Biarc analysis for skinning of circles

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Abstract

By circle skinning we have a discrete set of circles and we would like to find two curves, which touch each of them and satisfy some conditions. There exist methods to give a solution for this problem, but none of them use biarcs for the construction. Meek and Walton published a very deep analysis of biarcs in [1], and they divided them into several families.

Of course one of the basic problems is to find the mentioned curves for *two* circles. In this paper several necessary conditions are given to avoid intersections in this basic case between the skinning curve and the circles using a concrete family of biarcs from [1]. A method is publicated in [3] with which we can find the touching points for the skinning.

Keywords: skinning, biarcs, interpolation, circles

1. Introduction

We have several well-known methods to interpolate a set of points, it is very important in Computer Aided Geometric Design to solve this problem. Skinning is a special case of interpolations, where we have circles instead of points and we would like to find a pair of curves, which touch each of the circles and have certain preferences. There exist actual researches to get a solution for skinning problem, often in higher dimensions with spheres [7, 8, 3, 2]. The results can be very useful by covering problems, geometric design, designing tubular structures or molecular modeling. The mentioned methods use C^1 or G^1 continuous curves by the interpolation. The main idea of this paper is to use biarcs for the construction, and find the skinning curves made by joining biarcs.
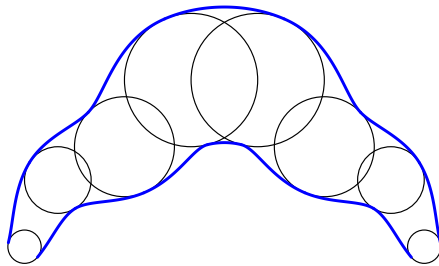
Figure 1: A skinning example

2. Localization of the touching points

Some researchers use an iterative method to find the touching points on the circles [7], others define them exactly before the curve interpolation [3]. In this paper the method is considered from [3]. The authors use Apollonius-circles for the construction, they find touching points on each circles with their neighbours. This method guarantees to find touching points which don't fit on other disks. This is a very powerful attribution, so we can use this method for our construction too.

3. Families of biarcs

If we join two circular arcs in G^1 continuity, we get a biarc. Biarcs have a long history, there exists a paper from 1937 [4], where biarcs was mentioned first time. We can create so-called *biarc curves* by joining biarcs [5], furthermore there are several methods to approximate a fixed curve with biarc curves [6]. This type of curves are very useful by design, because CNC machines can only cut lines and arcs. Meek and Walton published a very deep analysis of biarcs in [1], and they divided them into several families.

The authors construct a biarc from enhanced G^1 Hermite datas. This means we have two points, two unit tangent vectors and a total rotation of these vectors W, which can be greater, than 2π.

Of course one of the basic problems is to find a biarc for *two* circles. On the following picture (Figure 2) we can see the defining datas with some additional nominations supplemented with two circles. How can we determine θ to avoid intersections between arcs and circles?

The future goal is to analyse all of the families and determine necessary conditions to avoid intersections in each cases. In this paper we consider Case 1.3 (a) from [1]. In this case $r_A, r_B, \theta, W - \theta > 0$ and

$$2\pi < W < 4\pi,$$
$$2\pi + 2\alpha < W < 3\pi + \alpha,$$
$$-2\pi + W < \theta < 4\pi - W + 2\alpha.$$

(a) Enhanced G^1 Hermite data with a biarc

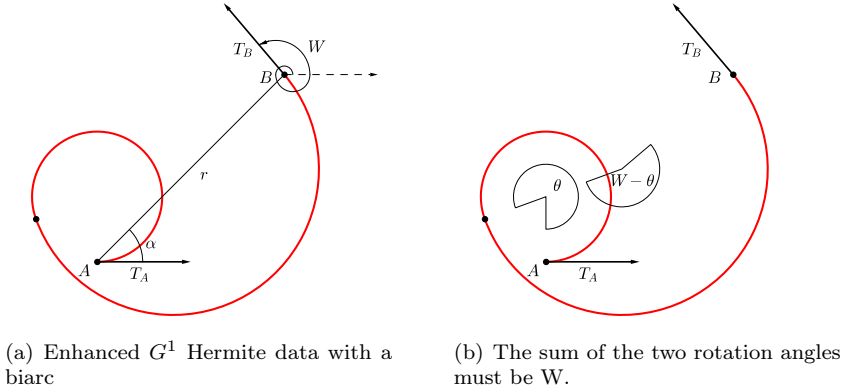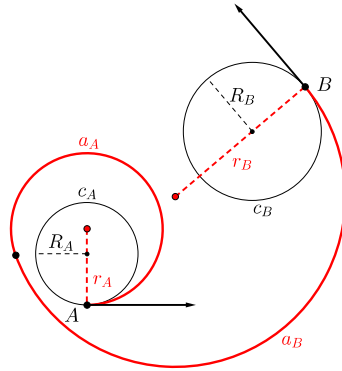(b) The sum of the two rotation angles must be W.

Figure 2

Figure 3

In [1] the authors mention positive and negative radii by the arcs, so we have to follow this convention by the circles. We can suppose that both radius of the circles are positiv, because we can determine the directions of the tangent vectors free by skinning problems. So the circles always can be placed at the "left side" of the tangent vectors. It is easy to see that we can avoid intersection between circle c_A and arc a_A with condition $r_A > R_A$. By similar arguments $r_B > R_B$ helps us to avoid intersection between a_B and c_B.

4. The calculation

4.1. $r_A > R_A$

We know that

$$r_A = \frac{r \cdot \sin\left(\frac{W+\theta}{2} - \alpha\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{W}{2}\right)} = r \cdot \frac{\sin\left(\frac{W}{2} - \alpha + \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{W}{2}\right)},$$

and

$$0 < \theta < 2\pi \Rightarrow \sin\frac{\theta}{2} > 0, \quad 2\pi < W < 4\pi \Rightarrow \sin\frac{W}{2} < 0, \quad 0 < \frac{\theta}{2} < \pi,$$

$$2\pi + 2\alpha < W < 3\pi + \alpha \quad \Rightarrow \quad \pi < \frac{W}{2} - \alpha < 2\pi \quad \Rightarrow \quad \sin\left(\frac{W}{2} - \alpha\right) < 0.$$

Now we should analyse the denominator to express θ from the following inequality:

$$r_A > R_A$$

$$\frac{r \cdot \sin\left(\frac{W+\theta}{2} - \alpha\right)}{2\sin\frac{\theta}{2}\sin\frac{W}{2}} > R_A$$

$$\frac{\sin\left(\left(\frac{W}{2} - \alpha\right) + \frac{\theta}{2}\right)}{\sin\frac{\theta}{2}} < \frac{2R_A \sin\frac{W}{2}}{r}$$

$$\frac{\sin\left(\frac{W}{2} - \alpha\right) \cdot \cos\frac{\theta}{2} + \cos\left(\frac{W}{2} - \alpha\right)\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} < \frac{2R_A \sin\frac{W}{2}}{r}$$

$$\sin\left(\frac{W}{2} - \alpha\right)\cot\frac{\theta}{2} < \frac{2R_A \sin\frac{W}{2}}{r} - \cos\left(\frac{W}{2} - \alpha\right)$$

$$\cot\frac{\theta}{2} > \frac{\frac{2R_A \sin\left(\frac{W}{2}\right)}{r} - \cos\left(\frac{W}{2} - \alpha\right)}{\sin\left(\frac{W}{2} - \alpha\right)}$$

For brevity let us set

$$K \doteq \frac{\frac{2R_A \sin\left(\frac{W}{2}\right)}{r} - \cos\left(\frac{W}{2} - \alpha\right)}{\sin\left(\frac{W}{2} - \alpha\right)}.$$

We distinguish two cases according as $K > 0$ or $K < 0$.

1. If $K > 0$

$$\cot\frac{\theta}{2} = K \quad \Rightarrow \quad \tan\frac{\theta}{2} = \frac{1}{K} \quad \Rightarrow \quad \theta = 2\arctan\frac{1}{K}$$

$$\boxed{0 < \theta < 2\arctan\frac{1}{K}}$$

2. If $K < 0$

$$\frac{\theta}{2} = \arctan\frac{1}{K} + \pi \quad \Rightarrow \quad \theta = 2\arctan\frac{1}{K} + 2\pi$$

$$\boxed{0 < \theta < 2\arctan\frac{1}{K} + 2\pi}$$

4.2. $r_B > R_B$

We know from [1] that

$$r_B = r \cdot \frac{\sin\left(\alpha - \frac{\theta}{2}\right)}{2\sin\left(\frac{W-\theta}{2}\right)\sin\frac{W}{2}},$$

and

$$W - 2\pi < \theta < W \quad \Rightarrow \quad \frac{W}{2} - \pi < \frac{\theta}{2} < \frac{W}{2} \quad \Rightarrow \quad 0 < \frac{W}{2} - \frac{\theta}{2} < \pi \quad \Rightarrow$$

$$\Rightarrow \quad \sin\left(\frac{W}{2} - \frac{\theta}{2}\right) > 0.$$

Now our inequation is the following:

$$r \cdot \frac{\sin\left(\alpha - \frac{\theta}{2}\right)}{2\sin\left(\frac{W-\theta}{2}\right)\sin\frac{W}{2}} > R_B$$

$$\frac{\sin\left(\alpha - \frac{\theta}{2}\right)}{\sin\left(\frac{W}{2} - \frac{\theta}{2}\right)} < \frac{2R_B\sin\frac{W}{2}}{r}$$

$$\frac{\sin\alpha\cos\frac{\theta}{2} - \cos\alpha\sin\frac{\theta}{2}}{\sin\frac{W}{2}\cos\frac{\theta}{2} - \cos\frac{W}{2}\sin\frac{\theta}{2}} < \frac{2R_B\sin\frac{W}{2}}{r}$$

If we introduce notation $X_B \doteq \frac{2R_B\sin\frac{W}{2}}{r}$, the inequation is

$$\sin\alpha\cos\frac{\theta}{2} - \cos\alpha\sin\frac{\theta}{2} < X_B\sin\frac{W}{2}\cos\frac{\theta}{2} - X_B\cos\frac{W}{2}\sin\frac{\theta}{2}.$$

1. If $\cos\frac{\theta}{2} > 0 \ (\Leftrightarrow 0 < \theta < \pi)$

$$\sin\alpha - \cos\alpha\tan\frac{\theta}{2} < X_B\sin\frac{W}{2} - X_B\cos\frac{W}{2}\tan\frac{\theta}{2}$$

$$\tan\frac{\theta}{2}\left(X_B\cos\frac{W}{2} - \cos\alpha\right) < X_B\sin\frac{W}{2} - \sin\alpha$$

(i) If $X_B \cos \frac{W}{2} - \cos \alpha > 0$

$$\tan \frac{\theta}{2} < \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha}$$

(a) If $X_B \sin \frac{W}{2} - \sin \alpha > 0$, since $0 < \frac{\theta}{2} < \frac{\pi}{2}$

$$\boxed{0 < \theta < 2 \arctan \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha}}$$

(b) If $X_B \sin \frac{W}{2} - \sin \alpha < 0$, we have a *contradiction* since $\tan \frac{\theta}{2} \not< 0$, if $0 < \frac{\theta}{2} < \frac{\pi}{2}$

(ii) If $X_B \cos \frac{W}{2} - \cos \alpha < 0$

$$\tan \frac{\theta}{2} > \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha}$$

(a) If $X_B \sin \frac{W}{2} - \sin \alpha < 0$

$$\boxed{2 \arctan \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha} < \theta < \pi}$$

(b) If $X_B \sin \frac{W}{2} - \sin \alpha > 0$

$$\boxed{0 < \theta < \pi}$$

2. If $\cos \frac{\theta}{2} < 0$ (this means that $\pi < \theta < 2\pi$)

$$\tan \frac{\theta}{2} \left(X_B \cos \frac{W}{2} - \cos \alpha \right) > X_B \sin \frac{W}{2} - \sin \alpha$$

(i) If $X_B \cos \frac{W}{2} - \cos \alpha > 0$

$$\tan \frac{\theta}{2} > \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha}$$

(a) If $X_B \sin \frac{W}{2} - \sin \alpha > 0$, we have a *contradiction*.
(b) If $X_B \sin \frac{W}{2} - \sin \alpha < 0$

$$\boxed{2 \arctan \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha} + 2\pi < \theta < 2\pi}$$

(ii) If $X_B \cos \alpha - \cos \alpha < 0$

$$\tan \frac{\theta}{2} < \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha}$$

(a) If $X_B \sin \frac{W}{2} - \sin \alpha < 0$

$$\boxed{\pi < \theta < 2\pi}$$

(b) If $X_B \sin \frac{W}{2} - \sin \alpha > 0$

$$\boxed{\pi < \theta < 2 \arctan \frac{X_B \sin \frac{W}{2} - \sin \alpha}{X_B \cos \frac{W}{2} - \cos \alpha} + 2\pi}$$

If we consider these conditions together, we get usable interval(s) to create appropriate biarc for two circles.

5. Conclusions

Now we have several conditions to avoid intersections by constructing a biarc from a special family for two circles. Of course we did not eliminate the cases, where we have intersections between a_A and c_B or a_B and c_A. The future goal is to extend the calculation for analysing these cases too, and the other cases from [1]. With these results together we can get a powerful basic to create skinning curves using biarcs.

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