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A new recursion relationship for Bernoulli Numbers

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Abstract

We give an elementary proof of a generalization of the Seidel-Kaneko and Chen-Sun formula involving the Bernoulli numbers.

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MSC: 11B68, 11B83

1. Introduction

The Bernoulli Numbers B_n, $n = 0, 1, 2, \dots$ are defined by the exponential generating function:

$$B(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{1.1}$$

As (1.1) implies that $B(-z) = z + B(z)$, we have:

$$(-1)^n B_n = B_n + \delta_1^n, \text{ for } n \geq 0. \tag{1.2}$$

where the notation δ_i^n is the classical Kronecker symbol which equals 1 if $n = i$ and 0 otherwise. Consequently, we have $B_1 = -\frac{1}{2}$, and $B_n = 0$, when n is odd and $n \geq 3$. Let us define $\epsilon_n := \frac{1 + (-1)^n}{2}$, thus:

$$\epsilon_n B_n = B_n + \frac{1}{2}\delta_1^n, \text{ for } n \geq 0. \tag{1.3}$$

Note that the Bernoulli polynomials can be defined by the following function:

$$B(x,z) := \frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}.$$

Thus, we have:

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!}\right).$$

Therefore the polynomial $B_n(x)$ satisfies the following equality:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k. \tag{1.4}$$

We note also that:

$$B(x+1,z) - B(x,z) = \sum_{n=0}^{\infty} \left(B_n(x+1) - B_n(x)\right)\frac{z^n}{n!} = ze^{xz}.$$

Consequently, we deduce the following property of $B_n(x)$:

$$B_n(x+1) - B_n(x) = nx^{n-1}, \text{ for } n \geq 1. \tag{1.5}$$

In this paper, we are extending the well-known following formulae involving Bernoulli Numbers. First, the Seidel formula (1877) [4], re-discovered later by Kaneko [3] (1995):

$$\sum_{k=0}^{n} \binom{n+1}{k} (n+k+1)\, B_{n+k} = 0, \text{ for } n \geq 1.$$

And secondly, the Chen-Sun formula [1] (2009):

$$\sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)\,(n+k+2)(n+k+1)B_{n+k} = 0. \tag{1.6}$$

Our main result consists on the following:

Theorem 1.1. *For given odd natural q and for natural number $n \geq 0$, we have the equality:*

$$\sum_{k=0}^{n+q} \binom{n+q}{k} (n+k+q)\,(n+k+q-1)\cdots(n+k+1)\, B_{n+k} = 0. \tag{1.7}$$

Obviously, this result gives the Seidel-Kaneko formula when $q = 1$, and the Chen-Sun formula when $q = 3$.

2. Proof of the main result

For a given odd number q and for an integer number $n \geq 0$, we consider the polynomials:

$$H(x) = \frac{1}{2} x^{n+q}(x-1)^{n+q},$$

and

$$K(x) = \sum_{k=0}^{n+q} \frac{\epsilon_{n+k}}{(n+q+k+1)} \binom{n+q}{k} \left(B_{n+q+k+1}(x) - B_{n+q+k+1}\right). \tag{2.1}$$

By the binomial theorem, we deduce:

$$H(x) = \frac{1}{2} \sum_{k=0}^{n+q} (-1)^{n+k+1} \binom{n+q}{k} x^{n+q+k}, \tag{2.2}$$

and

$$H(x+1) = \frac{1}{2} \sum_{k=0}^{n+q} \binom{n+q}{k} x^{n+q+k}. \tag{2.3}$$

Thus, by using the equality property (1.5), we verify that:

$$K(x+1) - K(x) = H(x+1) - H(x) = \sum_{k=0}^{n+q} \epsilon_{n+k} \binom{n+q}{k} x^{n+q+k}. \tag{2.4}$$

Moreover

$$K(0) = H(0) = 0. \tag{2.5}$$

Then, (2.2), (2.3), (2.4) and (2.5) imply:

$$K(x) = H(x).$$

If $[x^n]P(x)$ denotes the coefficient of x^n in the polynomial $P(x)$, we can write:

$$[x^{q+1}]K(x) = [x^{q+1}]H(x). \tag{2.6}$$

So, from (1.4)

$$[x^{q+1}]K(x) = \sum_{k=0}^{n} \frac{\epsilon_{n+k} B_{n+k}}{(n+q+k+1)} \binom{n+q}{k} \binom{n+q+k+1}{q+1}, \tag{2.7}$$

and from (2.2), we have:

$$[x^{q+1}]H(x) = \frac{1}{2} \binom{n+q}{1-n}. \tag{2.8}$$

From (1.3), we know that:

$$\epsilon_{n+k}B_{n+k} = B_{n+k} + \frac{1}{2}\delta_{1-n}^{k}. \tag{2.9}$$

Since

$$\begin{aligned}\sum_{k=0}^{n+q} \frac{\delta_{1-n}^{k}}{2(n+q+k+1)}\binom{n+q}{k}\binom{n+q+k+1}{q+1} &= \frac{1}{2(q+1)}\binom{n+q}{1-n}\binom{q+1}{q} \\ &= \frac{1}{2}\binom{n+q}{1-n}. \end{aligned} \tag{2.10}$$

We deduce, from (2.7), (2.9) and (2.10) that:

$$[x^{q+1}]K(x) = \sum_{k=0}^{n+q} \frac{B_{n+k}}{(n+q+k+1)}\binom{n+q}{k}\binom{n+q+k+1}{q+1} + \frac{1}{2}\binom{n+q}{1-n}. \tag{2.11}$$

It follows from (2.6), (2.8) and (2.11) that:

$$\sum_{k=0}^{n+q} \frac{1}{(n+q+k+1)}\binom{n+q}{k}\binom{n+q+k+1}{q+1}B_{n+k} = 0, \tag{2.12}$$

and by multiplying by $(q+1)!$, we obtain, finally, the aimed result which is:

$$\sum_{k=0}^{n+q}\binom{n+q}{k}(n+k+q)(n+k+q-1)\ldots(n+k+1)B_{n+k} = 0.$$

This ends our proof.

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