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Polynomials with special coefficients*

Ferenc Mátyás[a], Kálmán Liptai[a]
János T. Tóth[b], Ferdinánd Filip[b]

[a]Institute of Mathematics and Informatics
Eszterházy Károly College, Eger, Hungary

[b]Department of Mathematics
Selye János University, Komarno, Slovakia

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Dedicated to professor Béla Pelle on his 80^th^ birthday

Abstract

The aim of this paper is to investigate the zeros of polynomials

$$P_{n,k}(x) = K_{k-1}x^n + K_k x^{n-1} + \cdots + K_{n+k-2}x + K_{n+k-1},$$

where the coefficients K_i's are terms of a linear recursive sequence of k-order ($k \geqslant 2$).

Keywords: linear recurrences, zeros of polynomials with special coefficients

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1. Introduction

Let the linear recursive sequence $K = \{K_n\}_{n=0}^{\infty}$ of order k ($k \geqslant 2$) be defined by the initial values $K_0 = K_1 = \cdots = K_{k-2} = 0$ and $K_{k-1} = 1$, the nonnegative integral weights $A_1, A_2, \cdots, A_k \neq 0$ and the linear recursion

$$K_n = A_1K_{n-1} + A_2K_{n-2} + A_3K_{n-3} + \cdots + A_kK_{n-k} \quad (n \geqslant k). \tag{1.1}$$

According to the explicit form for K_n we can write that

$$K_n = p_1(n)\alpha_{1,k}^n + p_2(n)\alpha_{2,k}^n + \cdots + p_t(n)\alpha_{t,k}^n, \tag{1.2}$$

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where $\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{t,k}$ are the distinct zeros of the characteristic polynomial

$$f_k(x) = x^k - A_1x^{k-1} - A_2x^{k-2} - \dots - A_{k-1}x - A_k \tag{1.3}$$

of the sequence K, while $p_i(n)$'s ($1 \leqslant i \leqslant t \leqslant k$) are polynomials of n with at most degree $m_i - 1$, where m_i is the multiplicity of $\alpha_{i,k}$ ($\sum_{i=1}^{t} m_i = k$).

In the particular case $k = 2, K_0 = 0, K_1 = 1, A_1 = A_2 = 1$ we can get the Fibonacci-sequence $F = \{F_n\}_{n=0}^{\infty}$, while if $k = 3, A_1 = A_2 = A_3 = 1$ the sequence K is known as the Tribonacci-sequence $T = \{T_n\}_{n=0}^{\infty}$.

D. Garth, D. Mills and P. Mitchell [1] introduced the definition of the Fibonacci-coefficient polynomials $p_n(x) = F_1x^n + F_2x^{n-1} + \dots + F_nx + F_{n+1}$ and – among others – determined the number of the real zeros of $p_n(x)$. In [2] we investigated the zeros of the much more general polynomials

$$q_{n,i}(x) = R_ix^n + R_{i+t}x^{n-1} + R_{i+2t}x^{n-2} \dots + R_{i+(n-1)t}x + R_{i+nt},$$

where the sequence $R = \{R_n\}_{n=0}^{\infty}$ can be obtained from (1.1) if $k = 2$ and $i \geqslant 1, t \geqslant 1$ are fixed integers.

The aim of this paper is to investigate the number of the real zeros of the polynomials

$$P_{n,k}(x) = K_{k-1}x^n + K_kx^{n-1} + \dots + K_{n+k-2}x + K_{n+k-1}. \tag{1.4}$$

It is worth mentioning that the problem investigated in this paper can be extended for much more general sequences than K, which can be the topic of a further paper, as it was suggested by the anonymous referee. The authors would like to express their gratitude to the referee for his/her valuable comments.

2. Preliminary and known results

At first we are going to introduce the following notation. Using (1.3) and (1.4) put

$$Q_{n,k}(x) := f_k(x) \cdot P_{n,k}(x). \tag{2.1}$$

Lemma 2.1. *The polynomial $Q_{n,k}(x)$ has the following much more suitable form:*

$$\begin{aligned} Q_{n,k}(x) = {} & K_{k-1}x^{n+k} - K_{n+k}x^{k-1} - \\ & - (A_kK_{n+1} + A_{k-1}K_{n+2} + \dots + A_2K_{n+k-1})x^{k-2} - \\ & - \dots - (A_kK_{n+k-2} + A_{k-1}K_{n+k-1})x - A_kK_{n+k-1}. \end{aligned}$$

Proof. After the multiplication in (2.1) $Q_{n,k}(x)$ can be written as

$$\begin{aligned} Q_{n,k}(x) = {} & K_{k-1}x^{n+k} + (K_k - A_1K_{k-1})x^{n+k-1} \\ & + (K_{k+1} - A_1K_k - A_2K_{k-1})x^{n+k-2} + \\ & \vdots \end{aligned}$$

$$
\begin{aligned}
&+ (K_{2k-2} - A_1 K_{2k-3} - A_2 K_{2k-4} - \cdots - A_{k-1} K_{k-1}) x^{n+1} \\
&+ (K_{2k-1} - A_1 K_{2k-2} - A_2 K_{2k-3} - \cdots - A_{k-1} K_k - A_k K_{k-1}) x^n + \\
&\vdots \\
&+ (K_{n+k-1} - A_1 K_{n+k-2} - A_2 K_{n+k-3} - \cdots - A_{k-1} K_n - A_k K_{n-1}) x^k \\
&- (A_1 K_{n+k-1} + A_2 K_{n+k-2} + \cdots + A_{k-1} K_{n+1} + A_k K_n) x^{k-1} \\
&- (A_2 K_{n+k-1} + A_3 K_{n+k-2} + \cdots + A_{k-1} K_{n+2} + A_k K_{n+1}) x^{k-2} - \\
&\vdots \\
&- (A_{k-1} K_{n+k-1} + A_k K_{n+k-2}) x - A_k K_{n+k-1}.
\end{aligned}
$$

But, due to the definition (1.1) the coefficients of the terms x^j are 0 if $n+k-1 \geqslant j \geqslant k$, thus we get that

$$
\begin{aligned}
Q_{n,k}(x) = {} & K_{k-1} x^{n+k} - K_{n+k} x^{k-1} \\
& - (A_k K_{n+1} + A_{k-1} K_{n+2} + \cdots + A_2 K_{n+k-1}) x^{k-2} \\
& - \cdots - (A_k K_{n+k-2} + A_{k-1} K_{n+k-1}) x - A_k K_{n+k-1},
\end{aligned}
$$

which matches the statement of Lemma 2.1. □

Let us consider the distinct zeros $\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{t,k}$ of the characteristic polynomial $f_k(x)$ from (1.3). The root $\alpha_{1,k}$ is said to be the dominant root of $f_k(x)$ if $\alpha_{1,k} > \mid \alpha_{j,k} \mid$ for every $2 \leqslant j \leqslant t$ and the multiplicity of $\alpha_{1,k}$ is equal to 1, that is $m_1 = 1, \alpha_{1,k} \in \mathbf{R}$ and since $A_k \geqslant 1$ therefore $\alpha_{1,k} > 1$.

Lemma 2.2. *Let $\alpha_{1,k}$ be the dominant root of $f_k(x)$. Then*

$$\lim_{n \to \infty} \frac{K_n}{K_{n-1}} = \alpha_{1,k}.$$

Proof. This is a known result, or it can easily be proven if one uses (1.2), where now $p_1(n)$ is a nonzero real number. □

When the weights $A_1 = A_2 = \cdots = A_k = 1$ in (1.1), that is, when

$$f_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1, \tag{2.2}$$

then we prove the following result about the real zeros of this $f_k(x)$.

Lemma 2.3. *If $f_k(x)$ is of form* (2.2), *then*
(i) *the polynomial $f_k(x)$ has only one positive zero, e.g. $\alpha_{1,k}$,*
(ii) *$\alpha_{1,k}$ strictly increasingly tends to 2, if k tends to infinity,*
(iii) *if k is even, then the polynomial $f_k(x)$ has exactly one negative zero, e.g. $\alpha_{2,k}$,*
(iv) *if k is even, then $\alpha_{2,k}$ strictly decreasingly tends to -1, if k tends to infinity,*
(v) *if k is odd, then the polynomial $f_k(x)$ has no negative zero.*

Proof. Since $x = 1$ and $x = 0$ are not roots of the equation $x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$, therefore it can be rewritten into the following equivalent forms:

$$\begin{aligned} x^k &= x^{k-1} + x^{k-2} + \cdots + x + 1, \\ x^k &= \frac{x^k - 1}{x - 1}, \\ x^{k+1} &= 2x^k - 1, \\ 2 - x &= x^{-k}. \end{aligned} \tag{2.3}$$

Drawing the graphs of both sides of (2.3) in the same Descartes' coordinate system, one can obtained the desired statements (i)–(v). □

Remark 2.4. In the case of Tribonacci sequence the polynomial $f_3(x) = x^3 - x^2 - x - 1$ has dominant root, namely $\alpha_{1,3} = 1,839286755\ldots$, the two other zeros of $f_3(x)$ are non-real conjugate complex numbers of absolute value $0.737353\ldots$. While the characteristic polynomial of the Fibonacci sequence is $f_2(x) = x^2 - x - 1$, its positive and negative zeros are $\alpha_{1,2} = \frac{1+\sqrt{5}}{2}$ and $\alpha_{2,2} = \frac{1-\sqrt{5}}{2}$, respectively.

It will be suitable to apply the following lemma if we want to give bounds for the absolute value of (real and complex) zeros of the polynomial

$$P_{n,k}(x) = K_{k-1}x^n + K_k x^{n-1} + \cdots + K_{n+k-2}x + K_{n+k-1}.$$

Lemma 2.5. *If every coefficients of the polynomial $g(x) = a_0 + a_1x + \cdots + a_nx^n$ are positive numbers and the roots of equation $g(x) = 0$ are denoted by $z_1, z_2, \ldots, z_n$, then*

$$\gamma \leqslant |z_i| \leqslant \delta$$

hold for every $1 \leqslant i \leqslant n$, where γ is the minimal, while δ is the maximal value in the sequence

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}.$$

Proof. This lemma is known as Theorem of S. Kakeya [3]. □

3. Results and proofs

At first we deal with the number of the real zeros of the polynomial defined in (1.4), that is

$$P_{n,k}(x) = K_{k-1}x^n + K_k x^{n-1} + \cdots + K_{n+k-2}x + K_{n+k-1}.$$

Clearly, positive real zeros of $P_{n,k}(x)$ do not exist, since – under our conditions – all of the coefficients are positive. Thus we can restrict our investigation on the existence of negative real zeros.

Theorem 3.1. *Let d and h denote the number of the negative real zeros of the characteristic polynomial $f_k(x)$ defined in* (1.3)*, and the polynomial $P_{n,k}(x)$ defined in* (1.4)*, respectively. Then*
(i) $k-1-2j = h+d$ *for some* $j = 0, 1, 2, \dots, (k-2)/2$*, if k and n are even,*
(ii) $k-2j = h+d$ *for some* $j = 0, 1, 2, \dots, (k-2)/2$*, if k is even and n is odd,*
(iii) $k-1-2j = h+d$ *for some* $j = 0, 1, 2, \dots, (k-1)/2$*, if k is odd and n is even,*
(iv) $k-2j = h+d$ *for some* $j = 0, 1, 2, \dots, (k-1)/2$*, if k and n are odd.*

Proof. We will prove only the case (i), since the other three cases can similarly be proven. Let us consider the polynomial $Q_{n,k}(x)$ from (2.1). According to Lemma 2.1

$$\begin{aligned}Q_{n,k}(x) &= f_k(x)\dot{P}_{n,k}(x)\\ &= K_{k-1}x^{n+k} - K_{n+k}x^{k-1}\\ &\quad - (A_kK_{n+1} + A_{k-1}K_{n+2} + \cdots + A_2K_{n+k-1})x^{k-2} - \cdots\\ &\quad - (A_kK_{n+k-2} + A_{k-1}K_{n+k-1})x - A_kK_{n+k-1}.\end{aligned}$$

For using the Descartes' rule of signs we create the the polynomial $Q_{n,k}(-x)$, which – with the assumption k and n are even – is:

$$\begin{aligned}Q_{n,k}(-x) &= K_{k-1}x^{n+k} + K_{n+k}x^{k-1}\\ &\quad - (A_kK_{n+1} + A_{k-1}K_{n+2} + \cdots + A_2K_{n+k-1})x^{k-2} + \cdots\\ &\quad + (A_kK_{n+k-2} + A_{k-1}K_{n+k-1})x - A_kK_{n+k-1}.\end{aligned}$$

Since the number of changes of signs in the polynomial $Q_{n,k}(-x)$ is $k-1$ (which is odd), therefore the number of the negative real zeros of the polynomial $Q_{n,k}(x)$ may be $1, 3, 5, \dots, k-1$. From these negative real zeros d zeros belong to the polynomial $f_k(x)$, while the other h to the polynomial $P_{n,k}(x)$. This proves the statement of Theorem 3.1 (i). □

Corollary 3.2. *If the polynomial $f_k(x)$ is defined as in* (2.2)*, that is when $A_1 = A_2 = \cdots = A_k = 1$, then – according to Lemma 2.3 – $d = 1$, if k is even, while $d = 0$, if k is odd. This implies that in this case the number of the negative real zeros of the polynomial $P_{n,k}(x)$ is:*
(i) $h = k-2-2j$ *for some* $j = 0, 1, 2, \dots, (k-2)/2$*, if k and n are even,*
(ii) $h = k-1-2j$ *for some* $j = 0, 1, 2, \dots, (k-2)/2$*, if k is even and n is odd,*
(iii) $h = k-1-2j$ *for some* $j = 0, 1, 2, \dots, (k-1)/2$*, if k is odd and n is even,*
(iv) $h = k-2j$ *for some* $j = 0, 1, 2, \dots, (k-1)/2$*, if k and n are odd.*

Corollary 3.3. *In the case of Tribonacci sequence , for $f_k(x) = f_3(x) = x^3 - x^2 - x - 1$ we get the following result. The number of the negative real zeros of the polynomial $P_{n,3}(x)$ is*
(i) 0 *or* 2*, if n is even,*
(ii) 1 *or* 3*, if n is odd.*

For the absolute value of zeros of polynomial $P_{n,k}(x)$ defined in (1.4) we prove the next theorem:

Theorem 3.4. *Let z be any zero of polynomial $P_{n,k}(x)$ and let a and b denote the minimum and the maximum of the set*

$$\left\{\frac{K_{n+k-1}}{K_{n+k-2}}, \frac{K_{n+k-2}}{K_{n+k-3}}, \frac{K_{n+k-3}}{K_{n+k-4}}, \dots, \frac{K_{k+1}}{K_k}, \frac{K_k}{K_{k-1}}\right\},$$

respectively. Then

$$a \leqslant |z| \leqslant b.$$

Proof. Applying Lemma 2.5 one can obtain the statement. □

Remark 3.5. According to Lemma 2.2 if $\alpha_{1,k}$ denotes the dominant root of $f_k(x)$ then

$$\lim_{n\to\infty} \frac{K_n}{K_{n-1}} = \alpha_{1,k}.$$

E.g. for the Tribonacci sequence the above quotients of consecutive coefficients tend to 1,83928675 in an alternating way, where $a = 1$, and $b = 2$.

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Ferenc Mátyás, Kálmán Liptai
Institute of Mathematics and Informatics
Eszterházy Károly College
P.O. Box 43
H-3301 Eger
Hungary
e-mail: `matyas@ektf.hu`
`liptaik@ektf.hu`

János T. Tóth, Ferdinánd Filip
Department of Mathematics
Selye János University
P.O. Box 54
94501 Komarno
Slovakia
e-mail: `tothj@selyeuni.sk`
`filipf@selyeuni.sk`