big) \prod_{i=p+1}^{d} n_i^{\alpha_i}} \cdot \varepsilon\Big) < \infty\,;$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\big(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\log\big(\prod_{i=1}^{p} n_i\big) \prod_{i=p+1}^{d} n_i^{\alpha_i} \cdot \varepsilon}\big) < \infty.$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, *then* $E\,X^2 \frac{(\log^+|X|)^{p-1}}{\log^+\log^+|X|} < \infty$ *and* $E\,X = 0$.

Theorem 8.5. *Suppose that* $\alpha_1 = 1/2$, *that* $2 \le p < d$, *let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ *be i.i.d. random variables, and set* $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. *If*

$$E\,X^2 \frac{(\log^+|X|)^{d-2}}{\log^+\log^+|X|} < \infty, \quad E\,X = 0, \quad \textit{and} \quad \operatorname{Var} X = \sigma^2,$$

then, for $\varepsilon > \sigma\sqrt{2p}$,

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\big(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\log\big(\prod_{i=1}^{p} n_i\big) \cdot \log\big(\prod_{i=p+1}^{d} n_i\big) \cdot \varepsilon}\big) < \infty\,;$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\big(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^{p} n_i \cdot \log\log\big(\prod_{i=1}^{p} n_i\big) \cdot \log\big(\prod_{i=p+1}^{d} n_i\big) \cdot \varepsilon}\big) < \infty.$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, *then* $E\,X^2 \frac{(\log^+|X|)^{d-2}}{\log^+\log^+|X|}$ *and* $E\,X = 0$.

Theorem 8.6. *Suppose that* $\alpha_1 = 1/2$, *that* $2 \le p < q < d$, *let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ *be i.i.d. random variables, and set* $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. *If*

$$E|X|^2 \frac{(\log^+|X|)^{q-2}}{\log^+\log^+|X|} < \infty, \quad E\,X = 0, \quad \textit{and} \quad \operatorname{Var} X = \sigma^2,$$

then, for $\varepsilon > \sigma\sqrt{2p}$, *we have*

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\big(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^{q} n_i \log\log\big(\prod_{i=1}^{p} n_i\big) \cdot \log\big(\prod_{i=p+1}^{q} n_i\big)} \cdot \prod_{i=q+1}^{d} n_i^{\alpha_i} \cdot \varepsilon\big) < \infty\,;$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\big(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^{q} n_i \log\log\big(\prod_{i=1}^{p} n_i\big) \cdot \log\big(\prod_{i=p+1}^{q} n_i\big)} \cdot \prod_{i=q+1}^{d} n_i^{\alpha_i} \cdot \varepsilon\big) < \infty.$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, *then* $E|X|^2 \frac{(\log^+|X|)^{q-2}}{\log^+\log^+|X|}$ *and* $E\,X = 0$.

Remark 8.7. When $p = d = 1$ Theorem 8.3 reduces to Lai (1974), Theorem 3, and for $p = d \ge 2$ to Gut (1980), Theorems 3.4 and 3.5. When $p = d$ in Theorem 8.4 one rediscovers Gut (1980), Theorem 6.2.

Remark 8.8. The reason for strict inequalities between p, q, and d in the last two results is that there is no "continuity" in the moment assumptions between those theorems and the earlier ones.

Remark 8.9. The first and necessary moment condition in Theorem 8.3 implies, in particular, that the variance is finite *except for the case when* $r = 2$ and $p = 1$. However, one can show (cf. Gut (1980), p. 301) that an intermediate condition is sufficient when $r = 2$ and $(p =)\, d = 1$ in the symmetric case. For the complicated precise condition and for more on this exceptional case we refer to Spătaru (2001). A similar remark applies to the case $p = 1$, since the variance is automatically finite unless $p = 1$.

A related problem occurs in the LIL where the proof of the necessity is "easy" when $d \geq 2$ and "hard" when $d = 1$.

9. Two additional problems

9.1. Other weights

In all results of the H-R-E-S-B-K kind the probabilities have had polynomial weights so far. So, what happens if the weights grow faster than polynomially? But not fast enough for the moment generating function to exist?

A first result in this direction is due to Lanzinger (1998), and corresponds to the equivalence of the moment condition and the convergence of the first sum for $d = 1$ (in a two-sided and, thus, stronger form) in the following result.

Theorem 9.1. *Let* $0 < \alpha < 1$, *and suppose that* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ *are i.i.d. random variables with* $E\,X = 0$ *and partial sums* $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. *The following are equivalent:*

$$E\exp\{|X|^\alpha\}(\log^+|X|)^{d-1} < \infty;$$

$$\sum_{\mathbf{n}} \exp\{|\mathbf{n}|^\alpha\} \cdot |\mathbf{n}|^{\alpha-2} P(|S_{\mathbf{n}}| > |\mathbf{n}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1;$$

$$\sum_{\mathbf{n}} \exp\{|\mathbf{n}|^\alpha\} \cdot |\mathbf{n}|^{\alpha-2} P(\max_{\mathbf{k}\leq\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1;$$

$$\sum_{j=1}^{\infty} \exp\{j^\alpha\} \cdot j^{\alpha-2} P(\sup_{j\leq|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1.$$

There remains, in fact, an intermediate case, namely, when the weights are between polynomial and exponential in the following sense.

Theorem 9.2. *Let* $\alpha > 1$, *and suppose that* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ *are i.i.d. random variables with* $E\,X = 0$ *and partial sums* $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. *The following are equivalent:*

$$E\exp\{(\log|X|)^\alpha\}(\log^+|X|)^{d-1} < \infty;$$

$$\sum_{\mathbf{n}} \exp\{(\log|\mathbf{n}|)^{\alpha}\} \cdot \frac{(\log|\mathbf{n}|)^{\alpha-1}}{|\mathbf{n}|^2} P(|S_{\mathbf{n}}| > |\mathbf{n}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1;$$
$$\sum_{\mathbf{n}} \exp\{(\log|\mathbf{n}|)^{\alpha}\} \cdot \frac{(\log|\mathbf{n}|)^{\alpha-1}}{|\mathbf{n}|^2} P(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}|\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1;$$
$$\sum_{j=1}^{\infty} \exp\{(\log j)^{\alpha}\} \cdot \frac{(\log j)^{\alpha-1}}{j^2} P(\sup_{j\le|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 1.$$

Once again, we refer to the original source Gut and Stadtmüller (2011c) for proofs and further details.

9.2. Last exit times

A strong limit theorem tells us, i.a., that the number of exceedances of some kind is a.s. finite. For the LLN (with obvious notation) this means that $P(|S_n| > n\varepsilon \text{ i.o.}) = 0$ for any $\varepsilon > 0$. Now, given this, one may ask for the number of them or the last time an exceedance occurs, which is called the *last exit time*, denoted $L(\varepsilon)$. The LLN is thus equivalent to the statement $P(L(\varepsilon) < \infty) = 1$. When $d = 1$ it is (maybe) more natural to put interest in $N(\varepsilon) =$ the number of exceedances, but, due to the partial order of $\mathbf{Z}_+^d$ we shall stick to last exit times here.

The point is that there is an obvious connection to the previous results. Namely, letting $a_{\mathbf{n}}$ denote $|\mathbf{n}|^{\alpha}$, $|\mathbf{n}^{\boldsymbol{\alpha}}|$, $\sqrt{|\mathbf{n}|\log|\mathbf{n}|}$, or $\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}$, then, for

$$L_d(\varepsilon) = \sup\{|\mathbf{n}| : |S_{\mathbf{n}}| > a_{\mathbf{n}}\varepsilon\},$$

we always have

$$\{L_d(\varepsilon) \ge j\} = \{\sup_{j\le|\mathbf{k}|} |S_{\mathbf{k}}/a_{\mathbf{k}}| > \varepsilon\},$$

which implies, for example, that

$$E\big(L_d(\varepsilon)\big)^r \asymp \sum_{j=1}^{\infty} j^{r-1} P(\sup_{j\le|\mathbf{k}|} |S_{\mathbf{k}}/a_{\mathbf{k}}| > \varepsilon),$$

after which the appropriate result above provides the relevant conditions for a moment of a given order to exist.

We confine ourselves with providing two examples, and leave it to the reader to invent the conclusion of his/her favorite choice.

Theorem 9.3. *Let $\alpha_1 > 1/2$, $\alpha_1 r > 1$, and set $L_d(\varepsilon) = \sup\{|\mathbf{n}| : |S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon\}$. The following are equivalent:*

$$E|X|^r(\log^+|X|)^{p-1} < \infty \quad \textit{and, if} \quad r \ge 1,\ E\,X = 0,$$
$$E\big(L_d(\varepsilon)\big)^{\alpha_1 r-1} < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Theorem 9.3 is (of course) related to Theorem 8.2. When $p = d$ it reduces to Gut (1980), Theorem 8.1.

As a final remark we mention that, for $a_n = \sqrt{n \log\log n}$, Slivka (1969) showed that no finite moment exists for the corresponding counting variable, which immediately implies the same for the last exit times and, all the more, for $L_d(\varepsilon)$. However, it was shown in Gut (1980), Theorem 8.3, that logarithmic moments may exist. More precisely:

Theorem 9.4. *Let* $L_d(\varepsilon) = \sup\{|\mathbf{n}| : |S_{\mathbf{n}}| > \sqrt{|\mathbf{n}| \log\log |\mathbf{n}|\varepsilon}\}$*, and suppose that* $E\,X = 0$ *and* $\operatorname{Var} X = \sigma^2 < \infty$*. Then*
(a) $E\big(L_d(\varepsilon)\big)^r = +\infty$ *for all* $r > 0$ *and all* $\varepsilon > 0$*.*
(b) *If, in addition,* $EX^2 \frac{(\log^+ |X|)^d}{\log\log^+ |X|} < \infty$*, then* $E \log L_d(\varepsilon) < \infty$ *for* $\varepsilon > \sigma\sqrt{2(d+1)}$*.*

10. Martingales and the LLN for random fields

New problems appear in random field settings, because there exist *four* different definitions of martingales.

In the standard definition one defines a family of nested σ-algebras $\{\mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d\}$ and an adapted family $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d\}$ of random variables, which together constitute a martingale iff

$$E(X_{\mathbf{n}} \mid \mathcal{F}_{\mathbf{m}}) = X_{\mathbf{m}} \quad \text{for} \quad \mathbf{m} \le \mathbf{n}.$$

The martingale convergence theorem runs as follows.

Theorem 10.1. (a) *If* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d\}$ *is a martingale, such that*

$$\sup_{\mathbf{n}} E|X_{\mathbf{n}}|(\log^+ |X_{\mathbf{n}}|)^{d-1} < \infty,$$

then $X_{\mathbf{n}}$ *converges almost surely as* $\mathbf{n} \to \infty$*.*
(b) *The same is true if the index set is a sector* S_θ^d *in* $\mathbf{Z}_+^d$*.*

Now, introducing a random field $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d\}$ of i.i.d. random variables, it is known that the field $\{X_{\mathbf{n}} = \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k} \le \mathbf{n}} Y_{\mathbf{k}}\}$, where $\mathbf{n} \in \mathbf{Z}_+^d$ or $\mathbf{n} \in S_\theta^d$, of arithmetic means consitute reversed martingales to which Theorem 10.1 is applicable.

The LLN thus follows *immediately* from Theorem 10.1.

We may thus combine our knowledge about the law of large numbers and about martingales as follows:

- The LLN in $\mathbf{Z}_+^d$ holds iff $E\,M(|Y|) < \infty$ i.e., iff $E|Y|(\log^+ |Y|)^{d-1} < \infty$;
- The LLN in the sector S_θ^d holds iff $E\,M(|Y|) < \infty$ i.e., iff $E|Y| < \infty$;
- Martingale convergence holds in both cases iff $E|Y|(\log^+ |Y|)^{d-1} < \infty$.

The moral of the story is that for the *sector* the martingale proof yields a weaker result, since the LLN requires only finite mean. The explanation is that

- LLN: The decisive point concerning logarithms or not is the *size* of the index set.
- Martingales: Logarithms are present because of the *dimension* of the index set.

So, even though the martingale proof is an elegant so-called one-line proof it is inferior in cases such as the sector.

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