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Geometric properties and constrained modification of trigonometric spline curves of Han

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Dedicated to professor Béla Pelle on his 80^{th} birthday

Abstract

New types of quadratic and cubic trigonometrial polynomial curves have been introduced in [2] and [3]. These trigonometric curves have a global shape parameter λ. In this paper the geometric effect of this shape parameter on the curves is discussed. We prove that this effect is linear. Moreover we show that the quadratic curve can interpolate the control points at $\lambda = \sqrt{2}$. Constrained modification of these curves is also studied. A curve passing through a given point is computed by an algorithm which includes numerical computations. These issues are generalized for surfaces with two shape parameters. We show that a point of the surface can move along a hyperbolic paraboloid.

Keywords: trigonometric curve, spline curve, constrained modification

MSC: 68U07, 65D17

1. Introduction

In Computer Aided Geometric Design the most prevalently used curves are B-Spline and NURBS curves. Besides the quadratic and the cubic B-Spline and NURBS curves the trigonometric spline curves are another way to define curves above a new function space. Among the first ones, C-Bézier and uniform CB-spline

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curves are defined by means of the basis $\{\sin t, \cos t, t, 1\}$, which was generalized to $\{\sin t, \cos t, t^{k-3}, t^{k-4}, \ldots, t, 1\}$ (cf. [14, 15, 16, 1]). Wang et al. introduced NUAT B-spline curves ([13]) that are the non-uniform generalizations of CB-spline curves. The other basic type is the HB-spline curve, the basis of which is $\{\sinh t, \cosh t, t, 1\}$, and $\{\sinh t, \cosh t, t^{k-3}, t^{k-4}, \ldots, t, 1\}$ in higher order ([12, 9]). Li and Wang developed its non-uniform generalization ([8]). Trigonometric curves can produce several kinds of classical important curves explicitly due to their trigonometric basis functions, including circle and circular cylinder [14], ellipse [16], surfaces of revolution [11], cycloid [10], helix [12], hyperbola and catenary [9]. Two recently defined trigonometric curves are the quadratic [2] and cubic [3] trigonometric curves of Xuli Han. The aim of this paper is to discuss the geometric properties of these curves, including the effect of their shape parameters and the possibility of constrained modification of the curves by these parameters. The method of our study is similar to the papers discussing geometric properties and modification of other types of spline curves. Such research has been done e.g. for C-Bézier curves [6], FB-spline curves [4], GB-spline curves [7] and another quartic curve of Han [5].

In this paper the definition of quadratic and cubic trigonometric polynomial curves are presented in Section 2. The geometric effects of the shape parameter as well as its application for constrained modification are discussed for quadratic curve in Section 3. These results are extended for the cubic case in Section 4, and for quadratic trigonometric surfaces in Section 5.

2. Definition of the basis functions

The construction of the basis functions of the quadratic trigonometric curve is the following [2].

Definition 2.1. Given knots $u_0 < u_1 < \cdots < u_{n+3}$, let

$$\begin{aligned} \Delta u_i &:= u_{i+1} - u_i, & t_i(u) &:= \frac{\pi}{2}\left(\frac{u-u_i}{\Delta u_i}\right), \\ \alpha_i &:= \frac{\Delta u_i}{\Delta u_{i-1} + \Delta u_i}, & c(t) &:= \big(1-\sin(t)\big)\big(1-\lambda\sin(t)\big), \\ \beta_i &:= \frac{\Delta u_i}{\Delta u_i - \Delta u_{i+1}}, & d(t) &:= \big(1-\cos(t)\big)\big(1-\lambda\cos(t)\big), \end{aligned}$$

where $-1 < \lambda < 1$. Then the associated trigonometric polynomial basis functions are defined to be the following functions:

$$b_i(u) = \begin{cases} \beta_i d(t_i), & u \in [u_i, u_{i+1}), \\ 1 - \alpha_{i+1} c(t_{i+1}) + \beta_{i+1} d(t_{i+1}), & u \in [u_{i+1}, u_{i+2}), \\ \alpha_{i+2} c(t_{i+2}), & u \in [u_{i+2}, u_{i+3}), \\ 0, & u \notin [u_i, u_{i+3}), \end{cases}$$

for $i = 0, 1, \ldots, n$.

In [2] the author proves several theorems in terms of this new curve, but the most important ones are the following: if $\lambda = 0$, then the trigonometric polynomial curve is an arc of an ellipse and the basis function $b_i(u)$ has C^1 continuity at each of the knots.

The definition of the cubic trigonometric polynomial curve in [3] is as follows.

Definition 2.2. Given knots $u_0 < u_1 < \cdots < u_{n+4}$ and refer to $U = (u_0, u_1, \ldots, u_{n+4})$ as a knot vector. For $\lambda \in \mathbb{R}$ and all possible $i \in \mathbb{Z}^+$, let $\Delta_i = u_{i+1} - u_i$,

$$\alpha_i = \tfrac{\Delta_i}{\Delta_{i-1}+\Delta_i}, \qquad \beta_i = \tfrac{\Delta_i}{\Delta_i+\Delta_{i+1}}, \qquad \gamma_i = \tfrac{1}{\Delta_{i-1}+(2\lambda+1)\Delta_i+\Delta_{i+1}},$$

$$\begin{array}{ll}
a_i = \Delta_i \alpha_i \gamma_{i-1}, & d_i = \Delta_i \beta_i \gamma_{i+1}, \\
b_{i2} = \tfrac{1}{4\lambda+6}(\Delta_{i+1} - \Delta_i)\gamma_i, & c_{i1} = \tfrac{1}{4\lambda+6}(\Delta_{i-1} - \Delta_i)\gamma_i, \\
b_{i0} = \Delta_{i-1}\alpha_i\gamma_i - a_i + b_{i2}, & c_{i0} = [1 - (2\lambda+3)\alpha_i]\, c_{i1}, \\
b_{i1} = \tfrac{1}{2}\Delta_i\gamma_i + b_{i2}, & c_{i2} = \tfrac{1}{2}\Delta_i\gamma_i + c_{i1}, \\
b_{i3} = [1 - (2\lambda+3)\beta_i]\, b_{i2}, & c_{i3} = \Delta_{i+1}\beta_i\gamma_i + c_{i1} - d_i, \\
f_0(t) = (1-\sin(t))^2(1-\lambda\sin(t)), & f_1(t) = (1+\cos(t))^2(1+\lambda\cos(t)), \\
f_2(t) = (1+\sin(t))^2(1+\lambda\sin(t)), & f_3(t) = (1-\cos(t))^2(1-\lambda\cos(t)).
\end{array}$$

Given a knot vector U, let $t_j(u) = \frac{\pi}{2}\frac{u-u_j}{\Delta_j}$ $(j = 0, 1, \ldots, n+3)$, the associated trigonometric polynomial basis functions are defined to be the following functions:

$$B_i(u) = \begin{cases} d_i f_3(t_i), & u \in [u_i, u_{i+1}), \\ \sum_{j=0}^{3} c_{i+1,j} f_j(t_{i+1}), & u \in [u_{i+1}, u_{i+2}), \\ \sum_{j=0}^{3} b_{i+2,j} f_j(t_{i+2}), & u \in [u_{i+2}, u_{i+3}), \\ a_{i+3} f_0(t_{i+3}), & u \in [u_{i+3}, u_{i+4}), \\ 0, & u \notin [u_i, u_{i+4}), \end{cases}$$

for $i = 0, 1, \ldots, n$.

3. The quadratic case

3.1. Geometric effect of the shape parameter

The basis functions of the quadratic trigonometric polynomial curve have a shape parameter $\lambda \in (-1, 1)$, (see Section 1). For our works we restrict the domain of definition of the basis functions to one span. Let $u \in [u_i, u_{i+1})$, then

$$b_i(u) = \left(\frac{u_{i+1} - u_i}{u_{i+2} + u_i}\right)\left(1 - \cos\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right)\left(1 - \lambda\cos\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right),$$

$$b_{i-1}(u) = 1 - \left(\frac{u_{i+1} - u_i}{u_{i-1} + u_{i+1}}\right)\left(1 - \sin\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right)\left(1 - \lambda\sin\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right)$$

$$- \left(\frac{u_{i+1} - u_i}{u_{i+2} + u_i}\right)\left(1 - \cos\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right)\left(1 - \lambda\cos\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1-u_i}}\right)\right),$$

$$b_{i-2}(u) = \left(\frac{u_{i+1} - u_i}{u_{i-1} + u_{i+1}}\right)\left(1 - \sin\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right)\left(1 - \lambda\sin\left(\frac{\pi}{2}\frac{u - u_i}{u_{i+1} - u_i}\right)\right),$$

hence one arc of the quadratic trigonometric polynomial curve is

$$\mathbf{T}_i(u, \lambda) = b_{i-2}(u)\mathbf{p}_{i-2} + b_{i-1}(u)\mathbf{p}_{i-1} + b_i(u)\mathbf{p}_i,$$

where $\mathbf{p}_i$ are the control points.

Theorem 3.1. *The geometric effect of the shape parameter is linear, that is if $u_0 \in [u_i, u_{i+1})$ is fixed, then the curve point $\boldsymbol{T}_i(u_0, \lambda)$ moves along a line segment.*

Proof. Parts of basis functions not containing the shape parameter λ can be considered as constants:

$$k_1 = \left(\tfrac{u_{i+1}-u_i}{u_{i+2}+u_i}\right)\left(1 - \cos\left(\tfrac{\pi}{2}\tfrac{u_0-u_i}{u_{i-1}-u_i}\right)\right), \qquad k_2 = \cos\left(\tfrac{\pi}{2}\tfrac{u_0-u_i}{u_{i-1}-u_i}\right),$$
$$k_3 = \left(\tfrac{u_{i+1}-u_i}{u_{i+1}+u_{i-1}}\right)\left(1 - \sin\left(\tfrac{\pi}{2}\tfrac{u_0-u_i}{u_{i-1}-u_i}\right)\right), \qquad k_4 = \sin\left(\tfrac{\pi}{2}\tfrac{u_0-u_i}{u_{i-1}-u_i}\right).$$

By these constants the basis functions of the quadratic trigonometric polynomial curve can be expressed as:

$$\begin{aligned} b_{i-2}(u_0, \lambda) &= k_3 - \lambda k_3 k_4, \\ b_{i-1}(u_0, \lambda) &= 1 - k_3 + \lambda k_3 k_4 - k_1 + \lambda k_1 k_2, \\ b_i(u_0, \lambda) &= k_1 - \lambda k_1 k_2. \end{aligned}$$

Thus the path of the curve point at parameter u_0 is

$$\mathbf{T}(u_0, \lambda) = (k_3 - \lambda k_3 k_4)\mathbf{p}_{i-2} + (1 - k_3 + \lambda k_3 k_4 - k_1 + \lambda k_1 k_2)\mathbf{p}_{i-1} + (k_1 - \lambda k_1 k_2)\mathbf{p}_i,$$

which yields an equation of a line segment

$$\begin{aligned} \mathbf{T}(u_0, \lambda) = &\big(k_3\mathbf{p}_{i-2} + (1 - k_1 - k_3)\mathbf{p}_{i-2} + k_1\mathbf{p}_i\big) \\ &+ \lambda\left(k_3 k_4(\mathbf{p}_{i-1} - \mathbf{p}_{i-2}) + k_1 k_2(\mathbf{p}_{i-1} - \mathbf{p}_i)\right). \end{aligned}$$

The theorem follows. □

3.2. Common interpolation

Interpolation of points in general is possible by any of the spline curves by a reverse algorithm: considering the points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ to be interpolated by the curve, the new control points can be computed. This algorithm, however, generally requires time-consuming computation solving a system of equations. Thus it can be useful to study if the curve (if it includes a shape parameter) can directly interpolate the given points at a proportional value of the shape parameter.

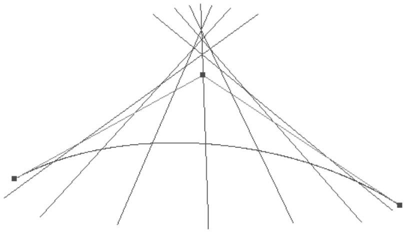

Figure 1: The geometric effect of the shape parameter with fixed parameter u_0. Straight lines show the λ-paths of the quadratic trigonometric curve.

Theorem 3.2. *The quadratic trigonometric polynomial curve interpolates the control points at $\lambda = \sqrt{2}$.*

Proof. Let the control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ so $u_{i-1} = 0$, $u_i = 0$, $u_{i+1} = 1$, $u_{i+2} = 1$ and the parameter where we search the interpolation is $u = \frac{u_i + u_{i+1}}{2}$ hence

$$\begin{aligned}
b_i(u) &= \left(1 - cos\left(\frac{\pi}{4}\right)\right)\left(1 - \sqrt{2}cos\left(\frac{\pi}{4}\right)\right) = 0,\\
b_{i-1}(u) &= 1 - \left(1 - cos\left(\frac{\pi}{4}\right)\right)\left(1 - \sqrt{2}cos\left(\frac{\pi}{4}\right)\right)\\
&\quad - \left(1 - sin\left(\frac{\pi}{4}\right)\right)\left(1 - \sqrt{2}sin\left(\frac{\pi}{4}\right)\right) = 1,\\
b_{i-2}(u) &= \left(1 - sin\left(\frac{\pi}{4}\right)\right)\left(1 - \sqrt{2}sin\left(\frac{\pi}{4}\right)\right) = 0.
\end{aligned}$$

The theorem follows. □

Figure 2: The quadratic trigonometric polynomial curve with $\lambda = \sqrt{2}$ interpolates the control points, having end tangents parallel to the sides of the control polygon.

3.3. Constrained modification

In Computer Aided Geometric Design a frequent problem is the constrained modification, when control points are given and we have another point $\mathbf{p}$ what we need to interpolate by the curve, possibly by altering the parameters of the curve. Our aim is to modify the given curve $\mathbf{T}(u, \lambda)$ by altering exclusively the shape parameter in a way, that the modified curve will pass through the given point, that is, for some parameters $\mathbf{T}(\bar{u}, \bar{\lambda}) = \mathbf{p}$. The first observation to create this interpolation with the quadratic trigonometric polynomial curve is to show how we could produce a segment with this curve.

Lemma 3.3. *Let the control points be $\mathbf{p}_{i-2}, \mathbf{p}_{i-1}, \mathbf{p}_i$. If $\lambda = -1$, then the quadratic trigonometric curve is a line segment between $\mathbf{p}_{i-2}$ and $\mathbf{p}_i$.*

Proof. Let the curve segment be

$$\mathbf{T}_i(u_0, -1) = \mathbf{p}_{i-1} + A(\mathbf{p}_{i-2} - \mathbf{p}_{i-1}) + B(\mathbf{p}_i - \mathbf{p}_{i-1}),$$

where $A = k_3 - \lambda k_3 k_4$, $B = k_1 - \lambda k_1 k_2$, and $A + B = 1$, therefore

$$\mathbf{T}_i(u_0, -1) = A\mathbf{p}_{i-2} + (1 - A)\mathbf{p}_i.$$

The theorem follows. $\square$

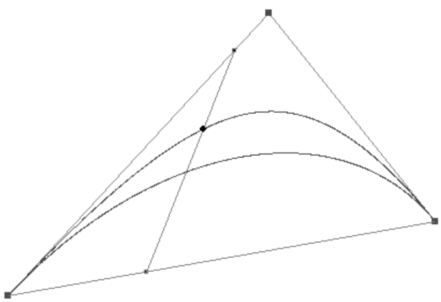

Figure 3: Constrained modification on the quadtaric trigonometric polynomial curve.

For a fixed parameter u_0 one can also find the intersection of the λ-path associated to u_0 and the control polygon. In the first case, when $0 \leqslant u_0 \leqslant 0,5$ we can find the parameter value λ_0 at which $B = 0$. Since $B = k_1 - \lambda k_1 k_2 = 0$, therefore $\lambda_0 = \frac{1}{k_2}$. So the intersection point is $\mathbf{T}(u_0, \frac{1}{k_2})$. In the second case, when $0,5 < u_0 \leqslant 1$ then $A = 0$ should hold. $A = k_3 - \lambda k_3 k_4$ yields $\lambda_0 = \frac{1}{k_4}$. Thus the point what we are looking for is $\mathbf{T}(u_0, \frac{1}{k_4})$. Since lambda paths are line segments, they can be described by the two points $\mathbf{T}_i(u_0, -1)$ and $\mathbf{T}_i(u_0, \lambda_0)$. Our next task is to find the value $\bar{u}$ for which the path $\mathbf{T}(\bar{u}, \lambda)$ passes trough $\mathbf{p}$, by elementary incidence computation. Finally the value of $\bar{\lambda}$ can be found by a numerical algorithm for which

$$\mathbf{T}(\bar{u}, \bar{\lambda}) = \mathbf{p}.$$

holds. From the algorithm it is clear that the admissible positions of $\mathbf{p}$ can be inside the convex hull of the control points.

4. Extension to the cubic case

The cubic trigonometric polynomial curve also has a shape parameter $\lambda \in \mathbb{R}$ (see Section 1), and we need the following remark from [3].

Remark 4.1. If $u_i \neq u_{i+1}$ $(3 \leqslant i \leqslant n)$, then for $u \in [u_i, u_{i+1}]$, the curve $T(u)$ can be represented by curve segment

$$\mathbf{T}(u) = B_{i-3}\mathbf{p}_{i-3} + B_{i-2}\mathbf{p}_{i-2} + B_{i-1}\mathbf{p}_{i-1} + B_i\mathbf{p}_i.$$

With a uniform knot vector, we have

$$B_{i-3}(u) = \tfrac{f_0(t)}{4\lambda+6}, \qquad B_{i-2}(u) = \tfrac{f_1(t)}{4\lambda+6}, \qquad B_{i-1}(u) = \tfrac{f_2(t)}{4\lambda+6}, \qquad B_i(u) = \tfrac{f_3(t)}{4\lambda+6},$$

where $t = \pi(u - u_i)/(2\Delta_i)$.

Theorem 4.2. *With a uniform knot vector if $u_0 \in [u_i, u_{i+1})$ is fixed, then the geometric effect of the shape parameter is linear, i.e. the λ-path of the point $\boldsymbol{T}(u_0, \lambda)$ of the curve are straight line segments.*

Proof. The derivatives of the basis functions with respect to λ are

$$\begin{aligned}
\frac{\delta B_{i-3}}{\delta\lambda} &= \frac{\left[-(1-\sin(t))^2\sin(t)\right](4\lambda+6) - 4(1-\sin(t))^2(1-\lambda\sin(t))}{(4\lambda+6)^2} = \\
&= -\frac{(-1+\sin(t))^2(2+3\sin(t))}{2(3+2\lambda)^2}, \\
\frac{\delta B_{i-2}}{\delta\lambda} &= \frac{\left[(1+\cos(t))^2\cos(t)\right](4\lambda+6) - 4(1+\cos(t))^2(1+\lambda\cos(t))}{(4\lambda+6)^2} = \\
&= \frac{(1+\cos(t))^2(-2+3\cos(t))}{2(3+2\lambda)^2}, \\
\frac{\delta B_{i-1}}{\delta\lambda} &= \frac{\left[(1+\sin(t))^2\sin(t)\right](4\lambda+6) - 4(1+\sin(t))^2(1+\lambda\sin(t))}{(4\lambda+6)^2} = \\
&= \frac{(1+\sin(t))^2(-2+3\sin(t))}{2(3+2\lambda)^2}, \\
\frac{\delta B_i}{\delta\lambda} &= \frac{\left[-(1-\cos(t))^2\cos(t)\right](4\lambda+6) - 4(1-\cos(t))^2(1-\lambda\cos(t))}{(4\lambda+6)^2} =
\end{aligned}$$

$$= -\frac{(-1+\cos(t))^2(2+3\cos(t))}{2(3+2\lambda)^2},$$

hence the derivative of the cubic trigonometric polynomial curve with respect to λ is

$$\begin{aligned}\frac{\delta \mathbf{T}}{\delta \lambda} &= \frac{1}{2(3+2\lambda)^2}\big((-1+sin(t))^2(2+3sin(t))\mathbf{p}_0 \\ &+ (1+\cos(t))^2(-2+3\cos(t))\mathbf{p}_1 + (1+\sin(t))^2(-2+3\sin(t))\mathbf{p}_2 \\ &+ (-1+\cos(t))^2(2+3\cos(t))\mathbf{p}_3\big).\end{aligned}$$

In the domain of $\mathbf{T}(u_0, \lambda)$ all the derivative vectors of the λ-paths with respect to λ point to the same direction, λ alters only the derivatives lengths. This proves the statement. $\square$

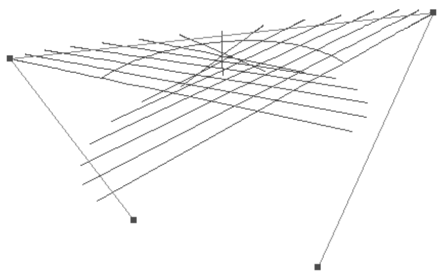

Figure 4: The geometric effect of the shape parameter is linear in the cubic case as well, λ-paths are line segments.

Since λ-paths belong to a fixed u_0 are line segments in the cubic case as well, constrained modification can be computed analogously to the quadratic case (cf. Fig. 5).

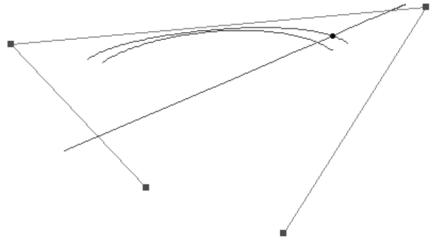

Figure 5: Constrained modification on the cubic trigonometric polynomial curve.

5. Extension to surfaces

In this section we present the quadratic trigonometric polynomial surface which is a tensor product surface obtained by the quadratic curve. The basis functions are as follows

$$b_i(u) = \left(\frac{u_{i+1}-u_i}{u_{i+2}+u_i}\right)\left(1-\cos\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right)\left(1-\lambda_u\cos\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right),$$

$$\begin{aligned} b_{i-1}(u) = 1 &- \left(\frac{u_{i+1}-u_i}{u_{i-1}+u_{i+1}}\right)\left(1-\sin\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right)\left(1-\lambda_u\sin\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right) \\ &- \left(\frac{u_{i+1}-u_i}{u_{i+2}+u_i}\right)\left(1-\cos\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right)\left(1-\lambda_u\cos\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right), \end{aligned}$$

$$b_{i-2}(u) = \left(\frac{u_{i+1}-u_i}{u_{i-1}+u_{i+1}}\right)\left(1-\sin\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right)\left(1-\lambda_u\sin\left(\frac{\pi}{2}\frac{u-u_i}{u_{i+1}-u_i}\right)\right),$$

$$b_i(v) = \left(\frac{v_{i+1}-v_i}{v_{i+2}+v_i}\right)\left(1-\cos\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right)\left(1-\lambda_v\cos\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right),$$

$$\begin{aligned} b_{i-1}(v) = 1 &- \left(\frac{v_{i+1}-v_i}{v_{i-1}+v_{i+1}}\right)\left(1-\sin\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right)\left(1-\lambda_v\sin\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right) \\ &- \left(\frac{v_{i+1}-v_i}{v_{i+2}+v_i}\right)\left(1-\cos\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right)\left(1-\lambda_v\cos\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right), \end{aligned}$$

$$b_{i-2}(v) = \left(\frac{v_{i+1}-v_i}{v_{i-1}+v_{i+1}}\right)\left(1-\sin\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right)\left(1-\lambda_v\sin\left(\frac{\pi}{2}\frac{v-v_i}{v_{i+1}-v_i}\right)\right),$$

where $-1 < \lambda_u, \lambda_v < 1$ are shape parameters, and the surface patch is defined as

$$\begin{aligned} \mathbf{T}(u,v) =& b_{i-2}(u)b_{i-2}(v)\mathbf{p}_j + b_{i-2}(u)b_{i-1}(v)\mathbf{p}_{j+1} + b_{i-2}(u)b_i(v)\mathbf{p}_{j+2} + \\ &+ b_{i-1}(u)b_{i-2}(v)\mathbf{p}_{j+3} + b_{i-1}(u)b_{i-1}(v)\mathbf{p}_{j+4} + b_{i-1}(u)b_i(v)\mathbf{p}_{j+5} + \\ &+ b_i(u)b_{i-2}(v)\mathbf{p}_{j+6} + b_i(u)b_{i-1}(v)\mathbf{p}_{j+7} + b_i(u)b_i(v)\mathbf{p}_{j+8}. \end{aligned}$$

The shape parameters λ_u, λ_v are independent so they modify the surface in separated ways. As we have seen for $\lambda = -1$ the quadratic curve is a line segment passing through $\mathbf{p}_{i-2}$ and $\mathbf{p}_i$. Consequently the quadratic trigonometric polynomial surface at $\lambda_u = -1$ and $\lambda_v = -1$ is a plane interpolating control points $\mathbf{p}_j, \mathbf{p}_{j+2}, \mathbf{p}_{j+6}, \mathbf{p}_{j+8}$.

Theorem 5.1. *The quadratic trigonometric polynomial surface interpolates the control points when* $\lambda_u = \sqrt{2}$ *and* $\lambda_v = \sqrt{2}$.

Proof. Considering the control points $\mathbf{p}_k$ $(k = 1, 2, \ldots, 9)$, knots $u_{i-1} = 0$, $u_i = 0$, $u_{i+1} = 1$, $u_{i+2} = 1$, $v_{i-1} = 0$, $v_i = 0$, $v_{i+1} = 1$, $v_{i+2} = 1$, the parameters at which the interpolation holds are $u = \frac{u_i+u_{i+1}}{2}$ and $v = \frac{v_i+v_{i+1}}{2}$ hence the statement follows from Theorem 3.2. □

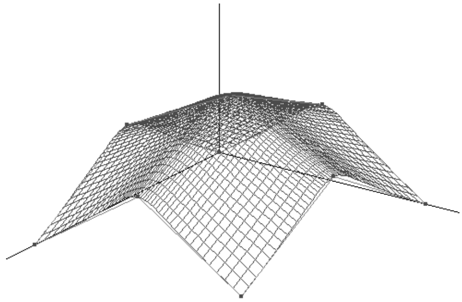

Figure 6: The quadratic trigonometric polynomial surface with $\lambda_u = \sqrt{2}$ and $\lambda_v = \sqrt{2}$.

5.1. Geometric effect of the shape parameters λ_u and λ_v on the quadratic trigonometric surface

Since shape parameters acts independently on the surface, if we fix either of them, the geometric effect of the other shape parameter is the same as in the curve case (see Section 3.1). Consequently in the case when both of the shape parameters are changing simultaneously, points of the surface can move on a doubly ruled surface, an example of which can be seen in Fig.7.

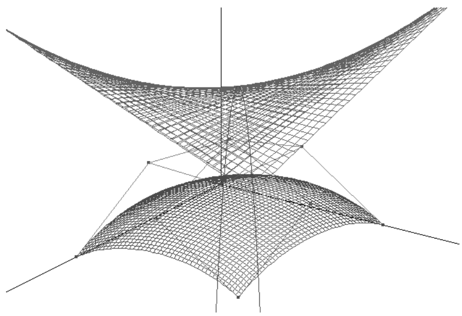

Figure 7: λ-paths of a surface point associated to $u_0 = 0,2$ and $v_0 = 0,2$ are on a hyperbolic paraboloid (one of its section curves is also shown).

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