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Laplace transform pairs of N-dimensions and second order linear partial differential equations with constant coefficients

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Abstract

In this paper, authors will present a new theorem and corollary on multi-dimensional Laplace transformations. They also develop some applications based on this results. The two-dimensional Laplace transformation is useful in the solution of partial differential equations. Some illustrative examples related to Laguerre polynomials are also provided.

Keywords: Two-dimensional Laplace transforms, second-order linear non-homogenous partial differential equations, Laguerre polynomials.

MSC: 44A30, 35L05

1. Introduction

In [3] R. S. Dahiya established several new theorems for calculating Laplace transform pairs of N-dimensions and two homogenous boundary value problems related to heat equations were solved. In [4] J. Saberi Najafi and R. S. Dahiya established several new theorems for calculating Laplace transforms of n-dimensions and in the second part application of those theorems to a number of commonly used special functions was considered, and finally, by using two dimensional Laplace transform, one-dimensional wave equation involving special functions was solved. Later in [1, 2] authors, established new theorems and corollaries involving systems of two-dimensional Laplace transforms containing several equations.

The generalization of the well-known Laplace transform

$$L[f(t);s]=\int_0^\infty e^{-st}f(t)\,\mathrm{d}t,$$

to n-dimensional is given by

$$L_n[f(\bar{t});\bar{s}] = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp(-\bar{s}\bar{t}) f(\bar{t}) P_n(\mathrm{d}\bar{t}),$$

where $\bar{t} = (t_1, t_2, \ldots, t_n)$, $\bar{s} = (s_1, s_2, \ldots, s_n)$, $\bar{s}\bar{t} = \sum_{i=1}^n s_i t_i$ and $P_n(\mathrm{d}\bar{t}) = \prod_{k=1}^n \mathrm{d}t_k$. In addition to the notations introduced above, we will use the following throughout this paper.

Let $\bar{t^{v}} = (t_1^v, t_2^v, \ldots, t_n^v)$ for any real exponent v and let $P_k(\bar{t})$ be the k-th symmetric polynomial in the components t_k of $\bar{t}$. Then

$$\begin{aligned} P_0(\bar{t^{v}}) &= 1, \\ P_1(\bar{t^{v}}) &= t_1^v + t_2^v + \ldots + t_n^v, \\ P_2(\bar{t^{v}}) &= \sum\nolimits_{i,j=1, i<j}^{n} t_i^v t_j^v, \\ &\vdots \\ P_n(\bar{t^{v}}) &= t_1^v t_2^v \ldots t_n^v. \end{aligned}$$

The inverse Laplace transform is given by

$$L^{-1}[F(\bar{s});\bar{t}] = \left(\frac{1}{2i\pi}\right)^n \int_{a-i\infty}^{a+i\infty} \int_{d-i\infty}^{d+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} e^{-\bar{s}\bar{t}} F(\bar{s}) P_n(\bar{s})\,\mathrm{d}\bar{s}.$$

2. The main theorem

Theorem 2.1. *Let*

$$g(s) = L[f(t);s], \quad F(s) = L[t^{-3/2} g(1/t);s], \quad H(s) = L[tf(t^4);s].$$

If $f(t)$, $t^{-3/2} g(\frac{1}{t})$ *and* $tf(t^4)$ *are continuous and integrable on* $(0,\infty)$, *then*

$$L_n\left[P_n(\bar{t}^{-1/2}) F\left(P_1^2(\bar{t}^{-1})\right); \bar{s}\right] = 4\pi^{(n+1)/2} \frac{H[2\sqrt{2} P_1(\bar{s}^{1/2})]}{P_n(\bar{s}^{1/2})},$$

where $n = 1, 2, \ldots, N$.

Proof. We have

$$g\left(\frac{1}{t}\right) = \int_0^\infty \exp\left(-\frac{u}{t}\right) f(u)\,\mathrm{d}u. \tag{2.1}$$

Multiply both sides of (2.1) by $t^{-3/2}\exp(-st)$, $\mathrm{Re}(s) > 0$ and integrate with respect to t on $(0,\infty)$ to get

$$\int_0^\infty \frac{e^{-st} g\left(t^{-1}\right)}{t^{3/2}}\,\mathrm{d}t = \int_0^\infty \int_0^\infty e^{-st} e^{-\frac{u}{t}} f\left(u\right) t^{-3/2}\,\mathrm{d}u\,\mathrm{d}t. \tag{2.2}$$

Since the integral on the right side of (2.2) is absolutely convergent, we may change the order of integration to obtain

$$\int_0^\infty \frac{e^{-st} g\left(t^{-1}\right)}{t^{3/2}} \,\mathrm{d}t = \int_0^\infty f\left(u\right) \int_0^\infty e^{-st-u/t} t^{-3/2} \,\mathrm{d}t \,\mathrm{d}u. \tag{2.3}$$

Evaluating the inner integral on the right side of (2.3), we get

$$F\left(s\right) = \sqrt{\pi} \int_0^\infty \frac{f\left(u\right) e^{-\sqrt{su}}}{\sqrt{u}} \,\mathrm{d}u.$$

Now, on setting $u = v^4$, replacing s by $P_1^2(\bar{t}^{-1})$ and then multiplying both sides of (2.3) by $P_n(\bar{t}^{-1/2})e^{-\bar{s}\bar{t}}$ and integrating with respect to $t_1, t_2, \ldots, t_n$ from 0 to ∞, leads to the statement. □

Corollary 2.2. *Letting $n = 2$ we get from Theorem 2.1, that*

$$L_2 \left\{ \frac{1}{\sqrt{xy}} F\left(\left(\frac{1}{x} + \frac{1}{y} \right)^2 \right); u, v \right\} = 4\pi^{3/2} \frac{H[2\sqrt{2}(\sqrt{u} + \sqrt{v})]}{\sqrt{uv}}. \tag{2.4}$$

As an application of the above theorem and corollary, some illustrative examples in two dimensions are also provided.

Example 2.3. Let $f(t) = \sin(\sqrt{t})$, then $F(s) = \frac{2\sqrt{\pi}}{1+4s}$,

$$H(s) = \frac{1}{2} + \frac{2\sqrt{\pi}}{8} \left\{ s \cos\left(\frac{s^2}{4}\right) \left(2S\left(\frac{s}{2\sqrt{\pi}}\right) - 1\right) + s \sin\left(\frac{s^2}{4}\right) \left(1 - 2C\left(\frac{s}{2\sqrt{\pi}}\right)\right) \right\},$$

hence

$$L_2 \left[\frac{(xy)^{\frac{3}{2}}}{4(x+y)^2 + x^2y^2}, u, v \right]$$
$$= \sqrt{\frac{\pi}{uv}} \left\{ \pi(\sqrt{u} + \sqrt{v}) \cos\left(2(\sqrt{u} + \sqrt{v})^2\right) \left(2S\left(\frac{2(\sqrt{u} + \sqrt{v})}{\sqrt{\pi}}\right) - 1\right) \right.$$
$$\left. + (\sqrt{u} + \sqrt{v}) \sin\left(2(\sqrt{u} + \sqrt{v})^2\right) \left(1 - 2C\left(\frac{2(\sqrt{u} + \sqrt{v})}{\sqrt{\pi}}\right)\right) + \sqrt{\pi} \right\},$$

where Fresnel's integrals are defined as following

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos(t)}{\sqrt{t}} \,\mathrm{d}t, \quad S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin(t)}{\sqrt{t}} \,\mathrm{d}t.$$

Example 2.4. If $f(t) = \ln(\alpha t)$ then

$$F(s) = \frac{1}{s} \{\ln(\alpha/s) - \gamma\} \quad \text{and} \quad H(s) = \frac{1}{s}\{\ln(\alpha) + 4(1 - \gamma - \ln(s))\}.$$

Using (2.4), we arrive at

$$L_2\left[\frac{\sqrt{xy}}{x+y}\left(\ln\left(\frac{4(x+y)^2}{\alpha(xy)^2}\right)-2\gamma\right),u,v\right]$$
$$=\pi\frac{4\ln(\sqrt{u}+\sqrt{v})-\ln(\alpha)+6\ln(2)+4(\gamma-1)}{2\sqrt{uv}(\sqrt{u}+\sqrt{v})^2}.$$

In the following example, we give an application of two-dimensional Laplace transforms and complex inversion formula for calculating some of the series related to Laguerre polynomials.

Example 2.5. We shall show that (see [6])

1. $\sum_{n=0}^{\infty} L_n(x)L_n(y)\lambda^n = \frac{1}{1-\lambda}e^{-\frac{\lambda(x+y)}{1-\lambda}} I_0\left(\frac{2\sqrt{\lambda xy}}{1-\lambda}\right)$,
2. $\sum_{n=0}^{\infty} L_n(t)L_n(\xi) = e^t\delta(t-\xi)$,

where $L_n(x)$ is Laguerre polynomial and $I_0(x)$ is modified Bessel 's function of order zero.

Solution.
1. It is well known that $L[L_n(x),p]=\frac{1}{p}\left(1-\frac{1}{p}\right)^n$. Taking two-dimensional Laplace transform of the left hand side, leads to the following

$$L_2\left[\sum_{n=0}^{\infty} L_n(x)L_n(y)\lambda^n,p,q\right]=\int_0^{\infty}\int_0^{\infty}\left(\sum_{n=0}^{\infty} L_n(x)L_n(y)\lambda^n e^{-px-qy}\right)\mathrm{d}x\,\mathrm{d}y.$$

Changing the order of summation and double integration to get

$$L_2\left[\sum_{n=0}^{\infty} L_n(x)L_n(y)\lambda^n,p,q\right]=\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty} L_n(x)L_n(y)\lambda^n e^{-px-qy}\,\mathrm{d}x\,\mathrm{d}y.$$

The value of the inner integral is

$$\sum_{n=0}^{\infty}\lambda^n\int_0^{\infty}\int_0^{\infty} L_n(x)L_n(y)e^{-px-qy}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\sum_{n=0}^{\infty}\lambda^n\left\{\frac{1}{pq}\left(1-\frac{1}{p}\right)^n\left(1-\frac{1}{q}\right)^n\right\}=\frac{1}{1-\lambda}\frac{1}{pq+k(p+q)-k},$$

where $k=\frac{\lambda}{1-\lambda}$. Using complex inversion formula for two-dimensional Laplace transform to obtain,

$$\sum_{n=0}^{\infty} L_n(x)L_n(y)\lambda^n$$
$$=\left(\frac{1}{2i\pi}\right)^2\int_{a-i\infty}^{a+i\infty}\int_{d-i\infty}^{d+i\infty} e^{px+qy}\frac{1}{1-\lambda}\frac{1}{pq+k(p+q)-k}\,\mathrm{d}p\,\mathrm{d}q$$

$$= \frac{1}{1-\lambda}\frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty}\left\{\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty}\frac{e^{px}}{pq+k(p+q)-k}dp\right\}e^{qy}\,\mathrm{d}q$$

$$= \frac{1}{1-\lambda}\frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty}\frac{e^{-\frac{kx(q-1)}{q+k}}}{q+k}e^{qy}\,\mathrm{d}q = \frac{1}{1-\lambda}e^{-\frac{\lambda(x+y)}{1-\lambda}}I_0\left(\frac{2\sqrt{\lambda xy}}{1-\lambda}\right).$$

2. Taking two-dimensional Laplace transform of the left hand side, leads to the following

$$L_2\left[\sum_{n=0}^{\infty}L_n(t)L_n(\xi),p,q\right] = \int_0^\infty\int_0^\infty\left(\sum_{n=0}^{\infty}L_n(t)L_n(\xi)e^{-pt-q\xi}\right)\mathrm{d}t\,\mathrm{d}\xi.$$

Changing the order of summation and double integration to get,

$$L_2\left[\sum_{n=0}^{\infty}L_n(t)L_n(\xi),p,q\right] = \sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty L_n(t)L_n(\xi)e^{-pt-q\xi}\,\mathrm{d}t\,\mathrm{d}\xi.$$

It is not difficult to show that the value of the inner integral is

$$\int_0^\infty\int_0^\infty L_n(t)L_n(\xi)e^{-pt-q\xi}\,\mathrm{d}t\,\mathrm{d}\xi = \frac{1}{pq}\left(1-\frac{1}{p}\right)^n\left(1-\frac{1}{q}\right)^n$$

and

$$\sum_{n=0}^{\infty}\frac{1}{pq}\left(1-\frac{1}{p}\right)^n\left(1-\frac{1}{q}\right)^n = \frac{1}{p+q-1}.$$

Using complex inversion formula for two-dimensional Laplace transforms to obtain,

$$\sum_{n=0}^{\infty}L_n(t)L_n(\xi) = \left(\frac{1}{2i\pi}\right)^2\int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty}\frac{e^{pt+q\xi}}{p+q-1}\,\mathrm{d}p\,\mathrm{d}q.$$

The above double integral may be re-written as follows,

$$\sum_{n=0}^{\infty}L_n(t)L_n(\xi) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{q\xi}\left\{\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{e^{pt}}{p-(1-q)}\,\mathrm{d}p\right\}\mathrm{d}q.$$

The value of the inner integral by residue theorem is equal to $e^{(1-q)t}$, upon substitution of this value in double integral we get,

$$\sum_{n=0}^{\infty}L_n(t)L_n(\xi) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{q\xi}e^{(1-q)t}\,\mathrm{d}q = e^t\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{-q(t-\xi)}\,\mathrm{d}q,$$

therefore

$$\sum_{n=0}^{\infty}L_n(t)L_n(\xi) = e^t\delta(t-\xi).$$

3. Solution to second-order linear partial differential equations with constant coefficients

The general form of second-order linear partial differential equation in two independent variables is given by (see [5]).

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = q(x, y), \quad 0 < x, y < \infty, \tag{3.1}$$

where A, B, C, D, E and F are given constant and $q(x, y)$ is source function of x and y or constant. We will use the following for the rest of this section (see [5, 6]). If

$$\begin{aligned} &u(x, 0) = f(x), \quad u(0, y) = g(y), \quad u_y(x, 0) = f_1(x), \\ &u_x(0, y) = g_1(y), \quad u(0, 0) = u_0 \end{aligned} \tag{3.2}$$

and if their one-dimensional Laplace transformations are $F(u),\ G(v),\ F_1(u)$ and $G_1(v)$, respectively, then

$$\begin{aligned} &L_2[u(x, y); u, v] = \int_0^\infty \int_0^\infty u(x, t)e^{-ux-vt}\, \mathrm{d}x\, \mathrm{d}t = U(u, v), \\ &L_2[u_{xx}; u, v] = u^2U(u, v) - uG(v) - G_1(v), \\ &L_2[u_{xy}; u, v] = uvU(u, v) - uF(u) - vG(v) - u(0, 0), \\ &L_2[u_{yy}; u, v] = v^2U(u, v) - uF(u) - F_1(u), \\ &L_2[u_x; u, v] = uU(u, v) - G(v), \\ &L_2[u_y; u, v] = vU(u, v) - F(u). \end{aligned} \tag{3.3}$$

Applying double Laplace transformation term wise to partial differential equations and the initial-boundary conditions in (3.2) and using (3.3), we obtain the transformed problem

$$\begin{aligned} U(u, v) = &\frac{1}{Au^2 + Cv^2 + Buv + Ev + Du + F}\{A(uG(v) + G_1(v)) \\ &+ B(uF(u) + vG(v) - u_0) + C(vF(u) + F_1(u)) \\ &+ DG(v) + EF(u) + Q(u, v)\}. \end{aligned} \tag{3.4}$$

Now, in the following examples we illustrate the above method.

Example 3.1. Letting $A = B = C = 0$, we get

$$Du_x + Eu_y + Fu = q(x, y), \quad 0 < x, y < \infty, \quad (E/D > 0).$$

With initial boundary conditions

$$u(x, 0) = f(x), \quad u(0, y) = g(y),$$

application of the relationship (3.4) gives

$$U(u,v)=\frac{DG(v)+EF(u)+Q(u,v)}{Ev+Du+F}. \tag{3.5}$$

The inverse double Laplace transform of (3.5) leads to the formal solution

$$u(x,y)=e^{-\frac{F}{D}x}g\left(y-\frac{E}{D}x\right)+e^{-\frac{F}{E}y}f\left(x-\frac{D}{E}y\right)$$
$$+\begin{cases}\frac{1}{D}\int_0^x e^{-\frac{F}{D}\xi}q(x-\xi,y-\frac{E}{D}\xi)\,\mathrm{d}\xi, & \text{if } y>\frac{E}{D}x,\\ \frac{1}{E}\int_0^y e^{-\frac{F}{E}\eta}q(x-\frac{D}{E}\eta,y-\eta)\,\mathrm{d}\eta, & \text{if } y<\frac{E}{D}x.\end{cases}$$

Example 3.2. If $C=E=D=0$, $A=1$, $B=\alpha$, $F=\beta$, then (3.1) reduces to

$$u_{xx}+\alpha u_{xy}+\beta u=q(x,y),\quad 0<x,y<\infty.$$

With the following initial conditions

$$u(0,y)=g(y),\quad u_x(0,y)=g_1(y),\quad u(x,0)=0,\quad u(0,0)=u_0$$

we obtain

$$U(u,v)=\frac{1}{u^2+\alpha uv+\beta}\{uG(v)+G_1(v)+\alpha(vG(v)-u_0)+Q(u,v)\}. \tag{3.6}$$

The inverse double Laplace transform of (3.6) yields (see [7])

$$u(x,y)=L_2^{-1}[U(u,v)]=L_2^{-1}\left[\frac{Q(u,v)}{u^2+\alpha uv+\beta}\right]+L_2^{-1}\left[\frac{uG(v)}{u^2+\alpha uv+\beta}\right]$$
$$+L_2^{-1}\left[\frac{G_1(v)}{u^2+\alpha uv+\beta}\right]+\alpha L_2^{-1}\left[\frac{vG(v)}{u^2+\alpha uv+\beta}\right]+\alpha u_0 L_2^{-1}\left[\frac{1}{u^2+\alpha uv+\beta}\right]$$

or equivalently

$$u(x,y)=\int_0^x\int_0^\xi J_0\left(2\sqrt{\beta\eta(x-\xi)}\right)q(\xi-\eta,y-\alpha\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$+g(y-\alpha x)+\frac{1}{\alpha}\int_0^{\alpha x}\sqrt{\frac{\beta\eta}{\alpha x-\eta}}J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}(x-\frac{\eta}{\alpha})}\right)g(y-\eta)\,\mathrm{d}\eta$$
$$+\frac{1}{\alpha}\int_0^{\alpha x}J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}(x-\frac{\eta}{\alpha})}\right)g_1(y-\eta)\,\mathrm{d}\eta+g(y)-g(y-\alpha x)$$
$$+\frac{1}{\alpha}\int_0^{\alpha x}\sqrt{\frac{\beta\eta}{\alpha x-\eta}}\left(2-\frac{\alpha x}{\eta}\right)J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}(x-\frac{\eta}{\alpha})}\right)g(y-\eta)\,\mathrm{d}\eta$$
$$+\begin{cases}0, & \text{if } y>\alpha x,\\ \alpha u_0 J_0(\frac{2}{\alpha}\sqrt{\beta y(\alpha x-\eta)}), & \text{if } y<\alpha x.\end{cases}$$

4. Conclusions

The multi-dimensional Laplace transform provides powerful method for analyzing linear systems. It is heavily used in solving differential and integral equations. The main purpose of this work is to develop a method of computing Laplace transform pairs of N-dimensions from known one-Dimensional Laplace transform and making continuous effort in expanding the transform tables and in designing algorithms for generating new inverses and direct transform from known ones. It is clear that the theorems of that type described here can be further generated for other type of functions and relations. These relations can be used to calculate new Laplace transform pairs.

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