Annales Mathematicae et Informaticae
35 *(2008) pp. 43–59*
http://www.ektf.hu/ami

Common fixed point theorems for pairs of single and multivalued D-maps satisfying an integral type

H. Bouhadjera, A. Djoudi

Laboratoire de Mathématiques Appliquées
Université Badji Mokhtar, Annaba, Algérie

Submitted 4 March 2008; Accepted 28 June 2008

Abstract

This contribution is a continuation of [1, 3, 14]. The concept of subcompatibility between single maps and between single and multivalued maps is used as a tool for proving existence and uniqueness of common fixed points on complete metric and symmetric spaces. Extensions of known results, in particularly results given by Djoudi and Aliouche, Elamrani and Mehdaoui, Pathak et al. are thereby obtained.

Keywords: Commuting and weakly commuting maps, compatible and compatible maps of type (A), (B), (C) and (P), weakly compatible maps, δ-compatible maps, subcompatible maps, D-maps, integral type, common fixed point theorems, metric space.

MSC: 47H10, 54H25

1. Introduction and preliminaries

Let $(\mathcal{X}, d)$ be a metric space and let $B(\mathcal{X})$ be the class of all nonempty bounded subsets of $\mathcal{X}$. For all A, B in $B(\mathcal{X})$, define

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

If $A = \{a\}$, we write $\delta(A, B) = \delta(a, B)$. Also, if $B = \{b\}$, it yields that $\delta(A, B) = d(a, b)$.

From the definition of $\delta(A, B)$, for all A, B, C in $B(\mathcal{X})$ it follows that

$$\delta(A, B) = \delta(B, A) \geqslant 0,$$
$$\delta(A, B) \leqslant \delta(A, C) + \delta(C, B),$$

$$\delta(A,A) = \mathrm{diam}A,$$
$$\delta(A,B) = 0 \quad \text{iff} \quad A = B = \{a\}.$$

In his paper [15], Sessa introduced the notion of weak commutativity which generalized the notion of commutativity.

Later on, Jungck [6] gave a generalization of weak commutativity by introducing the concept of compatibility.

Again, to generalize weakly commuting maps, the same author with Murthy and Cho [8] introduced the concept of compatible maps of type (A).

Extending type (A), Pathak and Khan [13] made the notion of compatible maps of type (B).

In [11], the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A).

In 1998, Pathak, Cho, Kang and Madharia [12] defined the notion of compatible maps of type (C) as another extension of compatible maps of type (A).

In his paper [7], Jungck generalized all the concepts of compatibility by giving the notion of weak compatibility (subcompatibility).

The authors of [9] extended the concept of compatible maps to the setting of single and multivalued maps by giving the notion of δ-compatible maps.

Also, the same authors [10] extended the definition of weak compatibility to the setting of single and multivalued maps by introducing the concept of subcompatible maps.

In their paper [2], Djoudi and Khemis introduced the notion of D-maps which is a generalization of δ-compatible maps.

Definition 1.1 ([4]). A sequence $\{A_n\}$ of nonempty subsets of $\mathcal{X}$ is said to be convergent to a subset A of $\mathcal{X}$ if:

(i) each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for $n \in \mathbb{N}$,

(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in $\mathcal{X}$ for which there exists a point a in A, depending on x, such that $d(x,a) < \epsilon$.

Lemma 1.2 ([4, 5]). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(\mathcal{X})$ converging to A and B in $B(\mathcal{X})$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 1.3 ([5]). *Let $\{A_n\}$ be a sequence in $B(\mathcal{X})$ and y be a point in $\mathcal{X}$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(\mathcal{X})$.*

Definition 1.4 ([15]). The self-maps f and g of a metric space $\mathcal{X}$ are said to be weakly commuting if $d(fgx, gfx) \leqslant d(gx, fx)$ for all $x \in \mathcal{X}$.

Definition 1.5 ([6, 8, 13, 12, 11]). The self-maps f and g of a metric space $\mathcal{X}$ are said to be

(1) compatible if
$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

(2) compatible of type (A) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0,$$

(3) compatible of type (B) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leqslant \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right],$$
$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leqslant \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right],$$

(4) compatible of type (C) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)\Big],$$
$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)\Big],$$

(5) compatible of type (P) if

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 1.6 ([7]). The self-maps f and g of a metric space $\mathcal{X}$ are called weakly compatible if $fx = gx$, $x \in \mathcal{X}$ implies $fgx = gfx$.

Definition 1.7 ([9]). The maps $f\colon \mathcal{X} \to \mathcal{X}$ and $F\colon \mathcal{X} \to B(\mathcal{X})$ are δ-compatible if

$$\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $fFx_n \in B(\mathcal{X})$, $fx_n \to t$ and $Fx_n \to \{t\}$ for some $t \in \mathcal{X}$.

Definition 1.8 ([10]). Maps $f\colon \mathcal{X} \to \mathcal{X}$ and $F\colon \mathcal{X} \to B(\mathcal{X})$ are subcompatible if they commute at coincidence points; i.e., for each point $u \in \mathcal{X}$ such that $Fu = \{fu\}$, we have $Ffu = fFu$.

Definition 1.9 ([2]). The maps $f\colon \mathcal{X} \to \mathcal{X}$ and $F\colon \mathcal{X} \to B(\mathcal{X})$ are said to be D-maps iff there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that for some $t \in \mathcal{X}$

$$\lim_{n\to\infty} fx_n = t \quad\text{and}\quad \lim_{n\to\infty} Fx_n = \{t\}.$$

Recently in 2007, Pathak et al. [14] established a general common fixed point theorem for two pairs of weakly compatible maps satisfying integral type implicit relations. The first main object of this paper is to prove a common fixed point theorem for a quadruple of maps satisfying certain integral type implicit relations. Our result extended the result of [14] to the setting of single and multivalued maps.

For this consideration we need the following:

Let $\Phi = \{\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue-integrable map which is summable$\}$ and let $\mathcal{F}$ be the set of all continuous functions $F\colon \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfying the following conditions:

(F_a) $\int_0^{F(u,0,0,u,u,0)} \varphi(t)dt \leqslant 0$ implies $u = 0$;

(F_b) $\int_0^{F(u,0,u,0,0,u)} \varphi(t)dt \leqslant 0$ implies $u = 0$.

The function F satisfies the condition (F_1) if $\int_0^{F(u,u,0,0,u,u)} \varphi(t)dt > 0$ for all $u > 0$.

2. Main results

Theorem 2.1. *Let f, g be self-maps of a metric space $(\mathcal{X}, d)$ and let $F, G\colon \mathcal{X} \to B(\mathcal{X})$ be two multivalued maps such that*

(1) *$F\mathcal{X} \subseteq g\mathcal{X}$ and $G\mathcal{X} \subseteq f\mathcal{X}$,*

(2)

$$\int_0^{F(\delta(Fx,Gy),d(fx,gy),\delta(fx,Fx),\delta(gy,Gy),\delta(fx,Gy),\delta(gy,Fx))} \varphi\left(t\right)dt \leqslant 0$$

for all x, y in $\mathcal{X}$, where $F \in \mathcal{F}$ and $\varphi \in \Phi$. If either

(3) *f and F are subcompatible D-maps; g and G are subcompatible and $F\mathcal{X}$ is closed, or*

(3′) *g and G are subcompatible D-maps; f and F are subcompatible and $G\mathcal{X}$ is closed.*

Then, f, g, F and G have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = Gt = \{ft\} = \{gt\} = \{t\}\,.$$

Proof. Suppose that f and F are D-maps, then, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $fx_n \to t$ and $Fx_n \to \{t\}$ for some $t \in \mathcal{X}$. Since $F\mathcal{X}$ is closed and $F\mathcal{X} \subseteq g\mathcal{X}$, then, there is a point $u \in \mathcal{X}$ such that $gu = t$. We show that $Gu = \{gu\} = \{t\}$. Using inequality (2), we have

$$\int_0^{F(\delta(Fx_n,Gu),d(fx_n,gu),\delta(fx_n,Fx_n),\delta(gu,Gu),\delta(fx_n,Gu),\delta(gu,Fx_n))} \varphi\left(t\right)dt \leqslant 0.$$

Since F is continuous, we get at infinity

$$\int_0^{F(\delta(gu,Gu),0,0,\delta(gu,Gu),\delta(gu,Gu),0)} \varphi\left(t\right)dt \leqslant 0$$

which implies, by using condition (F_a), $\delta(gu, Gu) = 0$; i.e., $Gu = \{gu\} = \{t\}$. Since the pair (g, G) is subcompatible, it follows that $Ggu = gGu$; i.e., $Gt = \{gt\}$. If $t \neq gt$, using (2) we have

$$\int_0^{F(\delta(Fx_n,Gt),d(fx_n,gt),\delta(fx_n,Fx_n),\delta(gt,Gt),\delta(fx_n,Gt),\delta(gt,Fx_n))} \varphi(t)\,dt \leqslant 0.$$

Taking limit as $n \to \infty$, we get

$$\int_0^{F(d(t,gt),d(t,gt),0,0,d(t,gt),d(gt,t))} \varphi(t)\,dt \leqslant 0,$$

which contradicts (F_1). Hence, $Gt = \{gt\} = \{t\}$. Since $G\mathcal{X} \subseteq f\mathcal{X}$, there is $v \in \mathcal{X}$ such that $\{t\} = Gt = \{fv\}$. If $Fv \neq \{t\}$, using (2) again, we have

$$\begin{aligned} &\int_0^{F(\delta(Fv,Gt),d(fv,gt),\delta(fv,Fv),\delta(gt,Gt),\delta(fv,Gt),\delta(gt,Fv))} \varphi(t)\,dt \\ = &\int_0^{F(\delta(Fv,t),0,\delta(t,Fv),0,0,\delta(t,Fv))} \varphi(t)\,dt \leqslant 0, \end{aligned}$$

which implies by using condition (F_b) that $\delta(Fv, t) = 0$, hence, $Fv = \{t\} = \{fv\}$. Since F and f are subcompatible, it follows that $Ffv = fFv$; i.e., $Ft = \{ft\}$. If $t \neq ft$, using (2) we have

$$\begin{aligned} &\int_0^{F(\delta(Ft,Gt),d(ft,gt),\delta(ft,Ft),\delta(gt,Gt),\delta(ft,Gt),\delta(gt,Ft))} \varphi(t)\,dt \\ = &\int_0^{F(d(ft,t),d(ft,t),0,0,d(ft,t),d(t,ft))} \varphi(t)\,dt \leqslant 0, \end{aligned}$$

which contradicts (F_1). Thus, $\{ft\} = \{t\} = Ft$.

We get the same conclusion if we use $(3')$ instead of (3).

The uniqueness of the common fixed point follows easily from conditions (2) and (F_1). □

Corollary 2.2. *Let f be a map from a metric space $(\mathcal{X}, d)$ into itself and let F be a map from $\mathcal{X}$ into $B(\mathcal{X})$. If*

(i) $F\mathcal{X} \subseteq f\mathcal{X}$,

(ii) f and F are subcompatible D-maps,

(iii)

$$\int_0^{F(\delta(Fx,Fy),d(fx,fy),\delta(fx,Fx),\delta(fy,Fy),\delta(fx,Fy),\delta(fy,Fx))} \varphi(t)\,dt \leqslant 0$$

for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and F is continuous and satisfies conditions (F_a) and (F_1) or (F_b) and (F_1). If $F\mathcal{X}$ is closed, then, f and F have a unique common fixed point in $\mathcal{X}$.

The next Theorem is a generalization of Theorem 2.1.

Theorem 2.3. *Let f, g be self-maps of a metric space $(\mathcal{X}, d)$ and let $F_n\colon \mathcal{X} \to B(\mathcal{X})$, where $n = 1, 2, \ldots$ be multivalued maps such that*

(i) $F_n\mathcal{X} \subseteq g\mathcal{X}$ and $F_{n+1}\mathcal{X} \subseteq f\mathcal{X}$,

(ii)

$$\int_0^{F(\delta(F_n x, F_{n+1} y), d(fx, gy), \delta(fx, F_n x), \delta(gy, F_{n+1} y), \delta(fx, F_{n+1} y), \delta(gy, F_n x))} \varphi(t)\, dt \leqslant 0$$

for all x, y in $\mathcal{X}$, where $F \in \mathcal{F}$ and $\varphi \in \Phi$. If either

(iii) f and F_n are subcompatible D-maps; g and F_{n+1} are subcompatible and $F_n\mathcal{X}$ is closed, or

(iii)′ g and F_{n+1} are subcompatible D-maps; f and F_n are subcompatible and $F_{n+1}\mathcal{X}$ is closed.

Then, f, g and F_n have a unique common fixed point $t \in \mathcal{X}$ such that

$$F_n t = \{ft\} = \{gt\} = \{t\}.$$

Now, let Ψ be the set of all maps $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is a Lebesgue-integrable which is summable, nonnegative and satisfies $\int_0^\epsilon \psi(t)dt > 0$ for each $\epsilon > 0$.

In [3], a common fixed point theorem for a pair of generalized contraction self-maps and a pair of multivalued maps in a complete metric space was obtained. Our second main subject is to complement and improve the result of [3] by relaxing the notion of δ-compatibility to subcompatibility, removing the assumption of continuity imposed on at least one of the four maps and deleting some conditions required on the functions Φ, a, b and c by using an integral type in a metric space instead of a complete metric space.

Theorem 2.4. *Let f, g be self-maps of a metric space $(\mathcal{X}, d)$ and let F, G be maps from $\mathcal{X}$ into $B(\mathcal{X})$ satisfying the following conditions*

(1′) f and g are surjective,

(2′)

$$\begin{aligned}\int_0^{\digamma(\delta(Fx,Gy))} \psi(t)dt \leqslant{}& a\left(d\left(fx, gy\right)\right) \int_0^{\digamma(d(fx,gy))} \psi(t)dt \\ &+ b\left(d\left(fx, gy\right)\right) \int_0^{\digamma(\delta(fx,Fx)) + \digamma(\delta(gy,Gy))} \psi(t)dt \\ &+ c\left(d\left(fx, gy\right)\right) \int_0^{\min\{\digamma(\delta(fx,Gy)), \digamma(\delta(gy,Fx))\}} \psi(t)dt\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\digamma\colon [0, \infty) \to [0, \infty)$ is an upper semi-continuous map such that $\digamma(t) = 0$ iff $t = 0$; $a, b, c\colon [0, \infty) \to [0, 1)$ are upper semi-continuous such that $a(t) + c(t) < 1$ for every $t > 0$ and $\psi \in \Psi$. If either

(3′) f and F are subcompatible D-maps; g and G are subcompatible, or

$(3'')$ *g and G are subcompatible D-maps; f and F are subcompatible.*
Then, f, g, F and G have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = Gt = \{ft\} = \{gt\} = \{t\}\,.$$

Proof. Suppose that f and F are D-maps, then, there is a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$. By condition $(1')$, there exist points u, v in $\mathcal{X}$ such that $t = fu = gv$. First, we show that $Gv = \{gv\} = \{t\}$. Using inequality $(2')$ we get

$$\begin{aligned}
&\int_0^{F(\delta(Fx_n,Gv))} \psi(t)dt \\
&\leqslant a\left(d\left(fx_n, gv\right)\right) \int_0^{F(d(fx_n,gv))} \psi(t)dt \\
&+ b\left(d\left(fx_n, gv\right)\right) \int_0^{F(\delta(fx_n,Fx_n))+F(\delta(gv,Gv))} \psi(t)dt \\
&+ c\left(d\left(fx_n, gv\right)\right) \int_0^{\min\{F(\delta(fx_n,Gv)),F(\delta(gv,Fx_n))\}} \psi(t)dt.
\end{aligned}$$

Taking the limit as $n \to \infty$, one obtains

$$\int_0^{F(\delta(gv,Gv))} \psi(t)dt \leqslant b\left(0\right) \int_0^{F(\delta(gv,Gv))} \psi(t)dt < \int_0^{F(\delta(gv,Gv))} \psi(t)dt$$

this contradiction implies that $Gv = \{gv\} = \{t\}$. Since the pair (g, G) is subcompatible, then, $Ggv = gGv$; i.e., $Gt = \{gt\}$. We claim that $Gt = \{gt\} = \{t\}$. Suppose not, then, by condition $(2')$ we have

$$\begin{aligned}
\int_0^{F(\delta(Fx_n,Gt))} \psi(t)dt &\leqslant a\left(d\left(fx_n, gt\right)\right) \int_0^{F(d(fx_n,gt))} \psi(t)dt \\
&+ b\left(d\left(fx_n, gt\right)\right) \int_0^{F(\delta(fx_n,Fx_n))+F(\delta(gt,Gt))} \psi(t)dt \\
&+ c\left(d\left(fx_n, gt\right)\right) \int_0^{\min\{F(\delta(fx_n,Gt)),F(\delta(gt,Fx_n))\}} \psi(t)dt.
\end{aligned}$$

When $n \to \infty$ we obtain

$$\begin{aligned}
\int_0^{F(\delta(t,Gt))} \psi(t)dt &= \int_0^{F(d(t,gt))} \psi(t)dt \\
&\leqslant \left[a\left(d\left(t, gt\right)\right) + c\left(d\left(t, gt\right)\right)\right] \int_0^{F(d(t,gt))} \psi(t)dt \\
&< \int_0^{F(d(t,gt))} \psi(t)dt
\end{aligned}$$

which is a contradiction. Hence, $\{gt\} = \{t\} = Gt$. Next, we claim that $Fu = \{fu\} = \{t\}$. If not, then, by $(2')$ we get

$$\begin{aligned}\int_0^{F(\delta(Fu,fu))}\psi(t)dt &= \int_0^{F(\delta(Fu,Gt))}\psi(t)dt\\ &\leqslant a\left(d\left(fu,gt\right)\right)\int_0^{F(d(fu,gt))}\psi(t)dt\\ &+ b\left(d\left(fu,gt\right)\right)\int_0^{F(\delta(fu,Fu))+F(\delta(gt,Gt))}\psi(t)dt\\ &+ c\left(d\left(fu,gt\right)\right)\int_0^{\min\{F(\delta(fu,Gt)),F(\delta(gt,Fu))\}}\psi(t)dt\\ &= b\left(0\right)\int_0^{F(\delta(fu,Fu))}\psi(t)dt < \int_0^{F(\delta(fu,Fu))}\psi(t)dt\end{aligned}$$

which is a contradiction. Thus, $Fu = \{fu\} = \{t\}$. Since F and f are subcompatible, then, $Ffu = fFu$; i.e., $Ft = \{ft\}$. Suppose that $ft \neq t$. Then, the use of $(2')$ gives

$$\begin{aligned}\int_0^{F(d(ft,t))}\psi(t)dt &= \int_0^{F(\delta(Ft,Gt))}\psi(t)dt\\ &\leqslant a\left(d\left(ft,gt\right)\right)\int_0^{F(d(ft,gt))}\psi(t)dt\\ &+ b\left(d\left(ft,gt\right)\right)\int_0^{F(\delta(ft,Ft))+F(\delta(gt,Gt))}\psi(t)dt\\ &+ c\left(d\left(ft,gt\right)\right)\int_0^{\min\{F(\delta(ft,Gt)),F(\delta(gt,Ft))\}}\psi(t)dt\\ &= \left[a\left(d\left(ft,t\right)\right) + c\left(d\left(ft,t\right)\right)\right]\int_0^{F(d(ft,t))}\psi(t)dt\\ &< \int_0^{F(d(ft,t))}\psi(t)dt\end{aligned}$$

this contradiction implies that $ft = t$ and hence $Ft = \{ft\} = \{t\}$. Therefore t is a common fixed point of both f, g, F and G.

The uniqueness of the common fixed point follows easily from condition $(2')$.

We get the same conclusion if we consider $(3'')$ in lieu of $(3')$. □

Remark 2.5. Theorem 3.1 of [3] becomes a special case of Theorem 2.4 with $\psi(x) = 1$.

If we put $f = g$ in Theorem 2.4, we get the next corollary.

Corollary 2.6. *Let $(\mathcal{X}, d)$ be a metric space and let $f\colon \mathcal{X} \to \mathcal{X}$; $F, G\colon \mathcal{X} \to B(\mathcal{X})$ be maps. Suppose that*

(*i*) *f is surjective,*
(*ii*)

$$\begin{aligned}\int_0^{\digamma(\delta(Fx,Gy))}\psi(t)dt \leqslant a\left(d\left(fx,fy\right)\right)&\int_0^{\digamma(d(fx,fy))}\psi(t)dt\\ +\,b\left(d\left(fx,fy\right)\right)&\int_0^{\digamma(\delta(fx,Fx))+\digamma(\delta(fy,Gy))}\psi(t)dt\\ +\,c\left(d\left(fx,fy\right)\right)&\int_0^{\min\{\digamma(\delta(fx,Gy)),\digamma(\delta(fy,Fx))\}}\psi(t)dt\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\digamma, \psi, a, b, c$ are as in Theorem 2.4. If either
(*iii*) *f and F are subcompatible D-maps; f and G are subcompatible, or*
(*iii*)$'$ *f and G are subcompatible D-maps; f and F are subcompatible.*
Then, f, F and G have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = Gt = \{ft\} = \{t\}.$$

For a single map $f\colon \mathcal{X} \to \mathcal{X}$ (resp. a multivalued map $F\colon \mathcal{X} \to B(\mathcal{X})$), $\mathcal{F}_f$ (resp. $\mathcal{F}_F$) will denote the set of fixed point of f (resp. F).

Theorem 2.7. *Let $F, G\colon \mathcal{X} \to B(\mathcal{X})$ be multivalued maps and let $f, g\colon \mathcal{X} \to \mathcal{X}$ be single maps on the metric space $\mathcal{X}$. If inequality $(2')$ holds for all x, y in $\mathcal{X}$, then,*

$$(\mathcal{F}_f \cap \mathcal{F}_g) \cap \mathcal{F}_F = (\mathcal{F}_f \cap \mathcal{F}_g) \cap \mathcal{F}_G.$$

Proof. We can check the above equality by using inequality $(2')$. □

Theorems 2.4 and 2.7 imply the next one.

Theorem 2.8. *Let f, g be self-maps of a metric space $(\mathcal{X}, d)$ and let F_n, where $n = 1, 2, \ldots$ be maps from $\mathcal{X}$ into $B(\mathcal{X})$ such that*
(*i*) *f and g are surjective,*
(*ii*)

$$\begin{aligned}&\int_0^{\digamma(\delta(F_nx,F_{n+1}y))}\psi(t)dt\\ &\leqslant a\left(d\left(fx,gy\right)\right)\int_0^{\digamma(d(fx,gy))}\psi(t)dt\\ &+b\left(d\left(fx,gy\right)\right)\int_0^{\digamma(\delta(fx,F_nx))+\digamma(\delta(gy,F_{n+1}y))}\psi(t)dt\\ &+c\left(d\left(fx,gy\right)\right)\int_0^{\min\{\digamma(\delta(fx,F_{n+1}y)),\digamma(\delta(gy,F_nx))\}}\psi(t)dt\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\digamma, \psi, a, b, c$ are as in Theorem 2.4. If either
(*iii*) *f and F_1 are subcompatible D-maps; g and F_2 are subcompatible, or*

(iii)′ g and F_2 are subcompatible D-maps; f and F_1 are subcompatible.
Then, f, g and F_n have a unique common fixed point $t \in \mathcal{X}$ such that

$$F_n t = \{ft\} = \{gt\} = \{t\} \quad \text{for} \quad n = 1, 2, \dots .$$

Let Ω be the family of all maps $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that ω is upper semicontinuous and $\omega(t) < t$ for each $t > 0$.

In [1], Djoudi and Aliouche proved a common fixed point theorem of Greguš type for four maps satisfying a contractive condition of integral type in a metric space using the concept of weak compatibility. Our aim henceforth is to extend this result to multivalued maps by using the concept of D-maps.

Theorem 2.9. *Let $(\mathcal{X}, d)$ be a metric space and let $f, g\colon \mathcal{X} \to \mathcal{X}$; $F_k\colon \mathcal{X} \to B(\mathcal{X})$ be single and multivalued maps, respectively. Suppose that*

(i) $F_k\mathcal{X} \subseteq g\mathcal{X}$ and $F_{k+1}\mathcal{X} \subseteq f\mathcal{X}$,

(ii)

$$\begin{aligned}
&\left(\int_0^{\delta(F_k x, F_{k+1} y)} \psi(t)dt\right)^p \\
&\leqslant \omega\left(a\left(\int_0^{d(fx,gy)} \psi(t)dt\right)^p + (1-a)\max\left\{\alpha\left(\int_0^{\delta(fx,F_k x)} \psi(t)dt\right)^p,\right.\right. \\
&\beta\left(\int_0^{\delta(gy,F_{k+1} y)} \psi(t)dt\right)^p, \left(\int_0^{\delta(fx,F_k x)} \psi(t)dt\right)^{\frac{p}{2}} \left(\int_0^{\delta(gy,F_k x)} \psi(t)dt\right)^{\frac{p}{2}}, \\
&\left(\int_0^{\delta(gy,F_k x)} \psi(t)dt\right)^{\frac{p}{2}} \left(\int_0^{\delta(fx,F_{k+1} y)} \psi(t)dt\right)^{\frac{p}{2}}, \\
&\left.\left.\frac{1}{2}\left(\left(\int_0^{\delta(fx,F_k x)} \psi(t)dt\right)^p + \left(\int_0^{\delta(gy,F_{k+1} y)} \psi(t)dt\right)^p\right)\right\}\right)
\end{aligned}$$

for all x, y in $\mathcal{X}$, where $k \in \mathbb{N}^* = \{1, 2, \dots\}$, $\omega \in \Omega$, $\psi \in \Psi$, $0 < a < 1$, $0 < \alpha, \beta \leqslant 1$ and p is an integer such that $p \geqslant 1$. If either

(iii) f and F_k are subcompatible D-maps; g and F_{k+1} are subcompatible and $F_k\mathcal{X}$ is closed, or

(iii)′ g and F_{k+1} are subcompatible D-maps; f and F_k are subcompatible and $F_{k+1}\mathcal{X}$ is closed.

Then, f, g and F_k have a unique common fixed point $t \in \mathcal{X}$ such that

$$F_k t = \{ft\} = \{gt\} = \{t\}.$$

Proof. Suppose that f and F_k are D-maps, then, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} F_k x_n = \{t\}$ for some $t \in \mathcal{X}$. Since $F_k\mathcal{X}$ is closed and $F_k\mathcal{X} \subseteq g\mathcal{X}$, then, there is $u \in \mathcal{X}$ such that $gu = t$. If $F_{k+1}u \neq \{gu\}$, using inequality (ii) we get

$$\left(\int_0^{\delta(F_kx_n,F_{k+1}u)}\psi(t)dt\right)^p$$
$$\leqslant \omega\left(a\left(\int_0^{d(fx_n,gu)}\psi(t)dt\right)^p\right.$$
$$+(1-a)\max\left\{\alpha\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^p,\beta\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p,\right.$$
$$\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(gu,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}},$$
$$\left(\int_0^{\delta(gu,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(fx_n,F_{k+1}u)}\psi(t)dt\right)^{\frac{p}{2}},$$
$$\left.\left.\frac{1}{2}\left(\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^p+\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p\right)\right\}\right).$$

Letting $n\to\infty$ we obtain

$$\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p$$
$$\leqslant \omega\left((1-a)\max\left\{\beta,\frac{1}{2}\right\}\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p\right)$$
$$<(1-a)\max\left\{\beta,\frac{1}{2}\right\}\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p<\left(\int_0^{\delta(gu,F_{k+1}u)}\psi(t)dt\right)^p$$

which is a contradiction. Then $F_{k+1}u=\{gu\}=\{t\}$. Since the pair (g,F_{k+1}) is subcompatible, we have $F_{k+1}gu=gF_{k+1}u$; i.e., $F_{k+1}t=\{gt\}$. If $t\neq gt$, using inequality (ii) we obtain

$$\left(\int_0^{\delta(F_kx_n,F_{k+1}t)}\psi(t)dt\right)^p$$
$$\leqslant \omega\left(a\left(\int_0^{d(fx_n,gt)}\psi(t)dt\right)^p\right.$$
$$+(1-a)\max\left\{\alpha\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^p,\beta\left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p,\right.$$
$$\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(gt,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}},$$

$$\left(\int_0^{\delta(gt,F_kx_n)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(fx_n,F_{k+1}t)}\psi(t)dt\right)^{\frac{p}{2}},$$
$$\frac{1}{2}\left(\left(\int_0^{\delta(fx_n,F_kx_n)}\psi(t)dt\right)^p+\left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p\right)\Bigg\}\Bigg).$$

At infinity we get

$$\left(\int_0^{d(t,gt)}\psi(t)dt\right)^p\leqslant\omega\left(\left(\int_0^{d(t,gt)}\psi(t)dt\right)^p\right)<\left(\int_0^{d(t,gt)}\psi(t)dt\right)^p$$

which is a contradiction. Therefore $F_{k+1}t=\{gt\}=\{t\}$. Since $F_{k+1}\mathcal{X}\subseteq f\mathcal{X}$, there exists $v\in\mathcal{X}$ such that $F_{k+1}t=\{t\}=\{fv\}$. We claim that $F_kv=\{fv\}$, suppose not, then by condition (ii) we have

$$\left(\int_0^{\delta(F_kv,F_{k+1}t)}\psi(t)dt\right)^p$$
$$\leqslant\omega\left(a\left(\int_0^{d(fv,gt)}\psi(t)dt\right)^p+(1-a)\max\left\{\alpha\left(\int_0^{\delta(fv,F_kv)}\psi(t)dt\right)^p,\right.\right.$$
$$\beta\left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p,\left(\int_0^{\delta(fv,F_kv)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(gt,F_kv)}\psi(t)dt\right)^{\frac{p}{2}},$$
$$\left(\int_0^{\delta(gt,F_kv)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(fv,F_{k+1}t)}\psi(t)dt\right)^{\frac{p}{2}},$$
$$\frac{1}{2}\left(\left(\int_0^{\delta(fv,F_kv)}\psi(t)dt\right)^p+\left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p\right)\Bigg\}\Bigg),$$

that is,

$$\begin{aligned}\left(\int_0^{\delta(F_kv,fv)}\psi(t)dt\right)^p&\leqslant\omega\left((1-a)\left(\int_0^{\delta(F_kv,fv)}\psi(t)dt\right)^p\right)\\&<(1-a)\left(\int_0^{\delta(F_kv,fv)}\psi(t)dt\right)^p\\&<\left(\int_0^{\delta(F_kv,fv)}\psi(t)dt\right)^p\end{aligned}$$

which is a contradiction. Hence $F_kv=\{fv\}=\{t\}$. Since the pair (f,F_k) is subcompatible, then, $F_kfv=fF_kv$; i.e., $F_kt=\{ft\}$. The use of (ii) gives

$$\left(\int_0^{\delta(F_kt,F_{k+1}t)}\psi(t)dt\right)^p$$

$$\leqslant \omega\left(a\left(\int_0^{d(ft,gt)}\psi(t)dt\right)^p + (1-a)\max\left\{\alpha\left(\int_0^{\delta(ft,F_kt)}\psi(t)dt\right)^p,\right.\right.$$

$$\beta\left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p, \left(\int_0^{\delta(ft,F_kt)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(gt,F_kt)}\psi(t)dt\right)^{\frac{p}{2}},$$

$$\left(\int_0^{\delta(gt,F_kt)}\psi(t)dt\right)^{\frac{p}{2}}\left(\int_0^{\delta(ft,F_{k+1}t)}\psi(t)dt\right)^{\frac{p}{2}},$$

$$\left.\left.\frac{1}{2}\left(\left(\int_0^{\delta(ft,F_kt)}\psi(t)dt\right)^p + \left(\int_0^{\delta(gt,F_{k+1}t)}\psi(t)dt\right)^p\right)\right\}\right),$$

i.e.,

$$\left(\int_0^{d(ft,t)}\psi(t)dt\right)^p \leqslant \omega\left(\left(\int_0^{d(ft,t)}\psi(t)dt\right)^p\right) < \left(\int_0^{d(ft,t)}\psi(t)dt\right)^p$$

this contradiction implies that $\{ft\} = \{t\} = F_kt$. Thus, t is a common fixed point of f, g and F_k.

The uniqueness of the common fixed point follows from inequality (ii).

If one uses condition $(iii)'$ instead of (iii), one gets the same conclusion. □

Theorem 2.10. *Let $(\mathcal{X}, d)$ be a metric space and let $f, g\colon \mathcal{X} \to \mathcal{X}$; $F_n\colon \mathcal{X} \to B(\mathcal{X})$ be single and multivalued maps such that*

(i) $F_n\mathcal{X} \subseteq g\mathcal{X}$ and $F_{n+1}\mathcal{X} \subseteq f\mathcal{X}$,

(ii)

$$\left(\int_0^{\delta(F_nx,F_{n+1}y)}\psi(t)dt\right)^p$$

$$\leqslant \omega\left(a\left(\int_0^{d(fx,gy)}\psi(t)dt\right)^p + (1-a)\max\left\{\int_0^{\delta(fx,F_nx)}\psi(t)dt,\right.\right.$$

$$\int_0^{\delta(gy,F_{n+1}y)}\psi(t)dt, \left(\int_0^{\delta(fx,F_nx)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(gy,F_nx)}\psi(t)dt\right)^{\frac{1}{2}},$$

$$\left.\left.\left(\int_0^{\delta(gy,F_nx)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(fx,F_{n+1}y)}\psi(t)dt\right)^{\frac{1}{2}}\right\}^p\right)$$

for all x, y in $\mathcal{X}$, where $\omega \in \Omega$, $\psi \in \Psi$, $0 < a < 1$ and p is an integer such that $p \geqslant 1$. If either

(iii) f and F_n are subcompatible D-maps; g and F_{n+1} are subcompatible and $F_n\mathcal{X}$ is closed, or

$(iii)'$ g and F_{n+1} are subcompatible D-maps; f and F_n are subcompatible and $F_{n+1}\mathcal{X}$ is closed.

Then, f, g and F_n have a unique common fixed point $t \in \mathcal{X}$ such that

$$F_n t = \{ft\} = \{gt\} = \{t\} \quad \text{for} \quad n = 1, 2, \dots .$$

Proof. It is similar to the proof of Theorem 2.9. □

Now, we prove a unique common fixed point theorem of Greguš type by using a strict contractive condition of integral type for two pairs of single and multivalued maps in a metric space.

Theorem 2.11. *Let f and g be self-maps of a metric space $(\mathcal{X}, d)$ and let $\{F_n\}$, $n = 1, 2, \dots$ be multivalued maps from $\mathcal{X}$ into $B(\mathcal{X})$ such that*
$(1'')$ *f and g are surjective,*
$(2'')$

$$\begin{aligned}&\int_0^{\delta(F_1x, F_ky)} \psi(t)dt \\ &< \alpha \int_0^{d(fx,gy)} \psi(t)dt + (1-\alpha)\max\left\{ a\int_0^{\delta(fx,F_1x)} \psi(t)dt, \right. \\ &b\int_0^{\delta(gy,F_ky)} \psi(t)dt, c\left(\int_0^{\delta(fx,F_1x)} \psi(t)dt\right)^{\frac{1}{2}} \left(\int_0^{\delta(gy,F_1x)} \psi(t)dt\right)^{\frac{1}{2}}, \\ &\left. d\left(\int_0^{\delta(gy,F_1x)} \psi(t)dt\right)^{\frac{1}{2}} \left(\int_0^{\delta(fx,F_ky)} \psi(t)dt\right)^{\frac{1}{2}} \right\}\end{aligned}$$

for all x, y in $\mathcal{X}$ and some $k > 1$ for which the right hand side is positive, where $\psi \in \Psi$, $0 < \alpha, a, b, c, d < 1$ and $\alpha + d(1-\alpha) < 1$. If either
$(3'')$ *f and F_1 are subcompatible D-maps; g and F_k are subcompatible, or*
$(3''')$ *g and F_k are subcompatible D-maps; f and F_1 are subcompatible.*
Then, f, g and $\{F_n\}$ have a unique common fixed point $t \in \mathcal{X}$ such that

$$F_n t = \{ft\} = \{gt\} = \{t\}, \quad \text{for} \quad n = 1, 2, \dots .$$

Proof. Suppose that condition $(3'')$ holds, then, there is a sequence $\{x_n\}$ in $\mathcal{X}$ such that $fx_n \to t$ and $F_1x_n \to \{t\}$ as $n \to \infty$ for some $t \in \mathcal{X}$. By condition $(1'')$, there are two elements u and v in $\mathcal{X}$ such that $t = fu = gv$. We show that $\{t\} = F_kv$. Indeed, using inequality $(2'')$ we get

$$\begin{aligned}&\int_0^{\delta(F_1x_n, F_kv)} \psi(t)dt \\ &< \alpha \int_0^{d(fx_n,gv)} \psi(t)dt + (1-\alpha)\max\left\{ a\int_0^{\delta(fx_n,F_1x_n)} \psi(t)dt, \right. \\ &b\int_0^{\delta(gv,F_kv)} \psi(t)dt, c\left(\int_0^{\delta(fx_n,F_1x_n)} \psi(t)dt\right)^{\frac{1}{2}} \left(\int_0^{\delta(gv,F_1x_n)} \psi(t)dt\right)^{\frac{1}{2}},\end{aligned}$$

$$d\left(\int_0^{\delta(gv,F_1x_n)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(fx_n,F_kv)}\psi(t)dt\right)^{\frac{1}{2}}\Bigg\}.$$

Taking limit as $n\to\infty$, we obtain

$$\int_0^{\delta(t,F_kv)}\psi(t)dt\leqslant b\,(1-\alpha)\int_0^{\delta(t,F_kv)}\psi(t)dt<\int_0^{\delta(t,F_kv)}\psi(t)dt$$

thus, we have $F_kv=\{t\}=\{gv\}$ and since g and F_k are subcompatible, we have $F_kgv=gF_kv$; that is, $F_kt=\{gt\}$. Again, by $(2'')$ we obtain

$$\begin{aligned}&\int_0^{\delta(F_1x_n,F_kt)}\psi(t)dt\\&<\alpha\int_0^{d(fx_n,gt)}\psi(t)dt+(1-\alpha)\max\Bigg\{a\int_0^{\delta(fx_n,F_1x_n)}\psi(t)dt,\\&b\int_0^{\delta(gt,F_kt)}\psi(t)dt,c\left(\int_0^{\delta(fx_n,F_1x_n)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(gt,F_1x_n)}\psi(t)dt\right)^{\frac{1}{2}},\\&d\left(\int_0^{\delta(gt,F_1x_n)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(fx_n,F_kt)}\psi(t)dt\right)^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

When $n\to\infty$, we get

$$\int_0^{d(t,gt)}\psi(t)dt\leqslant[\alpha+d\,(1-\alpha)]\int_0^{d(t,gt)}\psi(t)dt<\int_0^{d(t,gt)}\psi(t)dt$$

this contradiction implies that $\{t\}=\{gt\}=F_kt=\{fu\}$. We claim that $F_1u=\{t\}$. By condition $(2'')$ we have

$$\begin{aligned}&\int_0^{\delta(F_1u,t)}\psi(t)dt=\int_0^{\delta(F_1u,F_kt)}\psi(t)dt\\&<\alpha\int_0^{d(fu,gt)}\psi(t)dt+(1-\alpha)\max\Bigg\{a\int_0^{\delta(fu,F_1u)}\psi(t)dt,\\&b\int_0^{\delta(gt,F_kt)}\psi(t)dt,c\left(\int_0^{\delta(fu,F_1u)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(gt,F_1u)}\psi(t)dt\right)^{\frac{1}{2}},\\&d\left(\int_0^{\delta(gt,F_1u)}\psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(fu,F_kt)}\psi(t)dt\right)^{\frac{1}{2}}\Bigg\}\\&=(1-\alpha)\max\{a,c\}\int_0^{\delta(F_1u,t)}\psi(t)dt<\int_0^{\delta(F_1u,t)}\psi(t)dt\end{aligned}$$

this contradiction demands that $F_1u = \{t\} = \{fu\}$. Since f and F_1 are subcompatible, then, $F_1fu = fF_1u$; that is, $F_1t = \{ft\}$. Moreover, by $(2'')$ one may get

$$\int_0^{d(ft,t)} \psi(t)dt = \int_0^{\delta(F_1t,F_kt)} \psi(t)dt$$
$$< \alpha \int_0^{d(ft,gt)} \psi(t)dt + (1-\alpha)\max\left\{ a\int_0^{\delta(ft,F_1t)} \psi(t)dt, \right.$$
$$b\int_0^{\delta(gt,F_kt)} \psi(t)dt, c\left(\int_0^{\delta(ft,F_1t)} \psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(gt,F_1t)} \psi(t)dt\right)^{\frac{1}{2}},$$
$$\left. d\left(\int_0^{\delta(gt,F_1t)} \psi(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\delta(ft,F_kt)} \psi(t)dt\right)^{\frac{1}{2}}\right\}$$
$$= [\alpha + d(1-\alpha)]\int_0^{d(ft,t)} \psi(t)dt < \int_0^{d(ft,t)} \psi(t)dt$$

which is a contradiction. Thus, $\{ft\} = \{t\} = F_1t$. Therefore, $F_1t = F_kt = \{ft\} = \{gt\} = \{t\}$.

Uniqueness follows easily from condition $(2'')$. The proof is thus completed. □

Important remark. Every contractive or strict contractive condition of integral type automatically includes a corresponding contractive or strict contractive condition, not involving integrals, by setting $\varphi(t) = 1$ (resp. $\psi(t) = 1$) over $\mathbb{R}_+$. So, our results extend, generalize and complement several various results existing in the literature.

References

[1] Djoudi, A., Aliouche, A., Common fixed point theorems of Greguš type for weakly compatible mappings satisfying contractive conditions of integral type, *J. Math. Anal. Appl.* 329 (2007) no. 1, 31–45.

[2] Djoudi, A., Khemis, R., Fixed points for set and single valued maps without continuity, *Demonstratio Math.* 38 (2005) no. 3, 739–751.

[3] Elamrani, M., Mehdaoui, B., Common fixed point theorems for compatible and weakly compatible mappings, *Rev. Colombiana Mat.* 34 (2000) no. 1, 25–33.

[4] Fisher, B., Common fixed points of mappings and set-valued mappings, *Rostock. Math. Kolloq.* 18 (1981) 69–77.

[5] Fisher, B., Sessa, S., Two common fixed point theorems for weakly commuting mappings, *Period. Math. Hungar.* 20 (1989) no. 3, 207–218.

[6] Jungck, G., Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.* 9 (1986) no. 4, 771–779.

[7] JUNGCK, G., Common fixed points for noncontinuous nonself maps on nonmetric spaces, *Far East J. Math. Sci.* 4 (1996) no. 2, 199–215.

[8] JUNGCK, G., MURTHY, P.P., CHO, Y.J., Compatible mappings of type (A) and common fixed points, *Math. Japon.* 38 (1993) no. 2, 381–390.

[9] JUNGCK, G., RHOADES, B.E., Some fixed point theorems for compatible maps, *Internat. J. Math. Math. Sci.* 16 (1993) no. 3, 417–428.

[10] JUNGCK, G., RHOADES, B.E., Fixed points for set valued functions without continuity, *Indian J. Pure Appl. Math.* 29 (1998) no. 3, 227–238.

[11] PATHAK, H.K., CHO, Y.J., KANG, S.M., LEE, B.S., Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, *Matematiche (Catania)* 50 (1995) no. 1, 15–33.

[12] PATHAK, H.K., CHO, Y.J., KANG, S.M., MADHARIA, B., Compatible mappings of type (C) and common fixed point theorems of Greguš type, *Demonstratio Math.* 31 (1998) no. 3, 499–518.

[13] PATHAK, H.K., KHAN, M.S., Compatible mappings of type (B) and common fixed point theorems of Greguš type, *Czechoslovak Math. J.* 45(120) (1995) no. 4, 685–698.

[14] PATHAK, H.K., TIWARI, R., KHAN, M.S., A common fixed point theorem satisfying integral type implicit relations, *Appl. Math. E-Notes* 7 (2007) 222–228.

[15] SESSA, S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd) (N.S.)* 32(46) (1982) 149–153.

H. Bouhadjera
A. Djoudi
Laboratoire de Mathématiques Appliquées
Université Badji Mokhtar
B. P. 12, 23000, Annaba
Algérie
e-mail: `b_hakima2000@yahoo.fr`