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Some inequalities for q-polygamma function and ζ_q-Riemann zeta functions

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Abstract

In this paper, we present some inequalities for q-polygamma functions and ζ_q-Riemann Zeta functions, using a q-analogue of Holder type inequality.

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1. Introduction and preliminaries

In this section, we provide a summary of notations and definitions used in this paper. For details, one may refer to [3, 5].

For $n = 1, 2, \dots$ we denote by $\psi_n(x) = \psi^{(n)}(x)$ the polygamma functions as the n-th derivative of the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, where $\Gamma(x)$ denotes the usual gamma function.

Throughout this paper we will fix $q \in (0, 1)$. Let a be a complex number. The q-shifted factorials are defined by

$$(a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, \dots,$$

$$(a; q)_\infty = \lim_{n\to\infty}(a; q)_n = \prod_{k\geqslant 0}(1 - aq^k).$$

Jackson [4] defined the q-gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, \dots \tag{1.1}$$

It satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{1.2}$$

where for x complex $[x]_q = \frac{1-q^x}{1-q}$.

The q-gamma function has the following integral representation (see [2])

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t = \int_0^{\frac{\infty}{1-q}} t^{x-1} E_q^{-qt} d_q t, \quad x > 0.$$

where $E_q^x = \sum_{j=0}^\infty q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]_q!} = (1 + (1-q)x)_q^\infty$, which is the q-analogue of the classical exponential function.

The q-analogue of the ψ function is defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}, \quad x > 0. \tag{1.3}$$

The q-Jackson integral from 0 to a is defined by (see [4, 5])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^\infty f(aq^n) q^n. \tag{1.4}$$

For $a = \infty$ the q-Jackson integral is defined by (see [4, 5])

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^\infty f(q^n) q^n \tag{1.5}$$

provided that sums in (1.4) and (1.5) converge absolutely.

In [2] the q-Riemman zeta function is defined as follows (see Section 2.3 for the definitions)

$$\zeta_q(s) = \sum_{n=1}^\infty \frac{1}{\{n\}_q^s} = \sum_{n=1}^\infty \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}. \tag{1.6}$$

In relation to (1.3) and (1.6), K. Brahim [1], using a q-analogue of the generalized Schwarz inequality, proved the following Theorems.

Theorem 1.1. *For $n = 1, 2 \dots,$*

$$\psi_{q,n}(x)\psi_{q,m}(x) \geqslant \psi^2_{q,\frac{m+n}{2}}(x),$$

where $\psi_{q,n} = \psi_q^{(n)}$ is n-th derivative of ψ_q and $\frac{m+n}{2}$ is an integer.

Theorem 1.2. *For all $s > 1$,*

$$[s+1]_q \frac{\zeta_q(s)}{\zeta_q(s+1)} \geqslant q[s]_q \frac{\zeta_q(s+1)}{\zeta_q(s+2)}.$$

The aim of this paper is to present some inequalities for q-polygamma functions and q-zeta functions by using a q-analogue of Holder type inequality, similar to those in [1].

2. Main results

2.1. A lemma

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $a \in \mathbf{R}_+ \cup \{\infty\}$, let f and g be two nonnegative functions and let $p, t > 1$ such that $p^{-1} + t^{-1} = 1$. The following inequality holds*

$$\int_0^a f(x)g(x)d_qx \leqslant \Big(\int_0^a f^p(x)d_qx\Big)^{\frac{1}{p}} \Big(\int_0^a g^t(x)d_qx\Big)^{\frac{1}{t}}.$$

Proof. Let $a > 0$. By (1.4) we have that

$$\int_0^a f(x)g(x)d_qx = (1-q)a \sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n. \tag{2.1}$$

By the use of the Holder's inequality for infinite sums, we obtain

$$\Big(\sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n\Big) \leqslant \Big(\sum_{n=0}^{\infty} f^p(aq^n)q^n\Big)^{\frac{1}{p}} \cdot \Big(\sum_{n=0}^{\infty} g^t(aq^n)q^n\Big)^{\frac{1}{t}}. \tag{2.2}$$

Hence

$$\begin{aligned}(1-q)a&\Big(\sum_{n=0}^{\infty} f(aq^n)g(aq^n)q^n\Big)\\ &\leqslant ((1-q)a)^{\frac{1}{p}}\Big(\sum_{n=0}^{\infty} f^p(aq^n)q^n\Big)^{\frac{1}{p}} \cdot ((1-q)a)^{\frac{1}{t}}\Big(\sum_{n=0}^{\infty} g^t(aq^n)q^n\Big)^{\frac{1}{t}}.\end{aligned} \tag{2.3}$$

The result then follows from (2.1), (2.2) and (2.3). □

2.2. The q-polygamma function

From (1.1) one can derive the following series representation for the function $\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$:

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geqslant 1} \frac{q^{nx}}{1-q^n}, \quad x > 0, \tag{2.4}$$

which implies that

$$\psi_q(x) = -\log(1-q) + \frac{\log q}{1-q}\int_0^q \frac{t^{x-1}}{1-t} d_q t. \tag{2.5}$$

Theorem 2.2. *For $n = 2, 4, 6\dots$ set $\psi_{q,n}(x) = \psi_q^{(n)}(x)$ the n-th derivative of the function ψ_q. Then for $p, t > 1$ such that $\frac{1}{p} + \frac{1}{t} = 1$ the following inequality holds*

$$\psi_{q,n}\Big(\frac{x}{p} + \frac{y}{t}\Big) \leqslant \psi_{q,n}(x)^{\frac{1}{p}} \cdot \psi_{q,n}(y)^{\frac{1}{t}}. \tag{2.6}$$

Proof. From (2.5) we deduce that

$$\psi_{q,n}(x) = \frac{\log q}{1-q}\int_0^q \frac{(\log u)^n u^{x-1}}{1-u} d_q u, \tag{2.7}$$

hence

$$\psi_{q,n}\Big(\frac{x}{p} + \frac{y}{t}\Big) = \frac{\log q}{1-q}\int_0^q \frac{(\log u)^n u^{\frac{x}{p}+\frac{y}{t}-1}}{1-u} d_q u.$$

By Lemma 2.1 with $a = q$ we have

$$\begin{aligned}\psi_{q,n}\Big(\frac{x}{p} + \frac{y}{t}\Big) &= \frac{\log q}{1-q}\int_0^q \Big[\frac{(\log u)^n}{1-u}\Big]^{\frac{1}{p}} u^{\frac{x-1}{p}} \Big[\frac{(\log u)^n}{1-u}\Big]^{\frac{1}{t}} u^{\frac{y-1}{q}} d_q u \\ &\leqslant \Big(\frac{\log q}{1-q}\int_0^q \frac{(\log u)^n u^{x-1}}{1-u} d_q u\Big)^{\frac{1}{p}} \Big(\frac{\log q}{1-q}\int_0^q \frac{(\log u)^n u^{y-1}}{1-u} d_q u\Big)^{\frac{1}{t}} \\ &= (\psi_{q,n}(x))^{\frac{1}{p}} (\psi_{q,n}(y))^{\frac{1}{t}}\end{aligned}$$

where $f(u) = \Big(\frac{(\log u)^n}{1-u}\Big)^p u^{\frac{x-1}{p}}$ and $g(u) = \Big(\frac{(\log u)^n}{1-u}\Big)^t u^{\frac{y-1}{t}}$. □

For $p = t = 2$ in (2.6) one has the following result.

Corollary 2.3. *We have*

$$\psi_{q,n}\Big(\frac{x+y}{2}\Big) \leqslant \sqrt{\psi_{q,n}(x) \cdot \psi_{q,n}(y)}.$$

2.3. q-zeta function

For $x > 0$ we set $\alpha(x) = \frac{\log x}{\log q} - E\Big(\frac{\log x}{\log q}\Big)$ and $\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}}$, where $E\Big(\frac{\log x}{\log q}\Big)$ is the integer part of $\frac{\log x}{\log q}$.

In [2] the q-zeta function is defined as follows

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}.$$

There ([2]) it is proved that ζ_q is a q-analogue of the classical Riemman Zeta function, and for all $s \in \mathbf{C}$ such that $\Re(s) > 1$, and for all $u > 0$ one has

$$\zeta_q(s) = \frac{1}{\widetilde{\Gamma}_q(s)} \int_0^\infty u^{s-1} Z_q(u) d_q u,$$

where $Z_q(t) = \sum_{n=1}^\infty e_q^{-\{n\}_q t}$, $\widetilde{\Gamma}_q(t) = \frac{\Gamma_q(t)}{K_q(t)}$, and

$$K_q(t) = \frac{(1-q)^{-s}}{1+(1-q)^{-1}} \cdot \frac{(-(1-q);q)_\infty(-(1-q)^{-1};q)_\infty}{(-(1-q)q^s;q)_\infty(-(1-q)^{-1}q^{1-s};q)_\infty}.$$

Theorem 2.4. *For $\frac{1}{p} + \frac{1}{t} = 1$ and $\frac{x}{p} + \frac{y}{t} > 1$,*

$$\frac{\widetilde{\Gamma}_q\left(\frac{x}{p}+\frac{y}{t}\right)}{\widetilde{\Gamma_q}^{\frac{1}{p}}(x) \cdot \widetilde{\Gamma_q}^{\frac{1}{t}}(y)} \leqslant \frac{\zeta_q^{\frac{1}{p}}(x) \cdot \zeta_q^{\frac{1}{t}}(y)}{\zeta_q\left(\frac{x}{p}+\frac{y}{t}\right)}.$$

Proof. From Lemma 2.1 we have that

$$\begin{aligned}\int_0^\infty u^{\frac{x}{p}+\frac{y}{t}-1} Z_q(u) d_q u &= \int_0^\infty u^{\frac{x-1}{p}} \cdot (Z_q(u))^{\frac{1}{p}} u^{\frac{y-1}{t}} \cdot (Z_q(u))^{\frac{1}{t}} d_q u. \\ &\leqslant \left(\int_0^\infty u^{x-1} \cdot (Z_q(u)) d_q u\right)^{\frac{1}{p}} \cdot \left(\int_0^\infty u^{y-1} \cdot (Z_q(u)) d_q u\right)^{\frac{1}{t}}.\end{aligned}$$

For $f(u) = u^{\frac{x-1}{p}} \cdot (Z_q(u))^{\frac{1}{p}}$ and $g(u) = u^{\frac{y-1}{t}} \cdot (Z_q(u))^{\frac{1}{t}}$ we obtain that

$$\widetilde{\Gamma}_q\left(\frac{x}{p}+\frac{y}{t}\right) \cdot \zeta_q\left(\frac{x}{p}+\frac{y}{t}\right) \leqslant \widetilde{\Gamma}_q^{\frac{1}{p}}(x) \cdot \widetilde{\Gamma}_q^{\frac{1}{t}}(y) \cdot \zeta_q^{\frac{1}{p}}(x) \cdot \zeta_q^{\frac{1}{t}}(y),$$

which completes the proof. $\square$

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References

[1] Brahim, K., Turán-Type Inequalities for some q-special functions, *J. Ineq. Pure Appl. Math.*, 10(2) (2009) Art. 50.

[2] Fitouhi, A., Bettaibi, N., Brahim, K., The Mellin transform in quantum calculus, *Constructive Approximation*, 23(3) (2006) 305–323.

[3] Gasper, G., Rahman, M., Basic Hypergeometric Series, 2nd Edition, (2004), Encyclopedia of Mathematics and Applications, 96, Cambridge University Press, Cambridge.

[4] Jackson, F.H., On a q-definite integrals, *Quart. J. Pure and Appl. Math.*, 41 (1910) 193–203.

[5] Kac, V.G., Cheung, P., Quantum Calculus, Universitext, Springer-Verlag, New York, (2002).

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