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Solving certain quintics

Raghavendra G. Kulkarni

Bharat Electronics Ltd., India

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Abstract

In this paper we present a simple method for factoring a quintic equation into quadratic and cubic polynomial factors by using a novel decomposition technique, wherein the given quintic is compared with the another, which deceptively appears like a sextic equation.

1. Introduction

From the works of Abel (1826) and Galois (1832), we know that a general quintic equation can not be solved in radicals [1, 2]. With some condition imposed on it, the quintic becomes solvable in radicals, and is aptly called solvable quintic equation. In this paper we present a very simple method for solving certain type of solvable quintic equations. The method converts given quintic equation into a decomposable quintic equation in an elegant fashion. The condition to be satisfied by the coefficients of the quintic so that it becomes solvable is derived. We discuss the behavior of roots of such quintic equations. A procedure to synthesize these quintics is given. We solve one numerical example using the proposed method at the end of the paper.

2. The proposed method

We know that in an N-th degree polynomial equation, the $(N-1)$-th term can be eliminated by suitable change of variable. Therefore without loss of generality, we consider the following reduced quintic equation:

$$x^5 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{2.1}$$

for solving by the proposed method, where the coefficients, a_0, a_1, a_2, and a_3, are real. Let us consider another quintic equation (which deceptively appears like a sextic equation!) as shown below:

$$\frac{1}{4b_2}\left[(x^3+b_2x^2+b_1x+b_0)^2-(x^3-b_2x^2+c_1x+c_0)^2\right]=0, \tag{2.2}$$

where b_0, b_1, b_2, c_0, and c_1 are unknowns to be determined, and $b_2 \neq 0$. Notice that the term inside the square bracket in the above expression is in the form of $A^2 - B^2$, hence the expression (2.2) can be split into two factors (quadratic and cubic) as shown below.

$$\left[x^2+\left(\frac{b_1-c_1}{2b_2}\right)x+\left(\frac{b_0-c_0}{2b_2}\right)\right]\left[x^3+\left(\frac{b_1+c_1}{2}\right)x+\left(\frac{b_0+c_0}{2}\right)\right]=0 \tag{2.3}$$

Therefore our aim is to represent the given quintic (2.1) in the form of (2.2), so that it can be easily decomposed as shown in (2.3). To achieve this, the coefficients of quintic (2.1) are to be equated with that of quintic (2.2). However since the coefficients of (2.2) are not explicitly written, we expand and rearrange the the expression (2.2) in the descending powers of x, as shown below.

$$x^5+\left[\frac{b_1-c_1}{2b_2}\right]x^4+\left(\frac{b_0-c_0+b_2(b_1+c_1)}{2b_2}\right)x^3+$$
$$\left[\frac{b_1^2-c_1^2+2b_2(b_0+c_0)}{4b_2}\right]x^2+\left(\frac{b_0b_1-c_0c_1}{2b_2}\right)x+\left[\frac{b_0^2-c_0^2}{4b_2}\right]=0 \tag{2.4}$$

Now, equating the coefficients of (2.1) and (2.4), we obtain five equations in five unknowns, b_0, b_1, b_2, c_0, and c_1, as shown below.

$$b_1-c_1=0 \tag{2.5}$$

$$b_0-c_0+b_2(b_1+c_1)=2a_3b_2 \tag{2.6}$$

$$b_1^2-c_1^2+2b_2(b_0+c_0)=4a_2b_2 \tag{2.7}$$

$$b_0b_1-c_0c_1=2a_1b_2 \tag{2.8}$$

$$b_0^2-c_0^2=4a_0b_2 \tag{2.9}$$

Employing the elimination method, we attempt to determine the unknowns using above equations (2.5)–(2.9). Using (2.5) we eliminate c_1 from equations (2.6), (2.7), and (2.8) leading to following new equations respectively.

$$b_0-c_0+2b_1b_2=2a_3b_2 \tag{2.10}$$

$$b_0=2a_2-c_0 \tag{2.11}$$

$$b_1(b_0-c_0)=2a_1b_2 \tag{2.12}$$

Using (2.11) we eliminate b_0 from (2.9), (2.10), and (2.12) resulting in the following expressions.

$$c_0 = a_2 - \frac{a_0 b_2}{a_2} \tag{2.13}$$

$$a_2 - c_0 + b_1 b_2 = a_3 b_2 \tag{2.14}$$

$$b_1(a_2 - c_0) = a_1 b_2 \tag{2.15}$$

Now, we use (2.13) to eliminate c_0 from (2.14) and (2.15) and obtain the following new expressions.

$$b_1 = a_3 - \frac{a_0}{a_2} \tag{2.16}$$

$$b_1 = \frac{a_1 a_2}{a_0} \tag{2.17}$$

Notice an interesting situation here! We are now left with two equations (2.16) and (2.17), and both are expressions for the unknown b_1. Eliminating b_1 from (2.16) using (2.17) leaves us with an expression, which contains only the coefficients of given quintic (2.1) as shown below.

$$a_1 = \frac{a_0 a_3}{a_2} - \frac{a_0^2}{a_2^2} \tag{2.18}$$

Note that at this stage we have exhausted all the equations, and the unknown b_2 is yet to be determined. It appears that we have hit a dead end in the pursuit of decomposition of quintic. After thinking a while, we note that what really required to be determined are the coefficients of quadratic and cubic polynomial factors in (2.3), and therefore we attempt to find expressions for these coefficients. For this purpose, the expression (2.3) is rewritten as,

$$(x^2 + d_1 x + d_0)(x^3 + e_1 x + e_0) = 0, \tag{2.19}$$

where d_0, d_1, e_0, and e_1 are given by,

$$d_0 = \frac{b_0 - c_0}{2b_2}, \tag{2.20}$$

$$d_1 = \frac{b_1 - c_1}{2b_2}, \tag{2.21}$$

$$e_0 = \frac{b_0 + c_0}{2}, \tag{2.22}$$

$$e_1 = \frac{b_1 + c_1}{2}. \tag{2.23}$$

Using (2.12) and (2.17) we evaluate d_0 as: $d_0 = a_0/a_2$. From (2.5) we note that $d_1 = 0$. From (2.11), e_0 is determined as: $e_0 = a_2$. Again using (2.5) we determine e_1 as: $e_1 = a_1 a_2 / a_0$. Thus all the coefficients in (2.19) are determined and there is no need to determine the unknown b_2. When each of the polynomial factors in (2.19) is equated to zero and solved, we obtain all the five roots of the given quintic equation (2.1).

3. A discussion on such solvable quintic

The expression (2.18) is the condition for the coefficients of quintic (2.1) to satisfy in order that the quintic becomes solvable. Such solvable quintics can be synthesized by determining the coefficient a_1 using expression (2.18) from the remaining real coefficients, a_0, a_2, and a_3, which are chosen arbitrarily. In the numerical example given (at the end of the paper) we first synthesize the quintic equation and then solve it to determine the roots. How do the roots of such quintic behave? To find out the answer, we express the decomposed quintic (2.19) as below (using the expressions for the coefficients of quadratic and cubic polynomial factors).

$$[x^2 + (a_0/a_2)][x^3 + (a_1 a_2/a_0)x + a_2] = 0. \tag{3.1}$$

From the above expression it is clear that the sum of roots of quadratic factor is zero. This automatically sets the sum of roots of cubic factor to zero since the sum of roots of quintic (2.1) is zero as x^4 term is missing in (2.1).

4. Numerical example

Let us synthesize solvable quintic proposed in this paper. Consider the quintic equation as below.

$$x^5 - 18x^3 + 30x^2 + a_1 x + 30 = 0 \tag{4.1}$$

The coefficient a_1 is determined from (2.18) as: -19. The coefficient in the quadratic factor d_0 is evaluated as 1, and the coefficients in the cubic factor, e_0 and e_1, are determined as 30 and -19 [see expression (3.1) for the factored quintic]. Thus the factored quintic is expressed as:

$$(x^2 + 1)(x^3 - 19x + 30) = 0.$$

Equating each factor in the above quintic to zero and solving, we determine the roots of quintic (4.1) as: $\pm i, 2, 3, -5$, where $i = \sqrt{-1}$.

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Raghavendra G. Kulkarni
Senior Deputy General Manager (HMC-D & E)
Bharat Electronics Ltd., Jalahalli Post, Bangalore-560013, India
Member-MAA, Life Member-Ramanujan Mathematical Society
Senior Member-IEEE.
Phone: +91-80-22195270, Fax: +91-80-28382738.
e-mail: `rgkulkarni@ieee.org`