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A geometric proof to Cantor's theorem and an irrationality measure for some Cantor's series*

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Abstract

Generalizing a geometric idea due to J. Sondow, we give a geometric proof for the Cantor's Theorem. Moreover, it is given an irrationality measure for some Cantor series.

Keywords: Irrationality, irrationality measure, Cantor, Smarandache function.

MSC: Primary 11J72, Secondary 11J82

1. Introduction

In 2006, Jonathan Sondow gave a nice geometric proof that e is irrational. Moreover, he said that a generalization of his construction may be used to prove the Cantor's theorem. But, he did not do that in his paper, see [2]. So we give a geometric proof to Cantor's theorem using a generalization to Sondow's construction. After, it is given an irrationality measure for some Cantor series, for that we generalize the Smarandache function. Also we give an irrationality measure for e that is a bit better than the given one in [2].

2. Cantor's Theorem

Definition 2.1. Let $a_0, a_1, \dots, b_1, b_2, \dots$ be sequences of integers that satisfy the inequalities $b_n \geqslant 2$, and $0 \leqslant a_n \leqslant b_n - 1$ if $n \geqslant 1$. Then the convergent series

$$\theta := a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \dots \tag{2.1}$$

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is called *Cantor series.*

Example 2.2. The number e is a Cantor series. For see that, take $a_0 = 2, a_n = 1, b_n = n+1$ for $n \geqslant 1$.

We recall the following theorem due to Cantor [1].

Theorem 2.3 (Cantor). *Let θ be a Cantor series. Suppose that each prime divides infinitely many of the b_n. Then θ is irrational if and only if both $a_n > 0$ and $a_n < b_n - 1$ hold infinitely often.*

Proof. For proving the necessary condition, observe that if $a_n = 0$ for $n \geqslant n_0$, then the series is a finite sum, hence θ is rational. If $a_n > 0$ infinitely often, let us to construct a nested sequence of closed intervals I_n with intersection θ. Let $I_1 = [a_0 + \frac{a_1}{b_1}, a_0 + \frac{a_1+1}{b_1}]$. Proceeding inductively, we have two possibilities, the first one, if $a_n = 0$, so define $I_n = I_{n-1}$. When $a_n \neq 0$, divide the interval I_{n-1} into $b_n - a_n + 1$ ($\geqslant 2$) subintervals, the first one with length $\frac{a_n}{b_1 \cdots b_n}$ and the other ones with equal length, namely, $\frac{1}{b_1 \cdots b_n}$, and let the first one be I_n. By construction, $|I_n| \geqslant \frac{1}{b_1 \cdots b_n}$, for all $n \in \mathbb{N}$ and when $a_n \neq 0$, the length of I_n is exactly $\frac{1}{b_1 \cdots b_n}$. By hypothesis on a_n, there exist infinitely many $n \in \mathbb{N}$, such that $|I_n| = \frac{1}{b_1 \cdots b_n}$. Thus, we have

$$I_n = \left[a_0 + \tfrac{a_1}{b_1} + \ldots + \tfrac{a_n}{b_1 \cdots b_n}, a_0 + \tfrac{a_1}{b_1} + \ldots + \tfrac{a_n+1}{b_1 \cdots b_n}\right] = \left[\tfrac{A_n}{b_1 \cdots b_n}, \tfrac{A_n+1}{b_1 \cdots b_n}\right]$$

where $A_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$. Also $\theta \in I_n$ for all $n \geqslant 1$. In fact, by hypothesis it is easy see that $\theta > \frac{A_n}{b_1 \cdots b_n}$, for all $n \geqslant 1$. For the other inequality, note that $\frac{a_m}{b_m} \leqslant 1 - \frac{1}{b_m}$, for all $m \in \mathbb{N}$, therefore

$$b_1 \cdots b_n(\theta - (a_0 + \frac{a_1}{b_1} + \ldots + \frac{a_n}{b_1 \cdots b_n})) \leqslant 1. \tag{2.2}$$

Now if $a_n = b_n - 1$ for $n \geqslant n_0$, then θ is the right-hand endpoint of I_{n_0-1}, because each I_n contains that endpoint and the lengths of the I_n tend to zero. Hence again θ is rational. For showing the sufficient condition, note that if $a_m < b_m - 1$, then holds the strict inequality in (2.2), for each $n < m$. Since $a_n > 0$ holds infinitely often,

$$\bigcap_{n=1}^{\infty} I_n = \theta.$$

Suppose that $\theta = \frac{p}{q} \in \mathbb{Q}$. Each prime number divides infinitely many b_n, so there exist n_0 sufficiently large such that $q | b_1 \cdots b_{n_0}$ and $a_{n_0} \neq 0$. Hence $b_1 \cdots b_{n_0} = kq$ for some $k \in \mathbb{N}$. Take $N \geqslant n_0$, such that, $a_{N+1} < b_{N+1} - 1$. Hence θ lies in interior of I_N. Also $I_N = I_{n_0+k}$ for some $k \geqslant 0$. Suppose $I_N = I_{n_0}$. We can write $\theta = \frac{kp}{b_1 \cdots b_{n_0}}$, thus $\frac{A_{n_0}}{b_1 \cdots b_{n_0}} < \frac{kp}{b_1 \cdots b_{n_0}} < \frac{A_{n_0}+1}{b_1 \cdots b_{n_0}}$. But that is a contradiction. If $I_N = I_{n_0+k}$, for $k \geqslant 1$, then we write $\theta = \frac{kp b_{n_0+1} \cdots b_{n_0+k}}{b_1 \cdots b_{n_0+k}}$. But that is again a contradiction. Therefore, it follows the irrationality of θ. $\square$

3. Irrationality measure

The next step is to give an irrationality measure for some Cantor series. Now, we construct an uncountable family of functions, where one of them is exactly a well-known function for us.

Definition 3.1. Given $\sigma = (b_1, b_2, \dots) \in \mathbb{N}^\infty$, satisfying

$(*)$ For all p prime number, the set $\{n \in \mathbb{N} \mid p|b_n\}$ is infinite.

We define the function $D(\cdot, \sigma) : \mathbb{Z}^* \to \mathbb{N}$, by

$$D(q, \sigma) := \min\{n \in \mathbb{N} \mid q|b_1 \cdots b_n\}$$

Note that $D(\cdot, \sigma)$ is well defined, by condition $(*)$ and the well-ordering theorem.

In [2], J. Sondow showed that for all integers p and q with $q > 1$,

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q)+1)!}, \tag{3.1}$$

where $S(q)$ is the smallest positive integer such that $S(q)!$ is a multiple of q (the so-called Smarandache function, see [3]). Note that if $\eta = (1, 2, 3, \dots)$, then $D(q, \eta) = S(q)$. Since e is a Cantor series and $D(\cdot, \sigma)$ is a generalization of Smarandache function, it is natural to think in a generalization or an improvement to the inequality in (3.1).

Lemma 3.2. *Given $n \in \mathbb{N}$, we have*

$$\left| \theta - \frac{m}{b_1 \cdots b_n} \right| \geqslant \min\left\{ \left| \theta - \frac{A_n}{b_1 \cdots b_n} \right|, \left| \theta - \frac{A_n + 1}{b_1 \cdots b_n} \right| \right\} \tag{3.2}$$

for all $m \in \mathbb{Z}$.

Proof. Suppose that the result fail for some $m \in \mathbb{Z}$. So, $\frac{m}{b_1 \cdots b_n}$ lies in interior of I_n. Contradiction. Hence (3.2) holds for all $m \in \mathbb{Z}$. □

Proposition 3.3. *Suppose that a Cantor series θ, like in (2.1) and satisfying $(*)$, is an irrational number. For all integers $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$, with $D(q, \sigma) > 1$, let k be the smallest integer greater than $D(q, \sigma)$ such that the interval I_k lies in the interior of $I_{D(q,\sigma)}$. Then*

$$\left| \theta - \frac{p}{q} \right| > \frac{\min\{a_k, b_k - a_k - 1\}}{b_1 \cdots b_k} \tag{3.3}$$

where $\sigma = (b_1, b_2, \dots)$.

Proof. Let $\sigma = (b_1, b_2, \dots)$. Set $n = D(q,\sigma)$ and $m = \frac{pb_1 \cdots b_n}{q}$. Therefore m and n are integers and

$$\begin{aligned}\left|\theta - \frac{p}{q}\right| &= \left|\theta - \frac{m}{b_1 \cdots b_n}\right| \\ &\geqslant \min\left\{\left|\theta - \frac{A_n}{b_1 \cdots b_n}\right|, \left|\theta - \frac{A_n + 1}{b_1 \cdots b_n}\right|\right\} \qquad (3.4)\\ &> \frac{\min\{a_k, b_k - a_k - 1\}}{b_1 \cdots b_k}. \qquad (3.5)\end{aligned}$$

The inequalities (3.4) and (3.5) follow respectively by Lemma 3.2 and the hypothesis on k. □

The result below gives a slight improvement to (3.1).

Corollary 3.3. *If p and q are integers, with $q \neq 0$, then*

$$\left|e - \frac{p}{q}\right| > \frac{1}{(D(q,\sigma)+2)!}, \qquad (3.6)$$

where $\sigma = (2, 3, 4, \dots)$.

Proof. Since that $\min_{p\in\mathbb{Z}} |e - p| > 0.28 > \frac{1}{6}$, then (3.6) holds in the case $q = \pm 1$. In case $q \neq \pm 1$ the inequality also holds by Proposition 3.3 and Example 2.2. Moreover, in this case we have $S(q) - 1 \in \{n \in \mathbb{N} \mid q|(n+1)!\}$ and $D(q,\sigma) + 1 \in \{n \in \mathbb{N} \mid q|n!\}$. Thus $S(q) = D(q,\sigma) + 1$. Hence

$$\left|e - \frac{p}{q}\right| > \frac{1}{(D(q,\sigma)+2)!} = \frac{1}{(S(q)+1)!}.$$

□

Actually, the improvement happens only because (3.6) also holds for $q = \pm 1$.

Example 3.4. The number $\xi := \frac{1}{(1!)^5} + \frac{1}{(2!)^5} + \frac{1}{(3!)^5} + \dots = 1.031378\dots$ is irrational, moreover for $p, q \in \mathbb{Z}$ with $q \neq 0$, we have

$$\left|\xi - \frac{p}{q}\right| > \frac{1}{(D(q,\sigma)+2)!^5}$$

where $\sigma = (2^5, 3^5, \dots)$.

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