Annales Mathematicae et Informaticae
38 *(2011) pp. 3–13*
http://ami.ektf.hu

On general strong laws of large numbers for fields of random variables

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Submitted September 6, 2011 Accepted November 25, 2011

Abstract

A general method to prove strong laws of large numbers for random fields is given. It is based on the Hájek-Rényi type method presented in Noszály and Tómács [14] and in Tómács and Líbor [16]. Noszály and Tómács [14] obtained a Hájek-Rényi type maximal inequality for random fields using moments inequalities. Recently, Tómács and Líbor [16] obtained a Hájek-Rényi type maximal inequality for random sequences based on probabilities, but not for random fields. In this paper we present a Hájek-Rényi type maximal inequality for random fields, using probabilities, which is an extension of the main results of Noszály and Tómács [14] by replacing moments by probabilities and a generalization of the main results of Tómács and Líbor [16] for random sequences to random fields. We apply our results to establishing a logarithmically weighted sums without moment assumptions and under general dependence conditions for random fields.

Keywords: Strong laws of large numbers, Maximal inequality, Probability inequalities, Random fields.

MSC: Primary 60F15 60G60. Secondary 62H11 62H35.

1. Introduction and notations

We are concerned in this paper with strong laws of large numbers (SLLN) for random fields. Although this type of problems is entirely settled for sequence of independent real random variables (see for instance [9]) and general strong laws of large numbers for dependent real random variables based on Hájek-Rényi types inequalities. But as for random fields, they are still open. As a reminder, we recall that a family of random elements $(X_n)_{n\in T}$ is said to be a random field if the set is endowed with a partial order $(\leq)$, not necessarily complete. For example, and it is the case in this paper, T may be $\mathbf{N}^d$, where $d > 1$ is an integer and $\mathbf{N}$ is the set of nonnegative integers. For such a real random field $(X_n)_{n\in \mathbf{N}^d}$, we intend to contribute to assessing the more general SLLN's, that is finding general conditions under which there exists a real number μ and a family of normalizing positive numbers $(b_n)_{n\in \mathbf{N}^d}$, named here as a d-sequence, such that, for $S_{(0,\dots,0)} = 0$, and $S_n = \sum_{m\leq n} X_m$ for $n > \mathbf{0}$, one has

$$S_n/b_n \to \mu, \quad \text{a.s.}$$

In the case of random fields, the data may be heavily dependent and then Hájek-Rényi type maximal inequalities are needed to obtain strong laws of large numbers, like in the real case. It seems that providing such inequalities goes back to Móricz [11] and Klesov [8]. Based on such inequalities, many authors established strong laws of large numbers such as Nguyen et al. [13], Tómács [19], Lagodowski [10], Noszály and Tómács [14], Móricz [12], Klesov [8], Fazekas et al. [5], Fazekas [2], [4] and the literature cited herein.

One of the motivations of finding general strong laws of large numbers comes from that the finding, as proved by Cairoli [1], that classical maximal probability inequalities for random sequences are not valid in general for random fields. Besides, nonparametric estimation for random fields or spatial processes was given increasing and simulated attention over the last few years as a consequence of growing demands from applied research areas (see for instance Guyon [6]). This results in the serious motivation to extend the Hájek-Rényi type maximal inequality for probabilities for random sequences, what the cited above authors tackled.

Our objective is to give a nontrivial generalization of some fundamental results of these authors that will lead to positive answers to classical and non solved SLLN's. Before a more precise formulation of our problem, we need a few additional notation.

From now on d is a fixed positive integer. The elements of $\mathbf{N}^d$ will be written in font bold like $\mathbf{n}$ while their coordinate are written in the usual way like $\mathbf{n} = (n_1, \dots, n_d)$. $\mathbf{N}^d$ is endowed with the usual partial ordering, that is $\mathbf{n} = (n_1, \dots, n_d) \leq \mathbf{m} = (m_1, \dots, n_d)$ if and only if or each $1 \leq i \leq d$, one has $n_i \leq m_i$. Further $\mathbf{m} < \mathbf{n}$ means $\mathbf{m} \leq \mathbf{n}$ and $\mathbf{n} \neq \mathbf{m}$. We specially denote $(1, \dots, 1) \equiv \mathbf{1}$ and $(0, \dots, 0) \equiv \mathbf{0}$. All the limits, unless specification, are meant as $\mathbf{n} = (n_1, \dots, n_d) \to \infty$, that is equivalent to say that $n_i \to \infty$ for each $1 \leq i \leq d$. To finish, any family of real numbers $(b_{\mathbf{n}})_{\mathbf{n}\in A}$ indexed by a subset $\mathbf{N}^d$ is called a

d-sequence. We intensively use product type d-sequences. A d-sequence $(b_{\mathbf{n}})_{\mathbf{n}\in A}$ is of product type if it may be written in the form

$$b_{\mathbf{n}} = \prod_{1\le i\le d} b^{(i)}_{n_i}.$$

This product type d-sequence is unbounded and nondecreasing if and only if each sequence $b^{(i)}_{n_i}$ is unbounded and nondecreasing in n_i. Now with these minimum notation, we are able to state the results of Tómács, Líbor and their co-authors.

On one hand, it is known that the Hájek-Rényi type maximal inequality (see [3]) is an important tool for proving SLLN's for sequences. It is natural that Noszály and Tómács [14] used a generalization of this result for random fields in order to get SLLN's for such objects. They stated

Proposition 1.1. *Let r be a positive real number, $a_{\mathbf{n}}$ be a nonnegative d-sequence. Suppose that $b_{\mathbf{n}}$ is a positive, nondecreasing d-sequence of product type. Then*

$$\mathbb{E}(\max_{\ell\le\mathbf{n}}|S_\ell|^r) \le \sum_{\ell\le\mathbf{n}} a_\ell \quad \forall\mathbf{n}\in\mathbf{N}^d$$

implies

$$\mathbb{E}\left(\max_{\ell\le\mathbf{n}}|S_\ell|^r b_\ell^{-r}\right) \le 4^d\sum_{\ell\le\mathbf{n}} a_\ell b_\ell^{-r} \quad \forall\mathbf{n}\in\mathbf{N}^d.$$

From this, they were led to the following general SLLN for random fields.

Theorem 1.2. *Let $a_{\mathbf{n}}, b_{\mathbf{n}}$ be non-negative d-sequences and let $r>0$. Suppose that $b_{\mathbf{n}}$ is a positive, nondecreasing, unbounded d-sequence of product type. Let us assume that*

$$\sum_{\mathbf{n}} \frac{a_{\mathbf{n}}}{b^r_{\mathbf{n}}} < \infty$$

and

$$\mathbb{E}\left(\max_{\mathbf{m}\le\mathbf{n}}|S_{\mathbf{m}}|^r\right) \le \sum_{\mathbf{m}\le\mathbf{n}} a_{\mathbf{m}} \quad \forall\mathbf{n}\in\mathbf{N}^d.$$

Then

$$\lim_{\mathbf{n}\to\infty}\frac{S_{\mathbf{n}}}{b_{\mathbf{n}}} = 0 \qquad a.s.$$

On an other hand, Tómács and Líbor [16], introduced a Hájek-Rényi inequality for probabilities and, subsequently, strong laws of large numbers for random sequences but not for random fields. They obtained first:

Theorem 1.3. *Let r be a positive real number, a_n be a sequence of nonnegative real numbers. Then the following two statements are equivalent.*
(i) There exists $C>0$ such that for any $n\in\mathbf{N}$ and any $\varepsilon>0$

$$\mathbb{P}(\max_{\ell\le n}|S_\ell|\ge\varepsilon) \le C\varepsilon^{-r}\sum_{\ell\le n} a_\ell.$$

(ii) There exists $C > 0$ such that for any nondecreasing sequence $(b_n)_{n\in \mathbf{N}}$ of positive real numbers, for any $n \in \mathbf{N}$ and any $\varepsilon > 0$

$$\mathbb{P}\left(\max_{\ell\leq n}|S_\ell|b_\ell^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r}\sum_{\ell\leq n}a_\ell b_\ell^{-r}.$$

And next, they derived from it this SLLN.

Theorem 1.4. *Let a_n and b_n are non-negative sequences of real numbers and let $r > 0$. Suppose that b_n is a positive non-decreasing, unbounded sequence of positive real numbers. Let us assume that*

$$\sum_n \frac{a_n}{b_n^r} < \infty$$

and there exists $C > 0$ such that for any $n \in \mathbf{N}$ and any $\varepsilon > 0$

$$\mathbb{P}\left(\max_{m\leq n}|S_m| \geq \varepsilon\right) \leq C\ \varepsilon^{-r}\sum_{m\leq n} a_m.$$

Then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad a.s.$$

As said previously, this paper aims at generalizing the previous results in the following way. First, we give a random fields version for Tómács and Líbor [16] as a first generalization in Proposition 2.1. Next we show that our version of Hájek-Rényi type maximal inequality for probabilities for random fields is a generalization of that of Noszály and Tómács [14] and leads to a more general SLLN.

We apply our method for logarithmically weighted sums without any moment assumption and under general dependence conditions for random fields. This shows that the generalization is not trivial.

The paper is organized as follows. Section 2 is devoted to our main results, a Hájek-Rényi type maximal inequality for probabilities for random fields and automatically a strong law of large numbers are given. Section 3 includes their proofs. Section 4 including applications and illustration of our results, concludes the paper.

2. Results

We first give a Hájek-Rényi type maximal inequality for probabilities for random fields, as an extension of Proposition 1 in Noszály and Tómács [14] and of Theorem 2.1 in Tómács and Líbor [16].

Proposition 2.1. *Let r be a positive real number, $a_{\mathbf{n}}$ be a nonnegative d-sequence. Suppose that $b_{\mathbf{n}}$ is a positive, nondecreasing d-sequence of product type. Then the*

following two statements are equivalent.
(i) There exists $C > 0$ *such that for any* $\mathbf{n} \in \mathbf{N}^d$ *and any* $\varepsilon > 0$

$$\mathbb{P}(\max_{\ell \leq \mathbf{n}} |S_\ell| \geq \varepsilon) \leq C\varepsilon^{-r} \sum_{\ell \leq \mathbf{n}} a_\ell.$$

(ii) There exists $C > 0$ *such for any* $\mathbf{n} \in \mathbf{N}^d$ *and any* $\varepsilon > 0$

$$\mathbb{P}\left(\max_{\ell \leq \mathbf{n}} |S_\ell| b_\ell^{-1} \geq \varepsilon\right) \leq 4^d C \ \varepsilon^{-r} \sum_{\ell \leq \mathbf{n}} a_\ell b_\ell^{-r}.$$

We derive from this proposition a general strong law of large numbers for random fields which includes extensions of Theorem 3 in Noszály and Tómács [14] and of Theorem 2.4 in Tómács and Líbor [16]. But we need this lemma first.

Lemma 2.2 (Lemma 2 in Noszály and Tómács [14])**.** *Let* $a_\mathbf{n}$ *be a nonnegative d-sequence and let* $b_\mathbf{n}$ *be a positive, nondecreasing, unbounded d-sequence of product type. Suppose that* $\sum_\mathbf{n} \frac{a_\mathbf{n}}{b_\mathbf{n}^r} < \infty$ *with a fixed real* $r > 0$*. Then there exists a positive, nondecreasing, unbounded d-sequence* $\beta_\mathbf{n}$ *of product type for which*

$$\lim_\mathbf{n} \frac{\beta_\mathbf{n}}{b_\mathbf{n}} = 0 \quad \textit{and} \quad \sum_\mathbf{n} \frac{a_\mathbf{n}}{\beta_\mathbf{n}^r} < \infty.$$

Here is our general strong law of large numbers.

Theorem 2.3. *Let* $a_\mathbf{n}$ *be a non-negative d-sequence and let* $r > 0$*. Suppose that* $b_\mathbf{n}$ *is a positive, non-decreasing, unbounded d-sequence of product type. If*

$$\sum_\mathbf{n} \frac{a_\mathbf{n}}{b_\mathbf{n}^r} < \infty$$

and there exists $C > 0$ *such that for any* $\mathbf{n} \in \mathbf{N}^d$ *and any* $\varepsilon > 0$

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_\mathbf{m}| \geq \varepsilon\right) \leq C \ \varepsilon^{-r} \sum_{\mathbf{m} \leq \mathbf{n}} a_\mathbf{m}$$

then

$$\lim_{\mathbf{n} \to \infty} \frac{S_\mathbf{n}}{b_\mathbf{n}} = 0 \quad \textit{a.s.}$$

3. Proofs of the main results

We will need Lemma 2.2 and these two following lemmas.

Lemma 3.1. *Let $\{Y_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^d\}$ be a field of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for all $x \in \mathbf{R}$,*

$$\mathbb{P}\left(\sup_{\mathbf{k}} Y_{\mathbf{k}} > x\right) = \lim_{\mathbf{n}\to\infty} \mathbb{P}\left(\max_{\mathbf{k}\leq\mathbf{n}} Y_{\mathbf{k}} > x\right).$$

Proof. It is easy to see that, for all $x \in \mathbf{R}$.

$$\left(\sup_{\mathbf{k}} Y_{\mathbf{k}} > x\right) = \bigcup_{\mathbf{n}=\mathbf{1}}^{\infty} \left(\max_{\mathbf{k}\leq\mathbf{n}} Y_{\mathbf{k}} > x\right).$$

Hence, by the monotone convergence theorem for probabilities, we get the statement. □

Lemma 3.2. *Let $\{Y_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^d\}$ be a field of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\varepsilon_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ a nondecreasing field of real numbers. If*

$$\lim_{\mathbf{n}\to\infty} \mathbb{P}\left(\sup_{\mathbf{k}} Y_{\mathbf{k}} > \varepsilon_{\mathbf{n}}\right) = 0,$$

then $\sup_{\mathbf{k}} Y_{\mathbf{k}} < \infty$ a.s.

Proof. By using the monotone convergence theorem for probabilities, we have

$$\mathbb{P}\left(\bigcap_{\mathbf{n}=\mathbf{1}}^{\infty} \left(\sup_{\mathbf{k}} Y_{\mathbf{k}} > \varepsilon_{\mathbf{n}}\right)\right) = \lim_{\mathbf{n}\to\infty} \mathbb{P}\left(\sup_{\mathbf{k}} Y_{\mathbf{k}} > \varepsilon_{\mathbf{n}}\right) = 0$$

which is equivalent to $\mathbb{P}\left(\bigcup_{\mathbf{n}=\mathbf{1}}^{\infty} \left(\sup_{\mathbf{k}} \; Y_{\mathbf{k}} \leq \varepsilon_{\mathbf{n}}\right)\right) = 1$. This implies that there exists $\mathbf{n}_\omega \in \mathbf{N}^d$ for almost every $\omega \in \Omega$ such that $\sup_{\mathbf{k}} \; Y_{\mathbf{k}}(\omega) \leq \varepsilon_{\mathbf{n}_\omega} < \infty$. □

We need more notation for the proofs. In $\mathbf{N}^d$ the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{N}^d$, then $\langle \mathbf{n} \rangle = \prod_{i=1}^d n_i$. A numerical sequence $a_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d$ is called d-sequence. If $a_{\mathbf{n}}$ is a d-sequence then its difference sequence, i.e. the d-sequence $b_{\mathbf{n}}$ for which $\sum_{\mathbf{m}\leq\mathbf{n}} b_{\mathbf{m}} = a_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d$, will be denoted by $\Delta a_{\mathbf{n}}$ (i.e. $\Delta a_{\mathbf{n}} = b_{\mathbf{n}}$). We shall say that a d-sequence $a_{\mathbf{n}}$ is of product type if $a_{\mathbf{n}} = \prod_{i=1}^d a_{n_i}^{(i)}$, where $a_{n_i}^{(i)}$ $(n_i = 0, 1, 2, \ldots)$ is a (single) sequence for each $i = 1, \ldots, d$. Our consideration will be confined to normalizing constants of product type: $b_{\mathbf{n}}$ will always denote $b_{\mathbf{n}} = \prod_{i=1}^d b_{n_i}^{(i)}$, where $b_{n_i}^{(i)}$ $(n_i = 0, 1, 2, \ldots)$ is a nondecreasing sequence of positive numbers for each $i = 1, \ldots, d$. In this case we shall say that $b_{\mathbf{n}}$ is a positive nondecreasing d-sequence of product type. Moreover, if for each $i = 1, \ldots, d$ the sequence $b_{n_i}^{(i)}$ is unbounded, then $b_{\mathbf{n}}$ is called positive, nondecreasing, unbounded d-sequence of product type. As usual, $\log^+(x) := \max\{1, \log(x)\}$, $x > 0$ and $|\log \mathbf{n}| := \prod_{m=1}^d \log^+ n_m$.

Proof of Proposition 2.1. It is clear that (ii) implies (i) by taking $b_{m_j} = 1$ for all $\mathbf{m} \in \mathbf{N}^d$ and $1 \leq j \leq d$. Now we turn to (i) $\Longrightarrow$ (ii). We can assume without loss of generality that $b_{0,j} = 1$ for $1 \leq j \leq d$. If not, we would replace $b_{\mathbf{m}}$ by $\prod_{j=1}^{d} b_{m,j}/b_{0,j}$, $\mathbf{m} \in \mathbf{N}^d$ and (ii) would remain true with a new constant equal to $Cb_0^{-r} = C(\prod_{j=1}^{d} b_{\mathbf{0},j})^{-r}$. Now consider a fixed $\mathbf{n} \in \mathbf{N}^d$ and an arbitrary a real number $c > 1$. Remark by the monotonicity of $(b_{\mathbf{m}})$ that $b_{\mathbf{m}_j} \geq 1$ for all $\mathbf{m} \in N^d$ and that the sequence $(c^p)_{p \geq 0}$ forms a partition of $[1, +\infty[$. This implies that for any $\mathbf{m} \in \mathbf{N}^d$, for any $1 \leq j \leq d$, there exists a nonnegative integer i_j such that $c^{i_j} \leq b_{m_j} < c^{i_j+1}$. Thus for $\mathbf{i} = (i_1, \ldots, i_d)$, we have that $\mathbf{m} \in \mathcal{A}_{\mathbf{i}} = \{\mathbf{s} \in \mathbf{N}^d$ and $c^{i_j} \leq b_{\mathbf{s}_j} < c^{i_j+1}, j = 1, \ldots, d\}$. Since this holds for all $\mathbf{m} \in \mathbf{N}^d$, we get

$$\mathbf{N}^d = \bigcup_{i \in \mathbf{N}^d} \mathcal{A}_{\mathbf{i}}.$$

Let us restrict ourselves to $\mathbf{m} \leq \mathbf{n}$, and let us define

$$\mathcal{A}_{\mathbf{i},\mathbf{n}} = \{\mathbf{s} \in \mathbf{N}^d, \mathbf{s} \leq \mathbf{n} \quad \text{and} \quad c^{i_j} \leq b_{\mathbf{s}_j} < c^{i_j+1}, j = 1, \ldots, d\}.$$

Since $c^p \to \infty$ as $p \to \infty$ and for $\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}$, for $1 \leq j \leq d$, $b_{\mathbf{s}_j} \leq b_{\mathbf{n}_j} \leq \max\{b_{n_k}, 1 \leq k \leq d\} < \infty$, the sets $\mathcal{A}_{\mathbf{i},\mathbf{n}}$ are empty for large values of i. Then put $\mathbf{k_n} = \max\{\mathbf{i} : \mathcal{A}_{\mathbf{i},\mathbf{n}} \neq \emptyset\} < +\infty$ and we have

$$[0, n] = \bigcup_{\mathbf{i} \leq \mathbf{k_n}} \mathcal{A}_{\mathbf{i},\mathbf{n}}.$$

It is also noticeable that if $\mathbf{m} \leq \mathbf{s} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}$, then necessarily $\mathbf{m}$ is in some $\mathcal{A}_{\mathbf{i}',\mathbf{n}}$ with $\mathbf{i}' \leq \mathbf{i}$. As well let $\mathbf{m}_{\mathbf{i},\mathbf{n}} = \max \mathcal{A}_{\mathbf{i},\mathbf{n}} \leq \mathbf{n}$ and define $D_{\mathbf{i},\mathbf{n}} = \sum_{\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}} a_{\mathbf{m}}$ where, by convention, $D_{\mathbf{i},\mathbf{n}} = 0$ and $\mathbf{m}_{\mathbf{i},\mathbf{n}} = (0, \ldots, 0)$ when $\mathcal{A}_{\mathbf{i},\mathbf{n}} = \emptyset$. From all that, we have

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq \sum_{\mathbf{i} \leq \mathbf{k_n}} \mathbb{P}\left(\max_{\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right).$$

Since for $\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}$, $b_{\mathbf{m}} = \prod_{j=1}^{d} b_{m_j} \geq \prod_{j=1}^{d} c^{i_j}$ and $\mathcal{A}_{\mathbf{i},\mathbf{n}} \subset [0, m_{\mathbf{i},\mathbf{n}}]$, we get

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq \sum_{\mathbf{i} \leq \mathbf{k_n}} \mathbb{P}\left(\max_{\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq$$

$$\sum_{\mathbf{i} \leq \mathbf{k_n}} \mathbb{P}\left(\max_{\mathbf{m} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}} |S_m| \geq \varepsilon \prod_{j=1}^{d} c^{i_j}\right).$$

Now by applying (i) one arrives at

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{\mathbf{i} \leq \mathbf{k_n}} \prod_{j=1}^{d} c^{-ri_j} \sum_{\mathbf{m} \leq \mathbf{m}_{i,n}} a_m \leq$$

$$C\varepsilon^{-r} \sum_{\mathbf{i} \leq \mathbf{k_n}} \prod_{j=1}^{d} c^{-ri_j} \sum_{\mathbf{m} \leq \mathbf{i}} D_{\mathbf{m},\mathbf{n}}.$$

By the remark made above, $\mathbf{m} \leq \mathbf{m}_{i,n} \in \mathcal{A}_{\mathbf{i},\mathbf{n}}$ implies that $\mathbf{m}$ is in some $\mathcal{A}_{\mathbf{s},\mathbf{n}}$ where $\mathbf{s} \leq \mathbf{i}$ and then by the definition of the $D_{\mathbf{i},\mathbf{n}}$ on has $\sum_{\mathbf{m} \leq \mathbf{m}_{i,n}} a_m \leq \sum_{\mathbf{m} \leq \mathbf{i}} D_{\mathbf{m},\mathbf{n}}$ and next

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{\mathbf{i} \leq \mathbf{k_n}} \prod_{j=1}^{d} c^{-ri_j} \sum_{\mathbf{m} \leq \mathbf{i}} D_{\mathbf{m}},$$

which becomes by a straightforward manipulations on the ranges of the sums, and where $k_{\mathbf{n}}(j)$ stands for the j-th coordinate of $k_{\mathbf{n}}$,

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{m \leq \mathbf{k_n}} D_{\mathbf{m},\mathbf{n}} \sum_{m \leq \mathbf{i} \leq \mathbf{k_n}} \prod_{j=1}^{d} c^{-ri_j} \leq$$

$$C\varepsilon^{-r} \sum_{m \leq \mathbf{k_n}} D_{\mathbf{m},\mathbf{n}} \prod_{j=1}^{d} \sum_{m_j \leq i_j \leq k_{\mathbf{n}}(j)} c^{-ri_j} =$$

$$C\varepsilon^{-r} \sum_{m \leq \mathbf{k_n}} D_{\mathbf{m},\mathbf{n}} \prod_{j=1}^{d} \frac{c^{-rm_j} - c^{-r(k_{\mathbf{n}}(j)+1)}}{1 - c^{-r}} \leq C\varepsilon^{-r} \sum_{m \leq \mathbf{k_n}} D_{\mathbf{m},\mathbf{n}} \prod_{j=1}^{d} \frac{c^{-rm_j}}{1 - c^{-r}},$$

since $c > 1$ and $k_{\mathbf{n}}(j) + 1 > m_j$. Now, at this last but one step, we have

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \left(\frac{c^r}{1 - c^{-r}}\right)^d \sum_{m \leq \mathbf{k_n}} D_{\mathbf{m},\mathbf{n}} \prod_{j=1}^{d} c^{-r(m_j+1)} \leq$$

$$C\varepsilon^{-r} \left(\frac{c^r}{1 - c^{-r}}\right)^d \sum_{m \leq \mathbf{k_n}} \sum_{\mathbf{s} \in \mathcal{A}_{\mathbf{m},\mathbf{n}}} a_{\mathbf{s}} \prod_{j=1}^{d} c^{-r(m_j+1)}.$$

Finally, taking into account the fact that for $\mathbf{s} \in \mathcal{A}_{\mathbf{m},\mathbf{n}}$, $c^{m_j+1} \geq b_{s_j}$, $1 \leq j \leq d$, that is $\prod_{j=1}^{d} c^{r(m_j+1)} \geq b_{\mathbf{s}}^r$, we arrive at

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \left(\frac{c^r}{1 - c^{-r}}\right)^d \sum_{m \leq \mathbf{k_n}} \sum_{\mathbf{s} \in \mathcal{A}_{\mathbf{m},\mathbf{n}}} \frac{a_{\mathbf{s}}}{b_{\mathbf{s}}^r} \leq$$

$$C\varepsilon^{-r} \left(\frac{c^r}{1 - c^{-r}}\right)^d \sum_{\mathbf{m} \leq \mathbf{n}} \frac{a_{\mathbf{m}}}{b_{\mathbf{m}}^r}.$$

Since c is arbitrary $c > 1$ and $\min_{c>1} \frac{c^r}{1-c^{-r}} = 4$, we achieve the proof by

$$\mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} |S_{\mathbf{m}}| b_{\mathbf{m}}^{-1} \geq \varepsilon\right) \leq 4^d \ C \ \ \varepsilon^{-r} \sum_{\mathbf{m} \leq \mathbf{n}} \frac{a_{\mathbf{m}}}{b_{\mathbf{m}}^r}. \qquad \square$$

Proof of Theorem 2.3. Let $\beta_{\mathbf{n}}$ be the d-sequence obtained in the Lemma 2.2. According to Proposition 2.1

$$\mathbb{P}\left(\max_{\ell \leq \mathbf{m}} |S_\ell| \beta_\ell^{-1} \geq \varepsilon_{\mathbf{k}}\right) \leq 4^d C \varepsilon_{\mathbf{k}}^{-r} \sum_{\ell \leq \mathbf{m}} a_\ell \beta_\ell^{-r} \quad \forall \mathbf{m} \leq \mathbf{n}.$$

By this fact we get for any fixed $\mathbf{k} \in \mathbf{N}^d$

$$\mathbb{P}\left(\sup_{\ell \le \mathbf{m}} |S_\ell| \beta_\ell^{-1} \ge \varepsilon_{\mathbf{k}}\right) \le \lim_{\mathbf{m}\to\infty} \mathbb{P}\left(\max_{\ell \le \mathbf{m}} |S_\ell| \beta_\ell^{-1} \ge \varepsilon_{\mathbf{k}}\right) \le 4^d C \varepsilon_{\mathbf{k}}^{-r} \sum_{\mathbf{n}} a_{\mathbf{n}} \beta_{\mathbf{n}}^{-r},$$

where $\{\varepsilon_{\mathbf{k}}, \mathbf{k} \in \mathbf{N}^d\}$ a positive, nondecreasing, unbounded field of real numbers. So we have by Lemma 2.2

$$\lim_{\mathbf{k}\to\infty} \mathbb{P}\left(\sup_{\ell} |S_\ell| \beta_\ell^{-1} \ge \varepsilon_{\mathbf{k}}\right) = 0.$$

Using Lemma 3.1

$$\mathbb{P}\left(\sup_{\ell} |S_\ell| \beta_\ell^{-1} \ge \varepsilon_{\mathbf{k}} \text{ for all } \mathbf{k} \in \mathbf{N}^d\right) = 0.$$

So we have by Lemma 3.2 $\sup_\ell |S_\ell| \beta_\ell^{-1} < \infty$ a.s. Finally by Lemma 2.2

$$0 \le \frac{|S_{\mathbf{n}}|}{b_{\mathbf{n}}} = \frac{|S_{\mathbf{n}}|}{\beta_{\mathbf{n}}} \frac{\beta_{\mathbf{n}}}{b_{\mathbf{n}}} \le \sup_{\ell} |S_\ell| \beta_\ell^{-1} \frac{\beta_{\mathbf{n}}}{b_{\mathbf{n}}} \to 0 \quad \text{a.s.} \qquad \square$$

4. Conclusion

4.1. A first application: Logarithmically weighted sums

The following result is an extension of Theorem 7 in Noszály and Tómács [14] and of Theorem 4.2 in Fazekas et al. [5]. In this Theorem, we do not need any moment assumption in contrary of these above cited theorems.

Theorem 4.1. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbf{N}^d\}$ be a field of random variables. Let $r > 1$. We assume there exists $C > 0$ such that for any $\mathbf{m} \in \mathbf{N}^d$ and any $\varepsilon > 0$*

$$\mathbb{P}\left(\max_{\ell \le \mathbf{m}} \sum_{\mathbf{k} \le \ell} \frac{X_{\mathbf{k}}}{\langle \mathbf{k} \rangle} \ge \varepsilon\right) \le C \varepsilon^{-r} \sum_{\ell \le \mathbf{m}} \frac{1}{\langle \ell \rangle}.$$

Then

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \frac{X_{\mathbf{k}}}{\langle \mathbf{k} \rangle} \to 0 \quad (\mathbf{n} \to \infty) \quad a.s.$$

Proof. Let us apply Theorem 2.3 with $a_{\mathbf{n}} = \frac{1}{\langle \mathbf{n} \rangle}$ and $b_{\mathbf{n}} = |\log \mathbf{n}|$. The proof is achieved by remarking that for $r > 1$

$$\sum_{\mathbf{n}} \frac{a_{\mathbf{n}}}{b_{\mathbf{n}}^r} = \sum_{\mathbf{n}} \frac{1}{|\log \mathbf{n}|^r} \frac{1}{\langle \mathbf{n} \rangle} < \infty. \qquad \square$$

4.2. A second application

By using Markov's Inequality and applying our results (see Theorem 2.3), under the same assumptions in Noszály and Tómács [14], we rediscover their results.

Acknowledgement. The paper was finalized while the second author was visiting MAPMO, University of Orléans, France, in 2011. He expresses her warm thanks to responsibles of MAPMO for kind hospitality. The authors also thank the referee for his valuable comments and suggestions.

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