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Conditions for groups whose group algebras have minimal Lie derived length*

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Dedicated to professor Béla Pelle on his 80^{th} birthday

Abstract

Two independent research yielded two different characterizations of groups whose group algebras have minimal Lie derived lengths. In this note we show that the two characterizations are equivalent and we propose a simplified description for these groups.

1. Introduction

Let FG be the group algebra of a group G over a field F. As every associative algebra, FG can be viewed as a Lie algebra with the Lie multiplication defined by $[x, y] = xy - yx$, for all $x, y \in FG$. Let $\delta^{[0]}(FG) = \delta^{(0)}(FG) = FG$, and for $n \geqslant 0$ denote by $\delta^{[n+1]}(FG)$ the F-subspace of FG spanned by all elements $[x, y]$ with $x, y \in \delta^{[n]}(FG)$, and by $\delta^{(n+1)}(FG)$ the associative ideal of FG generated by all elements $[x, y]$ with $x, y \in \delta^{(n)}(FG)$. We say that FG is Lie solvable (resp. strongly Lie solvable) if there exists n such that $\delta^{[n]}(FG) = 0$ (resp. $\delta^{(n)}(FG) = 0$), and the least such n is called the Lie derived length (resp. strong Lie derived length) of FG and denoted by $\mathrm{dl}_L(FG)$ (resp. $\mathrm{dl}^L(FG)$).

Sahai [6] proved that

$$\omega(FG')^{2^n-1}FG \subseteq \delta^{(n)}(FG) \subseteq \omega(FG')^{2^{n-1}}FG \quad \text{for all } n > 0, \tag{1.1}$$

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from which it follows that FG is strongly Lie solvable if and only if either G is abelian or the augmentation ideal $\omega(FG')$ of the subalgebra FG' is nilpotent, that is the derived subgroup G' of G is a finite p-group and $\mathrm{char}(F) = p$. Obviously, $\delta^{[n]}(FG) \subseteq \delta^{(n)}(FG)$ for all n, thus every strongly Lie solvable group algebra FG is Lie solvable too, and $\mathrm{dl}_L(FG) \leqslant \mathrm{dl}^L(FG)$. However, according to a result of Passi, Passman and Sehgal (see e.g. in [5]), there exists a Lie solvable group algebra which is not strongly Lie solvable. They proved that a group algebra FG is Lie solvable if and only if one of the following conditions holds: (*i*) G is abelian; (*ii*) G' is a finite p-group and $\mathrm{char}(F) = p$; (*iii*) G has a subgroup of index 2 whose derived subgroup is a finite 2-group and $\mathrm{char}(F) = 2$. Note that for $\mathrm{char}(F) = 2$ the values of $\mathrm{dl}_L(FG)$ and $\mathrm{dl}^L(FG)$ can be different (see e.g. Corollary 1 of [1]).

Evidently, if FG is commutative, then $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = 1$. Shalev [8] proved that if FG is a non-commutative Lie solvable group algebra of characteristic p, then $\mathrm{dl}_L(FG) \geqslant \lceil \log_2(p+1) \rceil$, where $\lceil \log_2(p+1) \rceil$ denotes the upper integer part of $\log_2(p+1)$. Shalev also showed that there is no better lower bound than $\lceil \log_2(p+1) \rceil$, emphasizing that the complete characterization of groups for which this lower bound is exact "*may be a delicate task*". Clearly, for a non-commutative strongly Lie solvable group algebra FG the value of $\mathrm{dl}^L(FG)$ can also be estimated from below by the same integer $\lceil \log_2(p+1) \rceil$, and the question of characterizing groups for which this bound is achieved can be posed. Since we conjecture there is no group algebra FG over a field F of characteristic $p > 2$ such that $\mathrm{dl}_L(FG) \neq \mathrm{dl}^L(FG)$, we may expect that the answer will solve Shalev's original problem.

Levin and Rosenberger (see e.g. in [5]) described the group algebras of Lie derived length two. Moreover, they also proved that $\mathrm{dl}_L(FG) = 2$ if and only if $\mathrm{dl}^L(FG) = 2$. This answers both questions for the special cases $p = 2$ and 3. Assume that $p \geqslant 5$ and G' has order p^n. As it is well-known $\omega(FG')^{n(p-1)} \neq 0$, furthermore there exists an integer i such that $p < 2^i \leqslant 2p - 1$. Hence, for $n \geqslant 2$ we have

$$0 \neq \omega(FG')^{n(p-1)} \subseteq \omega(FG')^{2p-2} \subseteq \omega(FG')^{2^i-1},$$

and by (1.1), $\mathrm{dl}^L(FG) \geqslant i + 1 > \lceil \log_2(p+1) \rceil$. Let now $n = 1$, that is G' is of order p, and denote by $C_G(G')$ the centralizer of G' in G. In view of Theorem 1 of [1] (in which the authors determined the Lie derived length and the strong Lie derived length of group algebras of groups whose derived subgroup is cyclic of odd order) the value of $\mathrm{dl}^L(FG)$ depends on the order of the factor group $G/C_G(G')$ as follows. For $m \geqslant 0$, let

$$s(l,m) = \begin{cases} 1 & \text{if} \quad l = 0; \\ 2s(l-1,m)+1 & \text{if} \quad s(l-1,m) \text{ is divisible by } 2^m; \\ 2s(l-1,m) & \text{otherwise.} \end{cases}$$

If $G/C_G(G')$ has order $2^m p^r$, then $\mathrm{dl}^L(FG) = d+1$, where d is the minimal integer for which $s(d,m) \geqslant p$; otherwise $\mathrm{dl}^L(FG) = \lceil \log_2(2p) \rceil > \lceil \log_2(p+1) \rceil$. Hence we have obtained the following criterion for groups whose group algebras have minimal strong Lie derived length.

Theorem 1.1 (Balogh, Juhász [1]). *Let FG be a strongly Lie solvable group algebra of positive characteristic p. Then $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$ if and only if one of the following conditions holds:*

(i) $p = 2$ *and* G' *is central elementary abelian subgroup of order* 4*;*

(ii) G' *has order* p*,* $G/C_G(G')$ *has order* $2^m p^r$*, and the minimal integer* d *such that* $s(d,m) \geqslant p$ *satisfies the inequality* $2^d - 1 < p$*.*

An alternative characterization of these groups is obtained independently in [9] by using a different method. For $m \geqslant 0$ let

$$g(0,m) = 1, \quad \text{and} \quad g(l,m) = g(l-1,m) \cdot 2^{m+1} + 1 \quad \text{for all} \quad l \in \mathbb{N};$$

further, denote by $q_{n-m,m}$ and $\epsilon_{n-m,m}$ the quotient and the remainder of the Euclidean division of $n - m - 1$ by $m + 1$, respectively.

Theorem 1.2 (Spinelli [9]). *Let FG be a non-commutative strongly Lie solvable group algebra over a field F of positive characteristic p. Let n be the positive integer such that $2^n \leqslant p < 2^{n+1}$ and s, q (q odd) the non-negative integers such that $p - 1 = 2^s q$. The following statements are equivalent:*

(i) $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$*;*

(ii) p *and* G *satisfy one of the following conditions:*

(a) $p = 2$*,* G' *has exponent* 2 *and an order dividing* 4 *and* G' *is central;*

(b) $p \geqslant 3$ *and* G' *is central of order* p*;*

(c) $5 \leqslant p < 2^{n+2}/3$*,* G' *is not central of order* p *and* $|G/C_G(G')| = 2^m$ *with* $m \leqslant s$ *a positive integer such that* $p \leqslant 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1, m)$*.*

In the present paper the authors are going to dispel doubts about that the different conditions of the two above theorems could describe different classes of groups. We give a direct proof of the equivalence between them. According to [10], these same conditions describe completely the groups whose group algebras have minimal Lie derived length. Combining our results with the main theorem of [10], we propose the following simplified answer to Shalev's question.

Theorem 1.3. *Let FG be a Lie solvable group algebra over a field F of positive characteristic p. Then the following statements are equivalent:*

(i) $\mathrm{dl}_L(FG) = \lceil \log_2(p+1) \rceil$*;*

(ii) $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$*;*

(iii) either $p = 2$ *and* G' *is central elementary abelian subgroup of order* 2 *or* 4*; or* G' *has order* $p > 2$*,* $|G/C_G(G')| = 2^m$ *and* $\lceil \log_2(p+1) \rceil = \lceil \log_2(\frac{2^{m+1}-1}{2^m} p) \rceil$*.*

2. Proof of the equivalence

In the next lemma we concentrate on the series $s(l,m)$ and $g(l,m)$, and on the connection between them.

Lemma 2.1. *For all $m,n,i \geqslant 0$,*

(i) $2^i \leqslant s(i,m) < 2^{i+1}$;

(ii) $s(i,m+1) \leqslant s(i,m)$;

(iii) $g(i,m) = s((m+1)i,m)$;

(iv) $s(n,m) = 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1,m)$;

(v) $s(i,m) = \frac{2^{m+i+1}-2^{(m+1)\{\frac{i}{m+1}\}}}{2^{m+1}-1}$, *where* $\{\frac{i}{m+1}\}$ *is the fractional part of* $\frac{i}{m+1}$.

Proof.

(i) This is obvious for $i=0$, and assume that $2^i \leqslant s(i,m) < 2^{i+1}$, or equivalently, $2^{i+1} \leqslant 2s(i,m) < 2^{i+2}$ for some $i \geqslant 0$. Moreover, $2s(i,m)$ is even, so $2^{i+1} \leqslant 2s(i,m) < 2s(i,m)+1 < 2^{i+2}$. By definition,

$$2s(i,m) \leqslant s(i+1,m) \leqslant 2s(i,m)+1$$

and the statement (i) is true.

(ii) For a fixed m assume that l is the minimal integer for which $s(l,m+1) > s(l,m)$. Then we get that $2s(l-1,m+1) \geqslant s(l,m) \geqslant 2s(l-1,m)$. Being l minimal $s(l-1,m) = s(l-1,m+1)$. Since $s(l-1,m)$ cannot be divisible by 2^{m+1} so

$$s(l,m) \geqslant 2s(l-1,m) = 2s(l-1,m+1) = s(l,m+1)$$

which is a contradiction.

(iii) For $i=0$ the definitions say that $g(0,m) = s(0,m) = 1$. Assume that $i \geqslant 0$ and $g(i,m) = s((m+1)i,m)$. Then

$$g(i+1,m) = g(i,m)\cdot 2^{m+1}+1 = s((m+1)i,m)\cdot 2^{m+1}+1.$$

Since $g(j,m)$ is odd for all j, we conclude that $s((m+1)j,m)$ is also odd. Using the definition we get that

$$s((m+1)i,m)\cdot 2^{m+1}+1 = s((m+1)(i+1),m)$$

and the proof is complete.

(iv) According to the definition, $s(i,m)$ is odd whenever i is divisible by $m+1$, and

$$\begin{aligned} s(n,m) &= s((m+1)(q_{n-m,m}+1)+\epsilon_{n-m,m},m) \\ &= s((m+1)(q_{n-m,m}+1),m)\cdot 2^{\epsilon_{n-m,m}}, \end{aligned}$$

and by (iii),

$$2^{\epsilon_{n-m,m}} \cdot s((m+1)(q_{n-m,m}+1), m) = 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1, m).$$

(v) Denote by q and r the quotient and the remainder of the Euclidean division of i by $m+1$, respectively. It is easy to check that

$$\begin{aligned} s(i,m) &= 2^{q(m+1)+r} + 2^{(q-1)(m+1)+r} + \cdots + 2^r \\ &= 2^r \sum_{j=0}^{q} (2^{m+1})^j = \frac{2^{(m+1)(q+1)+r} - 2^r}{2^{m+1}-1}. \end{aligned}$$

Using $i = q(m+1)+r$ and $r = (m+1)\{\frac{i}{m+1}\}$ we have the desired formula. □

Let G be a group with derived subgroup of order p. As it is well-known, the automorphism group of G' is isomorphic to the unit group of the field of p elements. But this unit group is cyclic of order $p-1$, so the factor group $G/C_G(G')$, which is isomorphic to a subgroup of it, is cyclic and its order divides $p-1$.

Proof of the equivalence. Denote by $\mathfrak{A}$ the set of all groups G which satisfy the conditions (ii) of Theorem 1.2; by $\mathfrak{B}$ those for which (i) or (ii) of Theorem 1.1 hold. Assume first that $G \in \mathfrak{A}$. We distinguish the following cases.

1. G' is central elementary abelian subgroup of order 4. Then by Theorem 1.1(i), $G \in \mathfrak{B}$.
2. G' is central of order p. Then the factor group $G/C_G(G')$ is trivial, and $s(i,0) = 2^{i+1}-1$. It is clear that the minimal integer d such that $2^{d+1}-1 \geqslant p$ satisfies the inequality $2^d - 1 < p$, therefore $G \in \mathfrak{B}$ in this case.
3. G' is not central of order p. Suppose that $2^n \leqslant p < 2^{n+1}$. Then, by Theorem 1.2$(ii/c)$, $|G/C_G(G')| = 2^m$ with a positive integer m such that

   $$p \leqslant 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1, m).$$

   According to Lemma 2.1(iv), $s(n,m) = 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1, m)$, hence $p \leqslant s(n,m)$. At the same time, $2^n \leqslant p < 2^{n+1}$, so by Lemma 2.1$(i)$ we have that n is the minimal integer such that $p \leqslant s(n,m)$, and since $2^n - 1 < p$, it follows that $G \in \mathfrak{B}$.

We have just shown that $\mathfrak{A} \subseteq \mathfrak{B}$. To prove the converse inclusion we consider the following cases.

1. G' is central elementary abelian subgroup of order 4. Then by part (a) of Theorem 1.2(ii), G also belongs to $\mathfrak{A}$.
2. G' is cyclic of order p. Then by the assumption $|G/C_G(G')| = 2^m p^r$, but $|G/C_G(G')|$ must divide $p-1$, actually r is always zero, and if s, q (q is odd) are the non-negative integers such that $p - 1 = 2^s q$, then $m \leqslant s$.

(a) $m = 0$. Then G' is central, and by parts (a) and (b) of Theorem 1.2, we have $G \in \mathfrak{A}$.

(b) $m > 0$. Then G' is not central and p is odd. Assume that the minimal integer d such that $s(d,m) \geqslant p$ satisfies the inequality $2^d - 1 < p$. It follows that $2^d \leqslant p < 2^{d+1}$, so $n = d$. By Lemma 2.1(iv),

$$p \leqslant s(n,m) = 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m} + 1, m).$$

Furthermore, Lemma 2.1(ii) yields $p \leqslant s(n,m) \leqslant s(n,1) < 2^{n+2}/3$. Finally, we show that $p \geqslant 5$. Indeed, if p was equal to 3, then m should be equal to 1, and from $s(d,1) \geqslant 3$ it would follow that $d = 2$. But $2^2 - 1 \not< 3$, so this is an impossible case.

The proof is done. □

3. Remarks

First we mention that we can get rid of the recursive sequence $s(l,m)$ in Theorem 1 of [1]. Indeed, assume that $|G'| = p^n$, where p is an odd prime, $|G/C_G(G')| = 2^m p^r$ and d is the minimal integer for which $s(d,m) \geqslant p^n$. By Lemma 2.1(v), we have

$$\frac{2^{m+d} - 2^{(m+1)\{\frac{d-1}{m+1}\}}}{2^{m+1}-1} < p^n \leqslant \frac{2^{m+d+1} - 2^{(m+1)\{\frac{d}{m+1}\}}}{2^{m+1}-1}.$$

Since $(m+1)\{\frac{d-1}{m+1}\}, (m+1)\{\frac{d}{m+1}\} \in \{0,1,\dots,m\}$, so

$$\frac{2^{m+d}-1}{2^{m+1}-1} < p^n \leqslant \frac{2^{m+d+1}-1}{2^{m+1}-1},$$

and

$$d < \log_2\left(\frac{2^{m+1}-1}{2^m}p^n + \frac{1}{2^m}\right) \leqslant d+1.$$

Keeping in mind that d is an integer, we conclude that

$$d+1 = \lceil \log_2\left(\frac{2^{m+1}-1}{2^m}p^n + \frac{1}{2^m}\right)\rceil = \lceil \log_2\left(\frac{2^{m+1}-1}{2^m}p^n\right)\rceil.$$

Now, we can restate our Theorem 1 of [1] as follows.

Theorem 3.1. *Let G be a group with cyclic derived subgroup of order p^n, where p is an odd prime, and let F be a field of characteristic p. If $G/C_G(G')$ has order $2^m p^r s$, where $(2p,s) = 1$, then*

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2 2p^n \nu_m \rceil,$$

where $\nu_m = 1$ if $s > 1$, otherwise $\nu_m = 1 - \frac{1}{2^{m+1}}$.

This implies Theorem 1.3.

Let now FG be a group algebra over a field F of positive characteristic p with Lie (or strong Lie) derived length n. Then $p < 2^n$, furthermore, $p \geqslant 2^{n-1}$ if and only if (iii) of Theorem 1.3 holds. Using this fact, we make an attempt to give a characterization of group algebras of Lie derived length 3 over a field of characteristic $p > 3$. As we told above, p must be smaller than 8, and for the cases $p = 5$ and 7 (iii) of Theorem 1.3 must hold. It is easy to check that only the following (p, m) pairs are possible: $(7, 0), (5, 0), (5, 1)$. This proves the following statement.

Corollary 3.2. *Let FG be the group algebra of a group G over a field F of characteristic $p > 3$. Then $\mathrm{dl}_L(FG) = 3$ if and only if one of the following conditions holds: (i) $p = 7$ and G' is central of order 7; (ii) $p = 5$, G' has order 5, and either G' is central or $x^g = x^{-1}$ for every $x \in G'$ and $g \notin C_G(G')$.*

For an alternative proof and for the case $p = 3$ we refer the reader to [6, 7].

Finally, we would like to draw reader's attention to recent articles [2, 3, 4] about Lie derived lengths of group algebras.

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