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On the best estimations for dispersions of special ratio block sequences*

Ferdinánd Filip[a], Kálmán Liptai[b]
Ferenc Mátyás[b], János T. Tóth[a]

[a]Department of Mathematics, J. Selye University
[b]Institute of Mathematics and Informatics, Eszterházy Károly College

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Dedicated to professor Béla Pelle on his 80th birthday

Abstract

Properties of dispersion of block sequences were investigated by J. T. Tóth, L. Mišík, F. Filip [20]. The present paper is a continuation of the study of relations between the density of the block sequence and so called dispersion of the block sequence.

Keywords: dispersion, block sequence, (R)-density.

MSC: Primary 11B05.

1. Introduction

In this part we recall some basic definitions. Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $X \subset \mathbb{N}$ let $X(n) = \#\{x \in X; x \leqslant n\}$. In the whole paper we will assume that X is infinite. Denote by $R(X) = \{\frac{x}{y}; x \in X,\ y \in X\}$ the *ratio set of* X and say that a set X is (R)-dense if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. Let us notice that the concept of (R)-density was defined and first studied in papers [17] and [18].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. The sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1.1}$$

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of finite sequences derived from X is called *ratio block sequence* of the set X. Thus the block sequence is formed by blocks $X_1, X_2, \dots, X_n, \dots$ where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right); \quad n = 1, 2, \dots.$$

This kind of block sequences were studied in papers, [1] , [3] , [4] , [16] and [20]. Also other kinds of block sequences were studied by several authors, see [2], [6], [8], [12] and [19]. Let $Y = (y_n)$ be an increasing sequence of positive integers. A sequence of blocks of type

$$Y_n = \left(\frac{1}{y_n}, \frac{2}{y_n}, \dots, \frac{y_n}{y_n}\right)$$

was invetigated in [11] which extends a result of [5]. Authors obtained a complete theory for the uniform distribution of the related block sequence (Y_n).
For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\},$$

the maximum distance between two consecutive terms in the n-th block.
In this paper we will consider the characteristics (see [20])

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n),$$

called the *dispersion* of the block sequence (1.1) derived from X, and its relations to the previously mentioned asymptotic density of the original set X.

At the end of this section, let us mention the concept of a dispersion of a general sequence of numbers in the interval $\langle 0, 1\rangle$. Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\langle 0, 1\rangle$. For every $N \in \mathbb{N}$ let $x_{i_1} \leqslant x_{i_2} \leqslant \dots \leqslant x_{i_N}$ be reordering of its first N terms into a nondecreasing sequence and denote

$$d_N = \frac{1}{2}\max\left\{\max\{x_{i_{j+1}} - x_{i_j};\ j = 1, 2, \dots N-1\}, x_{i_1}, 1 - x_{i_N}\right\}$$

the dispersion of the finite sequence $x_0, x_1, x_2, \dots x_N$. Properties of this concept can be found for example in [10] where it is also proved that

$$\limsup_{N\to\infty} N d_N \geqslant \frac{1}{\log 4}$$

holds for every one-to-one infinite sequence $x_n \in \langle 0, 1\rangle$. Also notice that the density of the whole sequence $(x_n)_{n=0}^{\infty}$ is equivalent to $\lim\limits_{N\to\infty} d_N = 0$. Also notice that the analogy of this property for the concept of dispersion of block sequences defined in the present paper does not hold.

Much more on these and related topics can be found in monograph [13].

2. Results

When calculating the value $\underline{D}(X)$, the following theorems are often useful (See [20], Theorem 1, Corollary 1, respectively).

(A1) Let

$$X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}\,,$$

where $x_n < x_{n+1}$ and let $c_n < d_n < c_{n+1}$, for $n \in \mathbb{N}$, be positive integers. Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1, 2, \dots, n\}}{d_{n+1}}\,.$$

(A2) Let X be identical to the form of X in (A1). Suppose that there exists a positive integer n_0 such that for all integers $n > n_0$

$$c_{n+1} - d_n \leqslant c_{n+2} - d_{n+1}\,.$$

Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}}\,.$$

The basic properties of the dispersion $\underline{D}(X)$ and the relations between dispersion and (R)-density are investigated in the paper [TMF]. The next theorem states the upper bound for dispersions $\underline{D}(X)$ of (R)-dense sets where $1 \leqslant a = \lim\limits_{n\to\infty} \frac{d_n}{c_n} < \infty$ (See [20], Theorem 10).

(A3) Let $X = \bigcup_{n=1}^{\infty} \Big(c_n, d_n\Big\rangle \cap \mathbb{N}$ be an (R)-dense set where $c_n < d_n < c_{n+1}$ for all $n \in \mathbb{N}$ and suppose that the limit $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = a$ exists. Then

$$\underline{D}(X) \leqslant \min\left\{\frac{1}{a+1}, \max\left\{\frac{a-1}{a^2}, \frac{1}{a^2}\right\}\right\},$$

more precisely,

$$\underline{D}(X) \leqslant \begin{cases} \frac{1}{1+a} & \text{if } a \in \langle 1, \frac{1+\sqrt{5}}{2}) \\ \frac{1}{a^2} & \text{if } a \in \langle \frac{1+\sqrt{5}}{2}, 2) \\ \frac{a-1}{a^2} & \text{if } a \in \langle 2, \infty)\,. \end{cases}$$

The following theorem shows that in the third case (if $a \geqslant 2$), that the dispersion $\underline{D}(X)$ can be any number in the interval $\langle 0, \frac{a-1}{a^2}\rangle$, where $X = \bigcup_{n=1}^{\infty} \Big(c_n, d_n\Big\rangle \cap \mathbb{N}$ is (R)-dense and $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = a$. Thus the upper bound for $\underline{D}(X)$ is the best possible in the case $a \geqslant 2$ (See [4], Theorem 2).

(A4) Let $a \geqslant 1$ be a real number and k be an arbitrary natural number. Then for every $\alpha \in \langle 0, \frac{a^k-1}{a^{2k}}\rangle$ there exists an (R)-dense set

$$X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$$

where $c_n < d_n < c_{n+1}$ are positive integers for every $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \frac{d_n}{c_n} = a$ and $\underline{D}(X) = \alpha$.

In this paper we prove that in the second case $\left(\text{if } a \in \left\langle \frac{1+\sqrt{5}}{2}, 2\right)\right)$, the dispersion $\underline{D}(X)$ can be any number in the interval $\langle 0, \frac{1}{a^2}\rangle$, where $X = \bigcup_{n=1}^{\infty} \left(c_n, d_n\right\rangle \cap \mathbb{N}$ is (R)-dense and $\lim_{n\to\infty} \frac{d_n}{c_n} = a$. Thus the upper bound for $\underline{D}(X)$ is the best possible in the case $a \in \left\langle \frac{1+\sqrt{5}}{2}, 2\right)$. The following lemma will be useful.

Lemma 2.1. *Let the set*

$$M(X) = \{n \in \mathbb{N} : c_{n+1} - d_n = \max\{c_{i+1} - d_i : i = 1, 2, \dots, n\}\} =$$
$$= \{m_1 < m_2 < \dots < m_k < \dots\}$$

be infinite. Then

$$\underline{D}(X) = \liminf_{k\to\infty} \frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}.$$

Proof. Let $n \in \mathbb{N}$ be an arbitrary integer such that $n \geqslant m_1$. Then there is unique $k \in \mathbb{N}$ with $m_k \leqslant n < m_{k+1}$. >From the definition of the set $M(X)$ we obtain

$$\frac{\max\{c_{i+1} - d_i : i = 1, 2, \dots, n\}}{d_{n+1}} = \frac{c_{m_k+1} - d_{m_k}}{d_{n+1}} \geqslant \frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}.$$

Then obviously

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1, 2, \dots, n\}}{d_{n+1}} \geqslant \liminf_{k\to\infty} \frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}.$$

On the other hand, the sequence $\left(\frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}\right)_{k=1}^{\infty}$ is a subsequence of the sequence $\left(\frac{\max\{c_{i+1} - d_i : i=1,2,\dots,n\}}{d_{n+1}}\right)_{n\in\mathbb{N}}$, hence

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1, 2, \dots, n\}}{d_{n+1}} \leqslant \liminf_{k\to\infty} \frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}.$$

The last two inequalities imply

$$\underline{D}(X) = \liminf_{k\to\infty} \frac{c_{m_k+1} - d_{m_k}}{d_{m_{k+1}}}.$$

□

Theorem 2.2. *Let $a \in \langle \frac{1+\sqrt{5}}{2}, 2\big)$ be an arbitrary real number. Then for every $\alpha \in \langle 0, \frac{1}{a^2}\rangle$ there is an (R)-dense set*

$$X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}\,,$$

where $c_n < d_n < c_{n+1}$ are positive integers for every $n \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = a$ and $\underline{D}(X) = \alpha$.

Proof. Let $a \in \langle \frac{1+\sqrt{5}}{2}, 2)$. According to (A4), it is sufficient to prove Theorem 2.2 for $\frac{a-1}{a^2} < \alpha \leqslant \frac{1}{a^2}$. Define function $f(b) = \frac{b-1}{ab}$. Clearly f is continuous and increasing on the interval $\langle a, \infty)$. Moreover

$$f(a) = \frac{a-1}{a^2} \qquad \text{and} \qquad f(a^2) = \frac{a^2-1}{a^3}\,.$$

We have $\frac{a^2-1}{a^3} \geqslant \frac{1}{a^2}$ if $a \geqslant \frac{1+\sqrt{5}}{2}$. Thus there exists a real number $a < b \leqslant a^2$ such that

$$\frac{b-1}{ab} = \alpha.$$

Define a set $X \subset \mathbb{N}$ by

$$X = \bigcup_{n=1}^{\infty} \big(A_n \cup B_n\big) \cap \mathbb{N}\,,$$

where for every $n \in \mathbb{N}$

$$A_n = (a_{n,1}, b_{n,1}\rangle \cup (a_{n,2}, b_{n,2}\rangle \qquad \text{a} \qquad B_n = \bigcup_{k=1}^{n} (c_{n,k}, d_{n,k}\rangle\,.$$

Put $a_{1,1} = 1$ and for every $n \in \mathbb{N}$ and $k = 2, 3, \dots, n$
$b_{n,1} = [aa_{n,1}] + 1, \quad a_{n,2} = b_{n,1} + 1, \quad b_{n,2} = [aa_{n,2}] + 1\,,$
$c_{n,1} = [bb_{n,2}] + 1, \quad d_{n,1} = [ac_{n_1}] + 1, \quad c_{n,k} = [bd_{n,k-1}] + 1, \quad d_{n,k} = [ac_{n,k}] + 1\,,$
and $a_{n+1,1} = (n+1)d_{n,n}$.
Obviously for every $n \in \mathbb{N}$

$$a < \frac{b_{n,1}}{a_{n,1}} \leqslant a + \frac{1}{a_{n,1}} \quad \text{and} \quad a < \frac{b_{n,1}}{a_{n,1}} \leqslant a + \frac{1}{a_{n,1}}\,,$$

and for $k = 1, 2, \dots, n$

$$a < \frac{d_{n,k}}{c_{n,k}} \leqslant a + \frac{1}{a_{n,1}}\,.$$

First we prove that $\underline{D}(X) = \alpha$. We have the following inequalities:

$$c_{n+1,1} - b_{n+1,2} \geqslant bb_{n+1,2} - b_{n+1,2} \geqslant (b-1)b_{n+1,2} \geqslant (b-1)a^2 a_{n+1,1} \geqslant$$
$$\geqslant (a-1)a^2 a_{n+1,1} \geqslant aa_{n+1,1} > a_{n+1,1} > a_{n+1,1} - d_{n,n}$$

The inequality $a^2(a-1) \geqslant a$ follows from $a \geqslant \frac{1+\sqrt{5}}{2}$. Then

$$c_{n,2}-d_{n,1} \geqslant bd_{n,1}-d_{n,1} = (b-1)d_{n,1} \geqslant (b-1)ac_{n,1} \geqslant (a-1)ac_{n,1} \geqslant c_{n,1} > c_{n,1}-b_{n,2}$$

and for every $k = 2,3,\dots,n-1$

$$c_{n,k+1} - d_{n,k} \geqslant bd_{n,k} - d_{n,k} = (b-1)d_{n,k} \geqslant (b-1)ac_{n,k} \geqslant$$
$$\geqslant (a-1)ac_{n,k} \geqslant c_{n,k} > c_{n,k} - d_{n,k-1}.$$

Finally

$$a_{n+2,1} - d_{n+1,n+1} = (n+2)d_{n+1,n+1} - d_{n+1,n+1} >$$
$$> d_{n+1,n+1} > c_{n+1,n+1} > c_{n+1,n+1} - d_{n+1,n}\,.$$

From the above inequalities we have for a sufficiently large $n \in \mathbb{N}$ the following inequalities:

$$\begin{aligned} 1 = a_{n,2} - b_{n,1} < a_{n,1} - d_{n-1,n-1} < c_{n,1} - b_{n,2} < c_{n,2} - d_{n,1} < \dots \\ \dots < c_{n,n} - d_{n,n-1} < a_{n+1,1} - d_{n,n}\,. \end{aligned} \tag{2.1}$$

Now we use Lemma 2.1. From (2.1) one can see that it is sufficient to study the quotients:

a) $\frac{a_{n+1,1}-d_{n,n}}{b_{n+1,2}}$,

b) $\frac{c_{n,1}-b_{n,2}}{d_{n,1}}$,

c) $\frac{c_{n,k}-d_{n,k-1}}{d_{n,k}}$ for $k = 2,3,\dots,n$.

In case a)

$$\liminf_{n\to\infty} \frac{a_{n+1,1}-d_{n,n}}{b_{n+1,2}} = \liminf_{n\to\infty} \frac{(n-1)d_{n,n}}{na^2 d_{n,n}} = \frac{1}{a^2} \geqslant \alpha,$$

in case b)

$$\liminf_{n\to\infty} \frac{c_{n,1}-b_{n,2}}{d_{n,1}} = \liminf_{n\to\infty} \frac{(b-1)b_{n,2}}{abb_{n,2}} = \frac{b-1}{ab} = \alpha$$

and in case c)

$$\frac{c_{n,k}-d_{n,k-1}}{d_{n,k}} \leqslant \frac{(b-1)d_{n,k-1}+1}{abd_{n,k-1}} \leqslant \frac{b-1}{ab} + \frac{1}{abd_{n,k-1}} \leqslant \alpha + \frac{1}{abd_{n,1}}$$

and

$$\frac{c_{n,k}-d_{n,k-1}}{d_{n,k}} \geqslant \frac{(b-1)d_{n,k-1}}{abd_{n,k-1}+b+1} \geqslant$$
$$\geqslant \frac{b-1}{ab} - \frac{b-1}{ab}\,\frac{b+1}{abd_{n,k-1}+b+1} \geqslant \alpha - \frac{b^2-1}{d_{n,1}}\,.$$

From this it is obvious that $\underline{D}(X) = \alpha$.

It remains to prove that the set X is (R)-dense. We have $\frac{1}{a^2} \leqslant \frac{1}{b}$ and $\frac{1}{b^l a^{l+2}} \leqslant \frac{1}{b^{l+1}a^l}$ for every $l = 1,2,\dots,$ hence

$$\Big(\frac{1}{a^2},1\Big\rangle \cup \bigcup_{l=1}^{\infty}\Big(\frac{1}{b^l a^{l+2}},\frac{1}{b^l a^{l-1}}\Big\rangle = (0,1\rangle$$

and it is sufficient to prove that the ratio set of the set X is dense on intervals

$$\Big(\frac{1}{a^2},1\Big\rangle \qquad \text{and} \qquad \Big(\frac{1}{b^l a^{l+2}},\frac{1}{b^l a^{l-l}}\Big\rangle$$

for every $l = 1, 2, \dots$.

Now we prove that the ratio set of X is dense on $\Big(\frac{1}{a^2},1\Big\rangle$. Let $(e,f) \subset \Big(\frac{1}{a^2},1\Big\rangle$. Put $\varepsilon = f - e$. Consider the set

$$\begin{aligned}\Big\{\frac{a_{n,1}+1}{b_{n,2}} < \frac{a_{n,1}+2}{b_{n,2}} < \dots < \frac{b_{n,1}}{b_{n,2}} <& \\ < \frac{a_{n,2}+1}{b_{n,2}} < \frac{a_{n,2}+2}{b_{n,2}} < \dots < \frac{b_{n,2}-1}{b_{n,2}} < \frac{b_{n,2}}{b_{n,2}} = 1\Big\},&\end{aligned} \tag{2.2}$$

which is obviously a subset of the ratio set of X. The largest difference between consecutive terms of (2.2) is $\frac{2}{b_{n,2}}$. Then

$$\frac{a_{n,1}+1}{b_{n,2}} = \frac{a_{n,1}}{b_{n,2}} + \frac{1}{b_{n,2}} \leqslant \frac{a_{n,1}}{a^2 a_{n,1}} + \frac{1}{b_{n,2}} = \frac{1}{a^2} + \frac{1}{b_{n,2}}\,.$$

If we choose $n \in \mathbb{N}$ so that $\frac{2}{b_{n,2}} < \varepsilon$, then the interval (e,f) is not disjoint with (2.2), hence the ratio set of X is dense in the interval $\Big(\frac{1}{a^2},1\Big\rangle$.

Let $l \in \mathbb{N}$ be arbitrary. We prove that the ratio set of X is dense in the interval $\Big(\frac{1}{b^l a^{l+2}},\frac{1}{b^l a^{l-1}}\Big\rangle$. Let $(e,f) \subset \Big(\frac{1}{b^l a^{l+2}},\frac{1}{b^l a^{l-1}}\Big\rangle$. Put $\varepsilon = f - e$. Choose $n_1 \in \mathbb{N}$ so that $n_1 > l$ and $a_{n,1} + 1 > \frac{2}{\varepsilon}$ for every $n > n_1$. Consider the set

$$\begin{aligned}\Big\{\frac{b_{n,2}}{c_{n,l}+1} > \frac{b_{n,2}-1}{c_{n,l}+1} > \dots > \frac{a_{n,2}+1}{c_{n,l}+1} > \frac{b_{n,1}}{c_{n,l}+1} >& \\ > \frac{b_{n,1}-1}{c_{n,l}+1} > \dots > \frac{a_{n,1}+1}{c_{n,l}+1} > \frac{a_{n,1}+1}{c_{n,l}+2} > \dots > \frac{a_{n,1}+1}{d_{n,l}}\Big\},&\end{aligned} \tag{2.3}$$

which is obviously a subset of the ratio set of X. The largest difference between consecutive terms of (2.3) is $\leqslant \frac{2}{a_{n,1}+1}$. On the other hand,

$$\lim_{n\to\infty} \frac{b_{n,2}}{c_{n,l}+1} = \frac{1}{b^l a^{l-1}} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_{n,1}+1}{d_{n,l}} = \frac{1}{b^l a^{l+2}}\,.$$

Then there exists $n_2 \in \mathbb{N}$, such that for every $n > n_2$

$$\frac{b_{n,2}}{c_{n,l}+1} > \frac{1}{b^l a^{l-1}} - \varepsilon \qquad \text{and} \qquad \frac{a_{n,1}+1}{d_{n,l}} < \frac{1}{b^l a^{l+2}} + \varepsilon\,.$$

If we choose $n > \max\{n_1, n_2\}$, then the interval (e,f) is not disjoint with (2.3), hence the ratio set of X is dense in the interval $\big(\frac{1}{b^l, a^{l+2}},\frac{1}{b^l, a^{l-1}}\big\rangle$. This concludes the proof. $\square$

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Ferdinánd Filip, **János T. Tóth**
Department of Mathematics
J. Selye University
Bratislavská cesta 3322
945 01 Komárno
Slovakia
e-mail: filip.ferdinand@selyeuni.sk
toth.janos@selyeuni.sk

Kálmán Liptai, **Ferenc Mátyás**
Institute of Mathematics and Informatics
Eszterházy Károly College
H-3300 Eger
Leányka út 4.
Hungary
e-mail: liptaik@ektf.hu
matyas@ektf.hu