

On $\mathcal{I}_c^{(q)}$ -convergence

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Abstract

In this paper we will study the properties of ideals $\mathcal{I}_c^{(q)}$ related to the notion of \mathcal{I} -convergence of sequences of real numbers. We show that $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$ -convergence are equivalent. We prove some results about modified Olivier's theorem for these ideals. For bounded sequences we show a connection between $\mathcal{I}_c^{(q)}$ -convergence and regular matrix method of summability.

1. Introduction

In papers [9], and [10] the notion of \mathcal{I} -convergence of sequences of real numbers is introduced and its basic properties are investigated. The \mathcal{I} -convergence generalizes the notion of the statistical convergence (see[5]) and it is based on the ideal \mathcal{I} of subsets of the set \mathbb{N} of positive integers.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$. \mathcal{I} is called an admissible ideal of subsets of positive integers, if \mathcal{I} is additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$), hereditary (i.e. $A \in \mathcal{I}, B \subset A \Rightarrow B \in \mathcal{I}$), containing all singletons and it doesn't contain \mathbb{N} . Here we present some examples of admissible ideals. More examples can be found in the papers [7, 9, 10, 12].

- (a) The class of all finite subsets of \mathbb{N} form an admissible ideal usually denote by \mathcal{I}_f .
- (b) Let ϱ be a density function on \mathbb{N} , then the set $\mathcal{I}_\varrho = \{A \subseteq \mathbb{N} : \varrho(A) = 0\}$ is an admissible ideal. We will use the ideals $\mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_u$ related to asymptotic, logarithmic, uniform density, respectively. For those densities for definitions see [9, 10, 12, 13].

- (c) For any $q \in (0, 1)$ the set $\mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-q} < \infty\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty\}$ is usually denoted by \mathcal{I}_c . It is easy to see, that for any $q_1 < q_2; q_1, q_2 \in (0, 1)$

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)} \subsetneq \mathcal{I}_c \subsetneq \mathcal{I}_d \quad (1.1)$$

In this paper we will study the ideals $\mathcal{I}_c^{(q)}$. In particular the equivalence between $\mathcal{I}_c^{(q)}$, $\mathcal{I}_c^{(q)*}$, Olivier's like theorems for this ideals and characterization of $\mathcal{I}_c^{(q)}$ -convergent sequences by regular matrices.

2. The equivalence between $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$ -convergence

Let us recall the notion of \mathcal{I} -convergence of sequences of real numbers, (cf.[9, 10]).

Definition 2.1. We say that a sequence $x = (x_n)_{n=1}^\infty$ \mathcal{I} -converges to a number L and we write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal \mathcal{I} .

\mathcal{I} -convergence satisfies usual axioms of convergence i.e. the uniqueness of limit, arithmetical properties etc. The class of all \mathcal{I} -convergent sequences is a linear space. We will also use the following elementary fact.

Lemma 2.2. Let $\mathcal{I}_1, \mathcal{I}_2$ be admissible ideals such that $\mathcal{I}_1 \subset \mathcal{I}_2$. If $\mathcal{I}_1 - \lim x_n = L$ then $\mathcal{I}_2 - \lim x_n = L$.

In the papers [9, 10] there was defined yet another type of convergence related to the ideal \mathcal{I} .

Definition 2.3. Let \mathcal{I} be an admissible ideal in \mathbb{N} . A sequence $x = (x_n)_{n=1}^\infty$ of real numbers is said to be \mathcal{I}^* -convergent to $L \in \mathbb{R}$ (shortly $\mathcal{I}^* - \lim x_n = L$) if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots\}$ we have, $\lim_{k \rightarrow \infty} x_{m_k} = L$.

It is easy to prove, that for every admissible ideal \mathcal{I} the following relation between \mathcal{I} and \mathcal{I}^* -convergence holds:

$$\mathcal{I}^* - \lim x_n = L \Rightarrow \mathcal{I} - \lim x_n = L.$$

Kostyrko, Šalát and Wilczyński in [9] give an algebraic characterization of ideals \mathcal{I} , for which the \mathcal{I} and \mathcal{I}^* -convergence are equal; it turns out that these ideals are with the property (AP).

Definition 2.4. An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to \mathcal{I} there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty B_j \in \mathcal{I}$. ($A \Delta B = (A \setminus B) \cup (B \setminus A)$).

For some ideals it is already known whether they have property (AP)(see [9, 10, 12, 13]). Now, will show the equivalence between $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$ -convergence.

Theorem 2.5. *For any $0 < q \leq 1$ the ideal $\mathcal{I}_c^{(q)}$ has a property (AP).*

Proof. It suffices to prove that any sequences $(x_n)_{n=1}^\infty$ of real numbers such that $\mathcal{I}_c^{(q)} - \lim x_n = \xi$ there exist a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus M \in \mathcal{I}_c^{(q)}$ and $\lim_{k \rightarrow \infty} x_{m_k} = \xi$.

For any positive integer k let $\varepsilon_k = \frac{1}{2^k}$ and $A_k = \{n \in \mathbb{N} : |x_n - \xi| \geq \frac{1}{2^k}\}$. As $\mathcal{I}_c^{(q)} - \lim x_n = \xi$, we have $A_k \in \mathcal{I}_c^{(q)}$, i.e.

$$\sum_{a \in A_k} a^{-q} < \infty.$$

Therefore there exist an infinite sequence $n_1 < n_2 < \dots < n_k \dots$ of integers such that for every $k = 1, 2, \dots$

$$\sum_{\substack{a > n_k \\ a \in A_k}} a^{-q} < \frac{1}{2^k}$$

Let $H = \bigcup_{k=1}^\infty [(n_k, n_{k+1}) \cap A_k]$. Then

$$\begin{aligned} \sum_{a \in H} a^{-q} &\leq \sum_{\substack{a > n_1 \\ a \in A_1}} a^{-q} + \sum_{\substack{a > n_2 \\ a \in A_2}} a^{-q} + \dots + \sum_{\substack{a > n_k \\ a \in A_k}} a^{-q} + \dots < \\ &< \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \dots < +\infty \end{aligned}$$

Thus $H \in \mathcal{I}_c^{(q)}$. Put $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$. Now it suffices to prove that $\lim_{k \rightarrow \infty} x_{m_k} = \xi$. Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $m_k > n_{k_0}$. Then m_k belongs to some interval (n_j, n_{j+1}) where $j \geq k_0$ and doesn't belong to A_j ($j \geq k_0$). Hence m_k belongs to $\mathbb{N} \setminus A_j$, and then $|x_{m_k} - \xi| < \varepsilon$ for every $m_k > n_{k_0}$, thus $\lim_{k \rightarrow \infty} x_{m_k} = \xi$. \square

3. Olivier's like theorem for the ideals $\mathcal{I}_c^{(q)}$

In 1827 L. Olivier proved the results about the speed of convergence to zero of the terms of a convergent series with positive and decreasing terms.(cf.[8, 11])

Theorem A. *If $(a_n)_{n=1}^\infty$ is a non-increasing sequences and $\sum_{n=1}^\infty a_n < +\infty$, then $\lim_{n \rightarrow \infty} n \cdot a_n = 0$.*

Simple example $a_n = \frac{1}{n}$ if n is a square i.e. $n = k^2$, ($k = 1, 2, \dots$) and $a_n = \frac{1}{2^n}$ otherwise shows that monotonicity condition on the sequence $(a_n)_{n=1}^\infty$ can not be in general omitted.

In [14] T.Šalát and V.Toma characterized the class $\mathcal{S}(T)$ of ideals such that

$$\sum_{n=1}^{\infty} a_n < +\infty \Rightarrow \mathcal{I} - \lim_{n \rightarrow \infty} n \cdot a_n = 0 \quad (3.1)$$

for any convergent series with positive terms.

Theorem B. *The class $\mathcal{S}(T)$ consists of all admissible ideals $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ such that $\mathcal{I} \supseteq \mathcal{I}_c$.*

From inclusions (1.1) is obvious that ideals $\mathcal{I}_c^{(q)}$ do not belong to the class $\mathcal{S}(T)$.

In what follows we show that it is possible to modify the Olivier's condition $\sum_{n=1}^{\infty} a_n < +\infty$ in such a way that the ideal $\mathcal{I}_c^{(q)}$ will play the role of ideal \mathcal{I}_c in Theorem B.

Lemma 3.1. *Let $0 < q \leq 1$. Then for every sequence $(a_n)_{n=1}^\infty$ such that $a_n > 0$, $n = 1, 2, \dots$ and $\sum_{n=1}^{\infty} a_n^q < +\infty$ we have $\mathcal{I}_c^{(q)} - \lim n \cdot a_n = 0$.*

Proof. Let the conclusion of the Lemma 3.1 doesn't hold. Then there exists $\varepsilon_0 > 0$ such that the set $A(\varepsilon_0) = \{n : n \cdot a_n \geq \varepsilon_0\}$ doesn't belong to $\mathcal{I}_c^{(q)}$. Therefore

$$\sum_{k=1}^{\infty} m_k^{-q} = +\infty, \quad (3.2)$$

where $A(\varepsilon_0) = \{m_1 < m_2 < \dots < m_k < \dots\}$. By the definition of the set $A(\varepsilon_0)$ we have $m_k \cdot a_{m_k} \geq \varepsilon_0 > 0$, for each $k \in \mathbb{N}$. From this $m_k^q \cdot a_{m_k}^q \geq \varepsilon_0^q > 0$ and so for each $k \in \mathbb{N}$

$$a_{m_k}^q \geq \varepsilon_0^q \cdot m_k^{-q} \quad (3.3)$$

From (3.2) and (3.3) we get $\sum_{k=1}^{\infty} a_{m_k}^q = +\infty$, and hence $\sum_{n=1}^{\infty} a_n^q = +\infty$. But it contradicts the assumption of the theorem. \square

Let's denote by $\mathcal{S}_q(T)$ the class of all admissible ideals \mathcal{I} for which an analog Lemma 3.1 holds. From Lemma 2.2 we have:

Corollary 3.2. *If \mathcal{I} is an admissible ideal such that $\mathcal{I} \supseteq \mathcal{I}_c^{(q)}$ then $\mathcal{I} \in \mathcal{S}_q(T)$.*

Main result of this section is the reverse of Corollary 3.2.

Theorem 3.3. *For any $q \in (0, 1)$ the class $\mathcal{S}_q(T)$ consists of all admissible ideals such that $\mathcal{I} \supseteq \mathcal{I}_c^{(q)}$.*

Proof. It is sufficient to prove that for any infinite set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \in \mathcal{I}_c^{(q)}$ we have $M \in \mathcal{I}$, too. Since $M \in \mathcal{I}_c^{(q)}$ we have

$$\sum_{k=1}^{\infty} m_k^{-q} < +\infty.$$

Now we define the sequence $(a_n)_{n=1}^\infty$ as follows

$$\begin{aligned} a_{m_k} &= \frac{1}{m_k} \quad (k = 1, 2, \dots), \\ a_n &= \frac{1}{10^n} \quad \text{for } n \in \mathbb{N} \setminus M. \end{aligned}$$

Obviously $a_n > 0$ and $\sum_{n=1}^\infty a_n^q < +\infty$ by the definition of numbers a_n . Since $\mathcal{I} \in \mathcal{S}_q(T)$ we have

$$\mathcal{I} - \lim n \cdot a_n = 0.$$

This implies that for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n : n \cdot a_n \geq \varepsilon\} \in \mathcal{I},$$

in particular $M = A(1) \in \mathcal{I}$. □

4. $\mathcal{I}_c^{(q)}$ -convergence and regular matrix transformations

$\mathcal{I}_c^{(q)}$ -convergence is an example of a linear functional defined on a subspace of the space of all bounded sequences of real numbers. Another important family of such functionals are so called matrix summability methods inspired by [1, 6]. We will study connections between $\mathcal{I}_c^{(q)}$ -convergence and one class of matrix summability methods. Let us start by introducing a notion of regular matrix transformation (see [4]).

Let $\mathbf{A} = (a_{nk})$ ($n, k = 1, 2, \dots$) be an infinite matrix of real numbers. The sequence $(t_n)_{n=1}^\infty$ of real numbers is said to be \mathbf{A} -limitable to the number s if $\lim_{n \rightarrow \infty} s_n = s$, where

$$s_n = \sum_{k=1}^\infty a_{nk} t_k \quad (n = 1, 2, \dots).$$

If $(t_n)_{n=1}^\infty$ is \mathbf{A} -limitable to the number s , we write $\mathbf{A} - \lim_{n \rightarrow \infty} t_n = s$.

We denote by $F(\mathbf{A})$ the set of all \mathbf{A} -limitable sequences. The set $F(\mathbf{A})$ is called the convergence field. The method defined by the matrix \mathbf{A} is said to be *regular* provided that $F(\mathbf{A})$ contains all convergent sequences and $\lim_{n \rightarrow \infty} t_n = t$ implies $\mathbf{A} - \lim_{n \rightarrow \infty} t_n = t$. Then \mathbf{A} is called a *regular matrix*.

It is well-known that the matrix \mathbf{A} is regular if and only if satisfies the following three conditions (see [4]):

$$(A) \quad \exists K > 0, \forall n = 1, 2, \dots \quad \sum_{k=1}^\infty |a_{nk}| \leq K;$$

$$(B) \quad \forall k = 1, 2, \dots \quad \lim_{n \rightarrow \infty} a_{nk} = 0$$

$$(C) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^\infty a_{nk} = 1$$

Let's ask the question: Is there any connection between \mathcal{I} -convergence of sequence of real numbers and \mathbf{A} -limit of this sequence? It is well known that a sequence $(x_k)_{k=1}^{\infty}$ of real numbers \mathcal{I}_d -converges to real number ξ if and only if the sequence is strongly summable to ξ in Caesaro sense. The complete characterization of statistical convergence (\mathcal{I}_d -convergence) is described by Fridy-Miller in the paper [6]. They defined a class of lower triangular nonnegative matrices \mathcal{T} with properties:

$$\sum_{k=1}^n a_{nk} = 1 \quad \forall n \in \mathbb{N}$$

if $C \subseteq \mathbb{N}$ such that $d(C) = 0$, then $\lim_{n \rightarrow \infty} \sum_{k \in C} a_{nk} = 0$.

They proved the following assertion:

Theorem C. *The bounded sequence $x = (x_n)_{n=1}^{\infty}$ is statistically convergent to L if and only if $x = (x_n)_{n=1}^{\infty}$ is \mathbf{A} -summable to L for every \mathbf{A} in \mathcal{T} .*

Similar result for \mathcal{I}_u -convergence was shown by V. Baláž and T. Šalát in [1]. Here we prove analogous result for $\mathcal{I}_c^{(q)}$ -convergence. Following this aim let's define the class \mathcal{T}_q lower triangular nonnegative matrices in this way:

Definition 4.1. Matrix $\mathbf{A} = (a_{nk})$ belongs to the class \mathcal{T}_q if and only if it satisfies the following conditions:

- (I) $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{nk} = 1$
- (q) If $C \subset \mathbb{N}$ and $C \in \mathcal{I}_c^{(q)}$, then $\lim_{n \rightarrow \infty} \sum_{k \in C} a_{nk} = 0$, $0 < q \leq 1$.

It is easy to see that every matrix of class \mathcal{T}_q is regular. As the following example shows the converse does not hold.

Example 4.2. Let $C = \{1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots\}$ and $q = 1$. Obviously $C \in \mathcal{I}_c^{(1)} = \mathcal{I}_c$. Now define the matrix \mathbf{A} by:

$$\begin{aligned} a_{11} &= 1, a_{1k} = 0, \quad k > 1 \\ a_{nk} &= \frac{1}{2k \cdot \ln n}, \quad k \neq l^2, k \leq n \\ a_{nk} &= \frac{1}{l \ln n}, \quad k = l^2, k \leq n \\ a_{nk} &= 0, \quad k > n \end{aligned}$$

It is easy to show that \mathbf{A} is lower triangular nonnegative regular matrix but does not satisfy the condition (q) from Definition 4.1.

$$\sum_{\substack{k < n^2 \\ k \in C}} a_{n^2 k} = \frac{1}{\ln n^2} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \geq \frac{\ln n}{2 \ln n} = \frac{1}{2} \not\rightarrow 0$$

for $n \rightarrow \infty$. Therefore $\mathbf{A} \notin \mathcal{T}_1$.

Lemma 4.3. *If the bounded sequence $x = (x_n)_{n=1}^\infty$ is not \mathcal{I} -convergent then there exist real numbers $\lambda < \mu$ such that neither the set $\{n \in \mathbb{N} : x_n < \lambda\}$ nor the set $\{n \in \mathbb{N} : x_n > \mu\}$ belongs to ideal \mathcal{I} .*

As the proof is the same as the proof on Lemma in [6] we will omit it.

Next theorem shows connection between $\mathcal{I}_c^{(q)}$ -convergence of bounded sequence of real numbers and \mathbf{A} -summability of this sequence for matrices from the class \mathcal{T}_q .

Theorem 4.4. *Let $q \in (0, 1)$. Then the bounded sequence $x = (x_n)_{n=1}^\infty$ of real numbers $\mathcal{I}_c^{(q)}$ -converges to $L \in \mathbb{R}$ if and only if it is \mathbf{A} -summable to $L \in \mathbb{R}$ for each matrix $\mathbf{A} \in \mathcal{T}_q$.*

Proof. Let $\mathcal{I}_c^{(q)} - \lim x_n = L$ and $\mathbf{A} \in \mathcal{T}_q$. As \mathbf{A} is regular there exists a $K \in \mathbb{R}$ such that $\forall n = 1, 2, \dots \sum_{k=1}^\infty |a_{nk}| \leq K$.

It is sufficient to show that $\lim_{n \rightarrow \infty} b_n = 0$, where $b_n = \sum_{k=1}^\infty a_{nk} \cdot (x_k - L)$. For $\varepsilon > 0$ put $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. By the assumption we have $B(\varepsilon) \in \mathcal{I}_c^{(q)}$. By condition (q) from Definition 4.1 we have

$$\lim_{n \rightarrow \infty} \sum_{k \in B(\varepsilon)} |a_{nk}| = 0 \quad (4.1)$$

As the sequence $x = (x_n)_{n=1}^\infty$ is bounded, there exists $M > 0$ such that

$$\forall k = 1, 2, \dots : |x_k - L| \leq M \quad (4.2)$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} |b_n| &\leq \sum_{k \in B(\frac{\varepsilon}{2K})} |a_{nk}| |x_k - L| + \sum_{k \notin B(\frac{\varepsilon}{2K})} |a_{nk}| |x_k - L| \leq \\ &\leq M \sum_{k \in B(\frac{\varepsilon}{2K})} |a_{nk}| + \frac{\varepsilon}{2K} \sum_{k \notin B(\frac{\varepsilon}{2K})} |a_{nk}| \leq \\ &\leq M \sum_{k \in B(\frac{\varepsilon}{2K})} |a_{nk}| + \frac{\varepsilon}{2} \end{aligned} \quad (4.3)$$

By part (q) of Definition 4.1 there exists an integer n_0 such that for all $n > n_0$

$$\sum_{k \in B(\frac{\varepsilon}{2K})} |a_{nk}| < \frac{\varepsilon}{2M}$$

Together by (4.3) we obtain $\lim_{n \rightarrow \infty} b_n = 0$.

Conversely, suppose that $\mathcal{I}_c^{(q)} - \lim x_n = L$ doesn't hold. We show that there exists a matrix $\mathbf{A} \in \mathcal{T}_q$ such that $\mathbf{A} - \lim_{n \rightarrow \infty} x_n = L$ does not hold, too. If $\mathcal{I}_c^{(q)} - \lim x_n = L' \neq L$ then from the first part of proof it follows that $\mathbf{A} - \lim_{n \rightarrow \infty} x_n = L'$

$\neq L$ for any $\mathbf{A} \in \mathcal{T}_q$. Thus, we may assume that $(x_n)_{n=1}^\infty$ is not $\mathcal{I}_c^{(q)}$ -convergent, and by the above Lemma 4.3 there exist λ and μ ($\lambda < \mu$), such that neither the set $U = \{k \in \mathbb{N} : x_k < \lambda\}$ nor $V = \{k \in \mathbb{N} : x_k > \mu\}$ belongs to the ideal $\mathcal{I}_c^{(q)}$. It is clear that $U \cap V = \emptyset$. If $U \notin \mathcal{I}_c^{(q)}$ then $\sum_{i \in U} i^{-q} = +\infty$ and if $V \notin \mathcal{I}_c^{(q)}$ then $\sum_{i \in V} i^{-q} = +\infty$. Let $U_n = U \cap \{1, 2, \dots, n\}$ and $V_n = V \cap \{1, 2, \dots, n\}$.

Now we define the matrix $\mathbf{A} = (a_{nk})$ by the following way: Let $s_{(1)n} = \sum_{i \in U_n} i^{-q}$ for $n \in U$, $s_{(2)n} = \sum_{i \in V_n} i^{-q}$ for $n \in V$ and $s_{(3)n} = \sum_{i=1}^n i^{-q}$ for $n \notin U \cup V$. As $U, V \notin \mathcal{I}_c^{(q)}$ we have $\lim_{n \rightarrow \infty} s_{(j)n} = +\infty, j = 1, 2, 3$.

$$a_{nk} = \begin{cases} a_{nk} = \frac{k^{-q}}{s_{(1)n}} & n \in U \text{ and } k \in U_n, \\ a_{nk} = 0 & n \in U \text{ and } k \notin U_n, \\ a_{nk} = \frac{k^{-q}}{s_{(2)n}} & n \in V \text{ and } k \in V_n, \\ a_{nk} = 0 & n \in V \text{ and } k \notin V_n, \\ a_{nk} = \frac{k^{-q}}{s_{(3)n}} & n \notin U \cup V, \\ a_{nk} = 0 & k > n, \end{cases}$$

Let's check that $\mathbf{A} \in \mathcal{T}_q$. Obviously \mathbf{A} is a lower triangular nonnegative matrix. Condition (I) is clear from the definition of matrix \mathbf{A} . Condition (q): Let $B \in \mathcal{I}_c^{(q)}$ and $b = \sum_{k \in B} k^{-q} < +\infty$. Then

$$\sum_{k \in B} a_{nk} \leq \frac{1}{s_{(3)n}} \sum_{k \in B \cap \{1, \dots, n\}} k^{-q} \chi_B(k) \leq \frac{b}{s_{(3)n}} \rightarrow 0$$

for $n \rightarrow \infty$. Thus $\mathbf{A} \in \mathcal{T}_q$.

For $n \in U$

$$\sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{s_{(1)n}} \sum_{k=1}^n k^{-q} \chi_U(k) x_k < \frac{\lambda}{s_{(1)n}} \sum_{k=1}^n k^{-q} \chi_U(k) = \lambda$$

on other hand for $n \in V$

$$\sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{s_{(2)n}} \sum_{k=1}^n k^{-q} \chi_V(k) x_k > \frac{\mu}{s_{(2)n}} \sum_{k=1}^n k^{-q} \chi_V(k) = \mu.$$

Therefore $\mathbf{A} - \lim_{n \rightarrow \infty} x_n$ does not exist. □

Corollary 4.5. *If $0 < q_1 < q_2 \leq 1$, then $\mathcal{T}_{q_2} \subsetneq \mathcal{T}_{q_1}$.*

Proof. Let $B \in \mathcal{I}_c^{(q_2)} \setminus \mathcal{I}_c^{(q_1)}$ and let $(x_n) = \chi_B(n)$, $n = 1, 2, \dots$. Clearly $\mathcal{I}_c^{(q_2)} - \lim x_n = 0$ and $\mathcal{I}_c^{(q_1)} - \lim x_n$ does not exist. Let \mathbf{A} be the matrix constructed from the sequence $(x_n)_{n=1}^\infty$ as in the proof of Theorem 4.4. In particular $\mathbf{A} \in \mathcal{T}_{q_1}$ and $\mathbf{A} - \lim_{n \rightarrow \infty} x_n$ does not exist. Therefore $\mathbf{A} \notin \mathcal{T}_{q_2}$. □

Further we show some type well-known matrix which fulfills condition (I). Let $(p_j)_{j=1}^\infty$ be the sequence of positive real numbers. Put $P_n = p_1 + p_2 + \dots + p_n$.

Now we define matrix $\mathbf{A} = (a_{nk})$ in this way:

$$a_{nk} = \frac{p_k}{P_n} \quad k \leq n$$

$$a_{nk} = 0 \quad k > n.$$

This type of matrix is called Riesz matrix.

Especially we put $p_n = n^\alpha$, where $0 < \alpha < 1$. Then

$$a_{nk} = \frac{k^\alpha}{1^\alpha + 2^\alpha + \dots + n^\alpha} \quad k \leq n$$

$$a_{nk} = 0 \quad k > n.$$

This special class of matrix we denote by (\mathbf{R}, n^α) . It is clear that this matrix fulfills conditions (I) and (q). For this class of matrix is true following implication:

$$\mathcal{I}_c^{(q)} - \lim x_k = L \Rightarrow (\mathbf{R}, n^\alpha) - \lim x_k = L$$

where $(x_k)_{k=1}^\infty$ is a bounded sequence, $0 < q \leq 1$, $0 < \alpha < 1$. Converse does not hold. It is sufficient to choose the characteristic function of the set of all primes \mathbb{P} . Then $(\mathbf{R}, n^\alpha) - \lim x_k = 0$, but $\mathcal{I}_c^{(q)} - \lim x_k$ does not exist, because $\sum_{n \in \mathbb{P}} n^{-q} = +\infty$, where \mathbb{P} is a set of all primes. Hence the class (\mathbf{R}, n^α) of matrices belongs to $\mathcal{T} \setminus \mathcal{T}_q$.

Problem 4.6. If we take any admissible ideal \mathcal{I} and define the class $\mathcal{T}_{\mathcal{I}}$ of matrices by replacing the condition (I) in Definition 4.1 by condition: if $C \subset \mathbb{N}$ and $C \in \mathcal{I}$, \mathcal{I} admissible ideal on \mathbb{N} then $\lim_{n \rightarrow \infty} \sum_{k \in C} |a_{nk}| = 0$ then it is easy to see that the if part of Theorem 4.4 holds for \mathcal{T} too. The question is what about only if part.

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