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Large deviations for some normalized sums of exponentially distributed random variables*

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

We prove large deviation results for sequences of normalized sums which are defined in terms of triangular arrays of exponentially distributed random variables. We also present some examples: one of them might have applications in reliability theory because it concerns the spacings of i.i.d. exponentially distributed random variables; in another one we consider a sequence of logarithmically weighted means.

Keywords: large deviations, exponential distribution, Riemann-ζ function, triangular array, spacings, logarithmically weighted mean.

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1. Introduction

Throughout the paper we use the symbol $Z \sim \mathcal{E}(\lambda)$ to mean that a random variable Z has exponential distribution with parameter λ, i.e. Z has continuous density $f_Z(t) = \lambda e^{-\lambda t} 1_{(0,\infty)}(t)$. The aim is to study the convergence and to present results on large deviations for the sequence $(R_n)_{n\geq 1}$ defined by

$$R_n := \frac{\sum_{j=1}^{n} T_j^{(n)}}{\gamma_n},$$

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where: $(T_j^{(n)})_{n\geq j\geq 1}$ is a triangular array of exponentially distributed random variables i.e., for every $n \geq 1$, $T_1^{(n)}, \ldots, T_n^{(n)}$ are independent and $T_j^{(n)} \sim \mathcal{E}(\lambda_j^{(n)})$ for some $(\lambda_j^{(n)})_{j\leq n}$; we put $\gamma_n := \sum_{j=1}^n s_{j,n}$ for $s_{j,n} := \frac{1}{\lambda_j^{(n)}}$, and we assume in the whole paper that $\lim_{n\to\infty} \gamma_n = +\infty$.

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (we refer to [2] for this topic), and the basic concept of Large Deviation Principle (LDP from now on) consists of an upper bound for all closed sets and a lower bound for all open sets. Here we can prove the upper bound for all closed sets (Theorem 3.1) and the lower bound for a class of open sets (Theorem 3.2) which depends on a constant $c > 0$ appearing in the assumptions. It is worth noting that, if $c \geq 1$, this class of open sets coincides with all the open sets; therefore, as stated in Corollary 3.6 below, we have a full LDP if $c \geq 1$.

We remark that in our setting we obtain a linear rate function (see I in eq. (3.2) below). This situation is completely different from the classical one, in which all the random variables $(T_j^{(n)})_{n\geq j\geq 1}$ have the *same* exponential distribution, i.e. $\mathcal{E}(1)$ (see assumption (ii) in Theorem 3.1), and $\gamma_n = n$ (for all $n \geq 1$). In such a case $(R_n)_{n\geq 1}$ is a sequence of partial empirical means of i.i.d. random variables and, by the well-known Cramér Theorem (see e.g. Theorem 2.2.3 in [2]), the LDP holds with a strictly convex rate function.

We also give some illustrative examples. In Example 4.1 we have $\lambda_j^{(n)} = j$ for all $j = 1, \ldots, n$; in view of potential applications in reliability theory, we notice that (for every $n \geq 1$) the random variables $(T_j^{(n)})_{j\leq n}$ can be considered as the *spacings* of independent random variables with distribution $\mathcal{E}(1)$ (see Remark 4.2). Example 4.3 consists of a simple choice of $(\lambda_j^{(n)})_{n\geq j\geq 1}$ such that $\lim_{n\to\infty} \lambda_j^{(n)} = j$ for all $j \geq 1$. In some sense Example 4.4 comes up in natural way by considering a slight change of the values $(\lambda_j^{(n)})_{n\geq j\geq 1}$ in Example 4.3; an interesting feature is that the value $\zeta(2)$ (i.e. the Riemann-ζ function computed at 2) plays a crucial role in the computations; moreover we give a version of Example 4.4 which reveals a connection with the logarithmically weighted means as in the recent paper [3] (see Remark 4.5). The full LDP can be proved for Examples 4.1–4.3 only, since Corollary 3.6 can be applied only for those two examples.

The paper is organized as follows: in Section 2 we give some preliminary results and illustrate some facts about large deviations; in Section 3 we state our results; in Section 4 we present the examples; Section 5 contains the proofs.

2. Preliminaries on large deviations and first results

We start by giving some convergence results for the sequence $(R_n)_{n\geq 1}$.

Proposition 2.1. *Assume that*

$$\sup_{\substack{n\geq 1,\\ 1\leq j\leq n}} s_{j,n} = C < +\infty.$$

Then $R_n \longrightarrow 1$ in probability as $n \to \infty$.

Proof. Since $\mathbf{E}\big[T_j^{(n)}\big] = s_{j,n}$, we have

$$R_n - 1 = \frac{\sum_{j=1}^n \big(T_j^{(n)} - \mathbf{E}\big[T_j^{(n)}\big]\big)}{\gamma_n}$$

and, by Chebyshev inequality,

$$P\left(\left|\frac{\sum_{j=1}^n \big(T_j^{(n)} - \mathbf{E}\big[T_j^{(n)}\big]\big)}{\gamma_n}\right| > \epsilon\right)$$
$$\leq \frac{\mathbf{Var}\big(\sum_{j=1}^n T_j^{(n)}\big)}{\epsilon^2\gamma_n^2} = \frac{\sum_{j=1}^n s_{j,n}^2}{\epsilon^2\gamma_n^2} \leq C\left(\frac{\sum_{j=1}^n s_{j,n}}{\gamma_n}\right)\left(\frac{1}{\epsilon^2\gamma_n}\right) \to 0,$$

as $n \to \infty$. □

In some particular cases convergence in probability can be improved to almost sure convergence; this will be shown in the following

Proposition 2.2. *Let $(X_j)_{j\geq 1}$ be a sequence of i.i.d. random variables, with $X_j \sim \mathcal{E}(1)$ for every j. Assume that $T_j^{(n)} := s_{j,n}X_j$. If*

$$\frac{\sup_{1\leq j\leq n} s_{j,n}}{\gamma_n} = \mathrm{o}\Big(\frac{1}{\sqrt{n\log n}}\Big),$$

then $R_n \longrightarrow 1$ $P-a.s.$ as $n \to \infty$.

Proof. Since $R_n - 1 = \sum_{j=1}^n a_{j,n}\big(X_j - 1\big)$ with $a_{j,n} = \frac{s_{j,n}}{\gamma_n}$, the result follows from Corollary 4 of [5]. □

The main asymptotic results in this paper concern large deviations. We start by recalling the definition of LDP, for which we refer to [2] (pages 4–5). Let $\mathcal{X}$ be a topological space equipped with its completed Borel σ-field. A sequence of $\mathcal{X}$-valued random variables $(Z_n)_{n\geq 1}$ satisfies the LDP with speed function v_n and rate function I if: $\lim_{n\to\infty} v_n = +\infty$; the function $I\colon \mathcal{X} \to [0;\infty]$ is lower semi-continuous;

$$\limsup_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in F) \leq -\inf_{x\in F} I(x) \qquad \text{for all closed sets } F; \tag{2.1}$$

$$\liminf_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in G) \geq -\inf_{x\in G} I(x) \qquad \text{for all open sets } G. \tag{2.2}$$

A rate function I is said to be *good* if its level sets $\{\{x \in \mathcal{X} : I(x) \leq \eta\} : \eta \geq 0\}$ are compact.

Throughout the paper we always have $\mathcal{X} = \mathbb{R}$ and we consider applications of Gärtner–Ellis Theorem (see e.g. Theorem 2.3.6 in [2]). The application of this theorem for the sequence $(Z_n)_{n\geq 1}$ consists in checking the existence of the function $\Lambda\colon \mathbb{R} \to (-\infty, \infty]$ defined by

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{v_n} \log \mathbf{E}[e^{\theta v_n Z_n}].$$

Then, if 0 belongs to the interior of $\{\theta \in \mathbb{R} : \Lambda(\theta) < \infty\}$ and if we set

$$I(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda(\theta)\}, \tag{2.3}$$

we have: (a) the upper bound (2.1); (b) the lower bound

$$\liminf_{n\to\infty} \frac{1}{v_n} \log P(Z_n \in G) \geq -\inf_{x\in G\cap\mathcal{F}} I(x) \qquad \text{for all open sets } G, \tag{2.4}$$

where $\mathcal{F}$ is the set of exposed points (see e.g. Definition 2.3.3 in [2]); (c) if Λ is essentially smooth (see e.g. Definition 2.3.5 in [2]) and lower semi-continuous, the LDP holds with a good rate function. Thus, if Λ is not essentially smooth, Gärtner–Ellis Theorem may provide a trivial non-sharp lower bound for open sets in terms of the exposed points of the rate function. It is exactly what happens in our case (see Theorem 3.1). Indeed Theorem 2.3.6 (b–c) in [2] would lead to the non-sharp lower bound (2.4) with $\mathcal{F} = \{1\}$, and this coincides with the sharp lower bound (2.2) if and only if $1 \in G$.

We point out that Corollary 3.6 here below provides an example in which the LDP holds, i.e. a case where the lower bound (2.4) (in terms of the exposed points) can be improved obtaining the lower bound for all open sets (2.2). Other examples are the one presented in Remark (d) after the statement of Theorem 2.3.6 in [2] where we have again a linear rate function (it is slightly different from the rate function I in eq. (3.2) below), and Exercise 2.3.24 in [2].

3. Statements of the main results

In order to apply Gärtner–Ellis Theorem, the first thing to do is to check the existence of the limit

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{v_n} \log \mathbf{E}[\exp(\theta v_n R_n)] = \lim_{n\to\infty} \frac{1}{v_n} \log \mathbf{E}\left[\exp\left(\theta \frac{v_n}{\gamma_n} \sum_{j=1}^{n} T_j^{(n)}\right)\right] \tag{3.1}$$

for all $\theta \in \mathbb{R}$, where v_n is the speed. We start with the following result where $v_n = \gamma_n$.

Theorem 3.1. *Let the following assumptions hold:*
(i) for each $n \geq 1$, the function $j \mapsto \lambda_j^{(n)}$ $(j = 1, \dots, n)$ is non-decreasing and $\lim_{n\geq j\to\infty} \lambda_j^{(n)} = +\infty$;

(ii) $n \mapsto \lambda_1^{(n)}$ *is ultimately monotone and* $\lim_{n\to\infty} \lambda_1^{(n)} = 1$.
Then the limit $\Lambda(\theta)$ *in* (3.1) *exists for every* $\theta \in \mathbb{R} \setminus \{1\}$ *with* $v_n = \gamma_n$, *and we have*

$$\Lambda(\theta) = \begin{cases} \theta & for\ \theta < 1 \\ +\infty & for\ \theta > 1. \end{cases}$$

It is easy to check that, if the limit $\Lambda(\theta)$ in (3.1) exists for $\theta = 1$, we have $\Lambda(1) \in [1, \infty]$ and the function I in (2.3) becomes

$$I(x) = \begin{cases} x - 1 & \text{for } x \geq 1 \\ +\infty & \text{for } x < 1. \end{cases} \tag{3.2}$$

Moreover, the function Λ is not essentially smooth; hence Gärtner–Ellis Theorem cannot give the sharp lower bound (2.2). In the next result we obtain a weak form of the lower bound by considering eq. (1.2.8) in [2].

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. Assume moreover that:*
(i) $\gamma_n \geq c \log n + \mathrm{o}(\log n)$ *ultimately* ($c > 0$ *constant);*
(ii) for $n \geq j \geq 1$, $\lambda_j^{(n)} - \lambda_1^{(n)} \geq j - 1$;
(iii) for each $n \geq 1$, $j \mapsto \frac{\lambda_j^{(n)} - \lambda_1^{(n)}}{j-1}$ ($j = 2, \ldots, n$) *is non-decreasing.*
Then, for $x \geq 1/c$ *and for all open sets* G *such that* $x \in G$, *we have*

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log P(R_n \in G) \geq -I(x),$$

where I *is as in* (3.2).

Remark 3.3. Assumption (iii) of Theorem 3.2 holds for instance if, for each integer n, the (finite) sequence $j \mapsto \lambda_j^{(n)}$ is the restriction to $\mathbb{N} \cap [2, n]$ of a convex function $x \mapsto f_{(n)}(x)$ defined on $[1, n]$.

Remark 3.4. We notice for future reference that assumption (iii) of Theorem 3.2 implies that, for $i \neq j$,

$$\frac{\lambda_i^{(n)} - \lambda_1^{(n)}}{|\lambda_j^{(n)} - \lambda_i^{(n)}|} \leq \frac{i-1}{|j-i|}.$$

In fact, for $j > i$, it gives

$$\frac{\lambda_j^{(n)} - \lambda_1^{(n)}}{\lambda_i^{(n)} - \lambda_1^{(n)}} \geq \frac{j-1}{i-1},$$

hence, by assumption (i) of Theorem 3.1,

$$\frac{\lambda_i^{(n)} - \lambda_1^{(n)}}{|\lambda_j^{(n)} - \lambda_i^{(n)}|} = \frac{\lambda_i^{(n)} - \lambda_1^{(n)}}{\lambda_j^{(n)} - \lambda_i^{(n)}} = \frac{1}{\frac{\lambda_j^{(n)} - \lambda_1^{(n)}}{\lambda_i^{(n)} - \lambda_1^{(n)}} - 1} \leq \frac{1}{\frac{j-1}{i-1} - 1} = \frac{i-1}{j-i} = \frac{i-1}{|j-i|}.$$

The proof for $i < j$ is similar.

Remark 3.5. A careful look at the proofs shows that assumption (ii) of Theorem 3.2 could be relaxed as follows:
(ii)′ There exists a sequence $(a_n)_{n\geq 1}$, with $\lim_{n\to\infty} a_n = 1$, such that, for every integer n and for each $j = 2, \dots, n$

$$\lambda_j^{(n)} - \lambda_1^{(n)} \geq a_n(j-1).$$

It follows that, if $\big(\lambda_j^{(n)}\big)_{j\leq n}$ verifies *(ii)′*, the same happens for $\big(\widetilde{\lambda}_j^{(n)}\big)_{j\leq n}$ such that

$$\widetilde{\lambda}_j^{(n)} = d_n\big(\lambda_j^{(n)} + c_n\big),$$

where $(c_n)_{n\geq 1}$ is any sequence and $\lim_{n\to\infty} d_n = 1$.

It is obvious that the weaker form of the lower bound provided by Theorem 3.2 coincides with the lower bound (2.2) if $c \geq 1$. Thus, putting together the results of Theorems 3.1 and 3.2 and Gärtner Ellis Theorem, we get the following corollary.

Corollary 3.6. *Let the whole set of assumptions (i) and (ii) of Theorem 3.1 and (i), (ii) and (iii) of Theorem 3.2 hold. Moreover we assume that the limit $\Lambda(\theta)$ in* (3.1) *exists for $\theta = 1$ with $v_n = \gamma_n$. Then, if $c \geq 1$, $(R_n)_{n\geq 1}$ satisfies an LDP with speed $v_n = \gamma_n$ and rate function I as* (3.2).

4. Examples

In this section we present some examples checking for each of them that the assumptions of Theorems 3.1–3.2 hold. We remark that Corollary 3.6 is in force (and therefore the LDP holds) for Examples 4.1–4.3, where $c \geq 1$. Here is the first example.

Example 4.1. Let $(\lambda_j^{(n)})_{j\leq n}$ be defined by $\lambda_j^{(n)} := j$ for $j = 1, \dots, n$ and $n \geq 1$.

Remark 4.2. Let $\{X_n : n \geq 1\}$ be independent random variables such that $X_n \sim \mathcal{E}(1)$ for all $n \geq 1$ and, for every $n \geq 1$, consider the order statistics $X_{n,n} \leq \dots \leq X_{1,n}$ of $X_1, \dots, X_n$; then the *spacings* $(T_j)_{j\leq n}$ defined by

$$T_j^{(n)} := X_{j,n} - X_{j+1,n}, \qquad j = 1, \dots, n \qquad (\textit{where } X_{n+1,n} = 0),$$

meet the framework of Example 4.1 (see for instance [1], Ex. 4.1.5, p. 185).

In this case the assumptions of Theorems 3.1–3.2 can be easily checked. Here we only notice that assumption (i) of Theorem 3.2 holds with $c = 1$ since $\gamma_n = \sum_{j=1}^n \frac{1}{j} \geq \log(n+1)$. Finally we can apply Corollary 3.6 because we have $\Lambda(1) = 1$ with $v_n = \gamma_n$ (this can be easily checked and we omit the details).

In the next Example 4.3 we consider a particular choice of the values $(\lambda_j^{(n)})_{j\leq n}$. It is worth noting that $\lim_{n\to\infty} \lambda_j^{(n)} = j$, which are the parameters in Example 4.1.

Example 4.3. Let $(\lambda_j^{(n)})_{j\leq n}$ be defined by $\lambda_j^{(n)} := \frac{1}{\frac{1}{j}-\frac{1}{n+1}} = \frac{(n+1)j}{n+1-j}$ for $j=1,\ldots,n$ and $n\geq 1$.

In this case the assumptions of Theorems 3.1–3.2 can be checked as follows. The assumptions (i) and (ii) of Theorem 3.1 are obvious. As to (i) of Theorem 3.2 (again with $c=1$) we notice that

$$\gamma_n = \sum_{j=1}^{n}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{n+1} = \sum_{j=1}^{n}\frac{1}{j} - \frac{n}{n+1} \geq \log(n+1) - \frac{n}{n+1}.$$

Assumption (ii) of Theorem 3.2 holds since

$$\lambda_j^{(n)} - \lambda_1^{(n)} = \frac{n+1}{n+1-j}\cdot\frac{n+1}{n}(j-1) \geq j-1;$$

moreover, it is easily seen that the function $x \mapsto f_{(n)}(x) = \frac{(n+1)x}{n+1-x}$ is convex, and we deduce that also (iii) of Theorem 3.2 is verified, by Remark 3.3. Finally, as for Example 4.1, we can apply Corollary 3.6 because we have $\Lambda(1)=1$ with $v_n=\gamma_n$ (this can be easily checked and we omit the details).

In the previous Example 4.3 we had

$$\frac{1}{\lambda_j^{(n)}} = \frac{1}{j} - \frac{1}{n+1} = \int_j^{n+1} \frac{1}{x^2}\,dx.$$

A natural idea is to investigate what happens if we substitute the integral with the sum over integers, i.e. if we consider $\sum_{k=j}^{n}\frac{1}{k^2}$ instead of $\int_j^{n+1}\frac{1}{x^2}\,dx$. Since in such a case $\lim_{n\to\infty}\frac{1}{\sum_{k=1}^{n}\frac{1}{k^2}} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \simeq 0.608 \neq 1$, assumption (ii) of Theorem 3.1 is satisfied if we perform a "normalization"; this leads to the following

Example 4.4. Let $(\lambda_j^{(n)})_{j\leq n}$ be defined by $\lambda_j^{(n)} := \frac{\zeta(2)}{\sum_{k=j}^{n}\frac{1}{k^2}}$ for $j=1,\ldots,n$ and $n\geq 1$.

Remark 4.5. Let $(\lambda_j^{(n)})_{j\leq n}$ be as in Example 4.4 and let $(U_j)_{j\geq 1}$ be a sequence of independent random variables, and assume that they are uniformly distributed on $(0,1)$. Then we set

$$T_j^{(n)} := \frac{1}{\zeta(2)}\sum_{k=j}^{n}\frac{1}{k}F_k^{-1}(U_j) \qquad j=1,\ldots,n,$$

where $F_k^{-1}(u) = -\frac{1}{k}\log(1-u)$ (for $u\in(0,1)$) is the inverse of the distribution function of a random variable $Z\sim\mathcal{E}(k)$. This is a version of Example 4.4 because, for each fixed $n\geq 1$, $(T_1^{(n)},\ldots,T_n^{(n)})$ are independent (obvious) and, for all $j=1\ldots,n$, $T_j^{(n)} = \frac{1}{\zeta(2)}\sum_{k=j}^{n}\frac{1}{k^2}F_1^{-1}(U_j) = (\lambda_j^{(n)})^{-1}F_1^{-1}(U_j)$ with $F_1^{-1}(U_j)\sim\mathcal{E}(1)$, and therefore $T_j^{(n)} \sim \mathcal{E}(\lambda_j^{(n)})$. Finally we remark that R_n is a logarithmically weighted mean as in [3] because, if we set $X_k := \sum_{j=1}^k F_k^{-1}(U_j)$, we have

$$R_n = \frac{\sum_{j=1}^n T_j^{(n)}}{\gamma_n} = \frac{\sum_{j=1}^n \frac{1}{\zeta(2)} \sum_{k=j}^n \frac{1}{k} F_k^{-1}(U_j)}{\sum_{j=1}^n \frac{1}{\zeta(2)} \sum_{k=j}^n \frac{1}{k^2}}$$
$$= \frac{\sum_{k=1}^n \frac{1}{k} \sum_{j=1}^k F_k^{-1}(U_j)}{\sum_{k=1}^n \sum_{j=1}^k \frac{1}{k^2}} = \frac{\sum_{k=1}^n \frac{1}{k} X_k}{\sum_{k=1}^n \frac{1}{k}}.$$

Now we have to check all the conditions of Theorems 3.1–3.2 for Example 4.4. The assumptions of Theorem 3.1 are obvious. Assumption (i) of Theorem 3.2 holds since

$$\gamma_n = \frac{1}{\zeta(2)} \sum_{j=1}^n \sum_{k=j}^n \frac{1}{k^2} = \frac{1}{\zeta(2)} \sum_{k=1}^n \sum_{j=1}^k \frac{1}{k^2} = \frac{1}{\zeta(2)} \sum_{k=1}^n \frac{1}{k} \geq \frac{1}{\zeta(2)} \log(n+1).$$

Note that in this case we have $c = \frac{1}{\zeta(2)} < 1$ and Corollary 3.6 cannot be applied; for completeness we check $\Lambda(1) = 1$ with $v_n = \gamma_n$.

Proof of $\Lambda(1) = 1$ with $v_n = \gamma_n$ for Example 4.4. We have to check that

$$\lim_{n\to\infty} \frac{-\sum_{j=1}^n \log(1 - s_{j,n})}{\sum_{j=1}^n s_{j,n}} = 1$$

because $\gamma_n = \sum_{j=1}^n s_{j,n}$ and

$$\log \mathbf{E}\left[\exp\left(\sum_{j=1}^n T_j^{(n)}\right)\right] = \sum_{j=1}^n \log \mathbf{E}\left[e^{T_j^{(n)}}\right]$$
$$= \sum_{j=1}^n \log \frac{\lambda_j^{(n)}}{\lambda_j^{(n)} - 1} = -\sum_{j=1}^n \log(1 - s_{j,n}).$$

Moreover, since $-\log(1 - s_{j,n}) \geq s_{j,n}$, it is enough to check

$$\limsup_{n\to\infty} \frac{-\sum_{j=1}^n \log(1 - s_{j,n})}{\sum_{j=1}^n s_{j,n}} \leq 1$$

and, noting that

$$\sum_{j=1}^n s_{j,n} = \frac{1}{\zeta(2)} \sum_{j=1}^n \sum_{k=j}^n \frac{1}{k^2} = \frac{1}{\zeta(2)} \sum_{k=1}^n \sum_{j=1}^k \frac{1}{k^2} = \frac{1}{\zeta(2)} \sum_{k=1}^n \frac{1}{k} \sim \frac{1}{\zeta(2)} \log n,$$

this is equivalent to

$$\limsup_{n\to\infty} \frac{-\sum_{j=1}^n \log(1 - s_{j,n})}{\log n} \leq \frac{1}{\zeta(2)}. \tag{4.1}$$

Now, since $s_{j,n} \leq s_{j,\infty} = 1 - s_{1,j-1}$ and $x \in [0,1) \mapsto -\log(1-x)$ is an increasing function, we get $\frac{-\sum_{j=1}^{n}\log(1-s_{j,n})}{\log n} \leq \frac{-\sum_{j=1}^{n}\log(s_{1,j-1})}{\log n}$; thus (4.1) is implied by

$$\lim_{n\to\infty} \frac{-\sum_{j=1}^{n}\log(s_{1,j-1})}{\log n} = \frac{1}{\zeta(2)}$$

or, equivalently (by Cesaro Theorem), $\lim_{n\to\infty} -n\log(s_{1,n-1}) = \frac{1}{\zeta(2)}$; in conclusion (4.1) is implied by

$$\frac{1}{\zeta(2)} = \lim_{n\to\infty} n(1 - s_{1,n-1}) = \lim_{n\to\infty} \frac{n}{\zeta(2)} \sum_{k=n}^{\infty} \frac{1}{k^2},$$

which can be be checked noting that

$$\frac{1}{\zeta(2)} = \frac{n}{\zeta(2)} \int_{n}^{\infty} \frac{1}{x^2}dx \leq \frac{n}{\zeta(2)} \sum_{k=n}^{\infty} \frac{1}{k^2} \leq \frac{n}{\zeta(2)} \int_{n-1}^{\infty} \frac{1}{x^2}dx = \frac{1}{\zeta(2)}\frac{n}{n-1}. \qquad \square$$

We conclude with the proof of assumptions (ii)–(iii) of Theorem 3.2 for Example 4.4.

Proof of assumption (ii) of Theorem 3.2 for Example 4.4. The condition is obvious for $j = 1$ and, from now on, we assume that $j = 2, \ldots, n$. Since

$$\begin{aligned} \lambda_j^{(n)} - \lambda_1^{(n)} &= \zeta(2) \frac{\sum_{k=1}^{j-1} \frac{1}{k^2}}{\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)\left(\sum_{k=j}^{n} \frac{1}{k^2}\right)} \geq \frac{\sum_{k=1}^{j-1} \frac{1}{k^2}}{\left(\sum_{k=j}^{n} \frac{1}{k^2}\right)} \\ &= \frac{\sum_{k=1}^{j-1} \frac{1}{k^2}}{\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{j-1} \frac{1}{k^2}} \geq \frac{\sum_{k=1}^{j-1} \frac{1}{k^2}}{\zeta(2) - \sum_{k=1}^{j-1} \frac{1}{k^2}} = \frac{1}{\zeta(2)\left(\sum_{k=1}^{j-1} \frac{1}{k^2}\right)^{-1} - 1}, \end{aligned}$$

it suffices to show that the last quantity above is $\geq j-1$ or, in equivalent form, that

$$\frac{\zeta(2)}{\sum_{k=1}^{j-1} \frac{1}{k^2}} \leq \frac{j}{j-1}.$$

With some algebra, the inequality to be proved can be transformed into the equivalent one

$$\zeta(2) - \sum_{k=j}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{j-1} \frac{1}{k^2} \geq \zeta(2)\left(1 - \frac{1}{j}\right),$$

or, after simplification,

$$a_j := -\sum_{k=j}^{\infty} \frac{1}{k^2} + \frac{\zeta(2)}{j} \geq 0.$$

Since $\lim_{j\to\infty} a_j = 0$, it is enough to show that (a_j) is non-increasing, i.e. for every j

$$-\sum_{k=j+1}^{\infty}\frac{1}{k^2}+\frac{\zeta(2)}{j+1}\le-\sum_{k=j}^{\infty}\frac{1}{k^2}+\frac{\zeta(2)}{j},$$

and therefore

$$0\ge\sum_{k=j}^{\infty}\frac{1}{k^2}-\sum_{k=j+1}^{\infty}\frac{1}{k^2}+\zeta(2)\Big(\frac{1}{j+1}-\frac{1}{j}\Big)=\frac{1}{j^2}-\frac{\zeta(2)}{j(j+1)}.$$

Multiplying by $j^2(j+1)$ we get the equivalent inequality

$$\big(\zeta(2)-1\big)j\ge 1,$$

which is true since

$$\big(\zeta(2)-1\big)j\ge 2\big(\zeta(2)-1\big)\simeq 1.28. \qquad \square$$

Proof of assumption (iii) of Theorem 3.2 for Example 4.4. For $k\ge 1$ we set $s_k := \sum_{h=1}^{k}\frac{1}{h^2}$ and, for $n\ge 2$ and $j=1,\dots,n-1$, we set $d_j^{(n)} := \frac{s_j}{\left(s_n-s_j\right)j}$. Then we have

$$d_{j-1}^{(n)}=\frac{s_{j-1}}{(s_n-s_{j-1})(j-1)}=\frac{\frac{\lambda_j^{(n)}}{\lambda_1^{(n)}}-1}{j-1}=\frac{1}{\lambda_1^{(n)}}\cdot\frac{\lambda_j^{(n)}-\lambda_1^{(n)}}{j-1}\qquad (j=2,\dots,n);$$

therefore we need to prove that the finite sequence $\left(d_j^{(n)}\right)_j$ is non-decreasing, i.e.

$$d_{j-1}^{(n)}\le d_j^{(n)}\qquad (j=2,\dots,n-1).$$

After rearranging we see that this is equivalent to

$$s_n\le\frac{s_{j-1}s_j}{js_{j-1}-(j-1)s_j}\qquad (j=2,\dots,n-1); \tag{4.2}$$

moreover $s_n\uparrow\zeta(2)$ as $n\uparrow\infty$ and the right hand side in (4.2) tends to $\zeta(2)$ as $j\to\infty$; hence it suffices to show that the right hand side in (4.2) is a non-increasing function of j, i.e.

$$\frac{s_{j-1}s_j}{js_{j-1}-(j-1)s_j}\ge\frac{s_js_{j+1}}{(j+1)s_j-js_{j+1}}\qquad (j\ge 2).$$

We check this inequality with some algebra and by taking into account that $s_{j-1}=s_j-\frac{1}{j^2}$ and $s_{j+1}=s_j+\frac{1}{(j+1)^2}$; indeed we get the inequality

$$s_j\le\frac{2j}{j+1}=2\left(1-\frac{1}{j+1}\right),$$

which is obviously true, since

$$s_j=\sum_{h=1}^{j}\frac{1}{h^2}\le\sum_{h=1}^{j}\frac{2}{h(h+1)}=2\sum_{h=1}^{j}\left(\frac{1}{h}-\frac{1}{h+1}\right)=2\left(1-\frac{1}{j+1}\right). \qquad \square$$

5. The proofs

Recall the notations $s_{j,n} := \left(\lambda_j^{(n)}\right)^{-1}$ and $\gamma_n = \sum_{j=1}^n s_{j,n}$, which will be systematically used in the sequel.

Proof of Theorem 3.1. We give several proofs according to different values of θ.
• Let us consider first the case $\theta < 1$ (excluding the case $\theta = 0$, which is trivial). Fix $\delta \in (0, \frac{1}{2})$. Assumption (i) assures that exists j_0 such that, for $j_0 \le j \le n$, we have

$$\left|s_{j,n}\theta\right| < \delta.$$

We write

$$\begin{aligned}
&\frac{1}{\gamma_n}\log \mathbf{E}\left[\exp\left(\theta\sum_{j=1}^n T_j^{(n)}\right)\right] \\
&= \frac{1}{\gamma_n}\sum_{j=1}^n \log \mathbf{E}\left[\exp\left(\theta T_j^{(n)}\right)\right] = -\frac{\sum_{j=1}^n \log\left(1 - s_{j,n}\theta\right)}{\gamma_n} \\
&= \left(-\frac{\sum_{j=1}^{j_0}\log\left(1 - s_{j,n}\theta\right)}{\gamma_n}\right) + \left(-\frac{\sum_{j=j_0+1}^{n}\log\left(1 - s_{j,n}\theta\right)}{\gamma_n}\right) = A_n + B_n.
\end{aligned}$$

We shall prove that

(a) $\lim_{n\to\infty} A_n = 0$; (b) $\theta \le \liminf_{n\to\infty} B_n \le \limsup_{n\to\infty} B_n \le \theta + |\theta|\delta$.

Proof of (a). We treat separately the two cases (a$_1$) $\theta > 0$ and (a$_2$) $\theta < 0$.

Proof of (a$_1$). Since $\theta < 1$, there exists $\epsilon > 0$ such that $\theta < 1 - \epsilon < 1$. By assumption (ii), $\lambda_1^{(n)} > 1 - \epsilon$ ultimately, so that (i) implies that, for every $j \le n$,

$$s_{j,n}\theta \le s_{1,n}\theta \le \frac{\theta}{1-\epsilon} < 1.$$

Hence ultimately we have

$$0 \le A_n = -\frac{\sum_{j=1}^{j_0}\log\left(1 - s_{j,n}\theta\right)}{\gamma_n} \le -\frac{\sum_{j=1}^{j_0}\log\left(1 - \frac{\theta}{1-\epsilon}\right)}{\gamma_n} \to 0, \qquad n\to\infty.$$

Proof of (a$_2$). In this case we have $s_{j,n}\theta \in (-\delta, 0]$, and therefore $0 \le \log(1 - s_{j,n}\theta) = \log(1 + s_{j,n}|\theta|)$; moreover the sequence $\left(s_{1,n}\right)_n$, being convergent (to 1), is bounded by some positive real number C; hence for every $j \le n$ we have $s_{j,n} \le s_{1,n} \le C$, which gives

$$|A_n| = \frac{\sum_{j=1}^{j_0}\log\left(1 - s_{j,n}\theta\right)}{\gamma_n} = \frac{\sum_{j=1}^{j_0}\log\left(1 + s_{j,n}|\theta|\right)}{\gamma_n}$$

$$\leq \frac{\sum_{j=1}^{j_0} \left|\log\left(1+C|\theta|\right)\right|}{\gamma_n} \to 0, \quad n \to \infty.$$

Proof of (b). For $|x| < 1/2$ we have $x \leq -\log(1-x) \leq x + x^2$; hence

$$\theta \cdot \frac{\sum_{j=j_0+1}^{n} s_{j,n}}{\gamma_n} \leq B_n \leq \theta \cdot \frac{\sum_{j=j_0+1}^{n} s_{j,n}}{\gamma_n} + \theta^2 \cdot \frac{\sum_{j=j_0+1}^{n} s_{j,n}^2}{\gamma_n},$$

and it is enough to check
(b_1) $\lim_{n\to\infty} \frac{\sum_{j=j_0+1}^{n} s_{j,n}}{\gamma_n} = 1$ and (b_2) $\limsup_{n\to\infty} \frac{\sum_{j=j_0+1}^{n} s_{j,n}^2}{\gamma_n} \leq \frac{\delta}{|\theta|}$.
Proof of (b_1). We have

$$\frac{\sum_{j=j_0+1}^{n} s_{j,n}}{\gamma_n} = 1 - \frac{\sum_{j=1}^{j_0} s_{j,n}}{\gamma_n},$$

and (as we have seen before) $s_{j,n} \leq s_{1,n} \leq C$ for every $j \leq n$; we deduce that

$$0 \leq \frac{\sum_{j=1}^{j_0} s_{j,n}}{\gamma_n} \leq \frac{\sum_{j=1}^{j_0} C}{\gamma_n} \to 0, \qquad n \to \infty.$$

Proof of (b_2). By construction we have $s_{j,n}|\theta| < \delta$ for $n \geq j \geq j_0$; thus

$$0 \leq \frac{\sum_{j=j_0+1}^{n} s_{j,n}^2}{\gamma_n} \leq \frac{\delta}{|\theta|} \cdot \frac{\sum_{j=j_0+1}^{n} s_{j,n}}{\gamma_n} \leq \frac{\delta}{|\theta|} \cdot \frac{\sum_{j=1}^{n} s_{j,n}}{\gamma_n} = \frac{\delta}{|\theta|}.$$

• We pass to the case $\theta > 1$. Since $\lim_{n\to\infty} \lambda_1^{(n)} = 1$, there exists an integer n_0 such that, for every $n > n_0$, we have $\theta > \lambda_1^{(n)}$; hence

$$\frac{1}{\gamma_n} \log \mathbf{E}\left[\exp\left(\theta \sum_{j=1}^{n} T_j^{(n)}\right)\right] \geq \frac{1}{\gamma_n} \log \mathbf{E}\left[\exp\left(\theta T_1^{(n)}\right)\right] = +\infty. \qquad \square$$

Proof of Theorem 3.2. The inequality to be proved is trivial if $x < 1$ (because $I(x) = +\infty$) and if $x = 1$ it holds by Proposition 2.1 (because $I(x) = 0$); so, throughout this proof, we restrict our attention to the case $x > 1$. We choose $\epsilon > 0$ so small to have $(x-\epsilon, x+\epsilon) \subset G$; hence

$$P(R_n \in G) \geq P(x - \epsilon < R_n < x + \epsilon) \geq P(x < R_n < x + \epsilon).$$

The main proof consists in showing that we have

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log P(x < R_n < x + \epsilon) \geq 1 - x - \epsilon;$$

(in fact we easily get

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log P(R_n \in G) \geq 1 - x - \epsilon,$$

and let ϵ go to zero).

Let F and f be the distribution function and the density of $\sum_{j=1}^n T_j^{(n)}$ respectively. By Lagrange Theorem, there exists $\xi \in \big(x, x+\epsilon\big)$ such that

$$P(x < R_n < x+\epsilon) = F\big((x+\epsilon)\gamma_n\big) - F\big(x\gamma_n\big) = \epsilon \cdot \gamma_n \cdot f(\xi\gamma_n).$$

Passing to the logarithm and dividing by γ_n we get

$$\frac{1}{\gamma_n} \log P(x < R_n < x+\epsilon) = \frac{\log \epsilon}{\gamma_n} + \frac{\log \gamma_n}{\gamma_n} + \frac{\log\big(f(\xi\gamma_n)\big)}{\gamma_n},$$

and of course only the last summand has to be considered. According to a well known formula (see for instance [4], p. 308 and ff.), f has the form

$$\begin{aligned} f(t) &= (-1)^{n-1}\lambda_1^{(n)} \cdot \dots \cdot \lambda_n^{(n)} \sum_{j=1}^n \frac{e^{-\lambda_j^{(n)} t}}{\prod_{i\neq j}(\lambda_j^{(n)} - \lambda_i^{(n)})} \\ &= \lambda_1^{(n)} \cdot \dots \cdot \lambda_n^{(n)} \frac{e^{-\lambda_1^{(n)} t}}{\prod_{i\neq 1}(\lambda_i^{(n)} - \lambda_1^{(n)})} \cdot \left\{ 1 - \sum_{j=2}^n e^{-(\lambda_j^{(n)} - \lambda_1^{(n)})t} \cdot \prod_{i\neq 1,j} \frac{\lambda_1^{(n)} - \lambda_i^{(n)}}{\lambda_j^{(n)} - \lambda_i^{(n)}} \right\} \end{aligned}$$

(note that this formula is allowed because the values $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ are all different by the hypotheses). Then we take the logarithm and we get

$$\begin{aligned} \log f(t) = &\sum_{j=1}^n \log \lambda_j^{(n)} - \lambda_1^{(n)} t - \sum_{j=2}^n \log\big(\lambda_j^{(n)} - \lambda_1^{(n)}\big) \\ &+ \log\left\{ 1 - \sum_{j=2}^n e^{-(\lambda_j^{(n)} - \lambda_1^{(n)})t} \cdot \prod_{i\neq 1,j} \frac{\lambda_1^{(n)} - \lambda_i^{(n)}}{\lambda_j^{(n)} - \lambda_i^{(n)}} \right\}. \end{aligned}$$

Calculating in $t = \xi\gamma_n$ and dividing by γ_n we find

$$\begin{aligned} \frac{\log\big(f(\xi\gamma_n)\big)}{\gamma_n} = &\left(\frac{\log \lambda_1^{(n)}}{\gamma_n}\right) + \left(\frac{\sum_{j=2}^n \log \frac{\lambda_j^{(n)}}{\lambda_j^{(n)} - \lambda_1^{(n)}}}{\gamma_n}\right) + \left(-\lambda_1^{(n)}\xi\right) \\ &+ \left(\frac{1}{\gamma_n} \cdot \log\left\{ 1 - \sum_{j=2}^n e^{-(\lambda_j^{(n)} - \lambda_1^{(n)})\xi\gamma_n} \cdot \prod_{i\neq 1,j} \frac{\lambda_1^{(n)} - \lambda_i^{(n)}}{\lambda_j^{(n)} - \lambda_i^{(n)}} \right\}\right) \end{aligned}$$

$$=: A_n + B_n + C_n + D_n.$$

By the assumption (ii) of Theorem 3.1 we have $\lim_{n\to\infty} A_n = 0$ and $\lim_{n\to\infty} C_n = -\xi > -x-\epsilon$. So the proof will be complete if we show that (a) $\liminf_{n\to\infty} B_n \geq 1$ and (b) $\lim_{n\to\infty} D_n = 0$.

Proof of (a). For every pair x, y, with $0 < x < y$ the inequality

$$\log \frac{y}{y-x} \geq \frac{x}{y},$$

(which comes from $\log(1+t) \leq t$ putting $t = -\frac{x}{y}$), applied to $y = \lambda_j^{(n)}$ and $x = \lambda_1^{(n)}$ gives

$$B_n = \frac{\sum_{j=2}^n \log \frac{\lambda_j^{(n)}}{\lambda_j^{(n)}-\lambda_1^{(n)}}}{\gamma_n} \geq \frac{\sum_{j=2}^n \frac{\lambda_1^{(n)}}{\lambda_j^{(n)}}}{\gamma_n} = \lambda_1^{(n)} \frac{\sum_{j=2}^n \frac{1}{\lambda_j^{(n)}}}{\gamma_n} \to 1,$$

by assumption (ii) of Theorem 3.1.

Proof of (b). It suffices to show that $\lim_{n\to\infty} a_n = 0$, where

$$a_n := -\sum_{j=2}^n e^{-(\lambda_j^{(n)}-\lambda_1^{(n)})\xi\gamma_n} \cdot \prod_{i\neq 1,j} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_i^{(n)}-\lambda_j^{(n)}}.$$

To begin with, we write

$$\begin{aligned} -\prod_{i\neq 1,j} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_i^{(n)}-\lambda_j^{(n)}} &= -\prod_{i=2}^{j-1} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_i^{(n)}-\lambda_j^{(n)}} \cdot \prod_{i=j+1}^{n} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_i^{(n)}-\lambda_j^{(n)}} \\ &= (-1)^{j-1} \prod_{i=2}^{j-1} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_j^{(n)}-\lambda_i^{(n)}} \cdot \prod_{i=j+1}^{n} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{\lambda_i^{(n)}-\lambda_j^{(n)}} = (-1)^{j-1} \prod_{i\neq 1,j} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{|\lambda_i^{(n)}-\lambda_j^{(n)}|}; \end{aligned}$$

by assumption (ii) of Theorem 3.2 and Remark 3.4, we have

$$|a_n| \leq \sum_{j=2}^n e^{-(\lambda_j^{(n)}-\lambda_1^{(n)})\xi\gamma_n} \cdot \prod_{i\neq 1,j} \frac{\lambda_i^{(n)}-\lambda_1^{(n)}}{|\lambda_i^{(n)}-\lambda_j^{(n)}|} \leq \sum_{j=2}^n e^{-(j-1)\xi\gamma_n} \cdot \prod_{i\neq 1,j} \left(\frac{i-1}{|i-j|}\right);$$

hence, by assumption (i) of Theorem 3.2,

$$|a_n| \leq \sum_{j=2}^n \left(\frac{1}{\mathrm{e}^{b_n}}\right)^{j-1} \cdot \left(\prod_{i\neq 1,j} \frac{i-1}{|i-j|}\right),$$

where $b_n := c\xi \log n + \mathrm{o}(\log n)$. Now

$$\prod_{i\neq 1,j} \frac{i-1}{|i-j|} = \frac{(n-1)!}{j-1} \frac{1}{\prod_{i=2}^{j-1}(j-i)} \frac{1}{\prod_{i=j+1}^{n}(i-j)} = \frac{(n-1)!}{(j-1)!(n-j)!} = \binom{n-1}{j-1};$$

thus

$$|a_n| \leq \sum_{j=2}^{n} \binom{n-1}{j-1} \left(\frac{1}{\mathrm{e}^{b_n}}\right)^{j-1} = \sum_{j=1}^{n-1} \binom{n-1}{j} \left(\frac{1}{\mathrm{e}^{b_n}}\right)^{j} = \left(1 + \frac{1}{\mathrm{e}^{b_n}}\right)^{n-1} - 1.$$

Now we show that

$$\lim_{n\to\infty} \left(1 + \frac{1}{\mathrm{e}^{b_n}}\right)^{n-1} = 1,$$

or equivalently

$$\lim_{n\to\infty} (n-1) \log \left(1 + \frac{1}{\mathrm{e}^{b_n}}\right) = 0.$$

In fact, since $c\xi > 1$ (because $\xi > x$ and $x \geq 1/c$), we have

$$b_n - \log(n-1) = c\xi \log n + \mathrm{o}(\log n) - \log(n-1) \to +\infty, \qquad n \to +\infty,$$

whence

$$\lim_{n\to\infty} (n-1) \log \left(1 + \frac{1}{\mathrm{e}^{b_n}}\right) = \lim_{n\to\infty} \frac{n-1}{\mathrm{e}^{b_n}} = 0. \qquad \square$$

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