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Further generalizations of the Fibonacci-coefficient polynomials*

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Abstract

The aim of this paper is to investigate the zeros of the general polynomials

$$q_n^{(i,t)}(x) = \sum_{k=0}^{n} R_{i+kt}x^{n-k} = R_i x^n + R_{i+t}x^{n-1} + \cdots + R_{i+(n-1)t}x + R_{i+nt},$$

where $i \geqslant 1$ and $t \geqslant 1$ are fixed integers.

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MSC: 11C08, 13B25

1. Introduction

The the second order linear recursive sequence

$$R = \{R_n\}_{n=0}^{\infty}$$

is defined by the following manner: let $R_0 = 0$, $R_1 = 1$, A and B be fixed positive integers. Then for $n \geqslant 2$

$$R_n = AR_{n-1} + BR_{n-2}. \tag{1.1}$$

According to the known Binet-formula, for $n \geqslant 0$

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

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where α and β are the zeros of the characteristic polynomial $x^2 - Ax - B$ of the sequence R. We can suppose that $\alpha > 0$ and $\beta < 0$.

In the special case $A = B = 1$ we can get the wellknown Fibonacci-sequence, that is, with the usual notation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 2).$$

According to D. Garth, D. Mills and P. Mitchell [1] the definition of the Fibonacci-coefficient polynomials $p_n(x)$ is the following:

$$p_n(x) = \sum_{k=0}^{n} F_{k+1}x^{n-k} = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}.$$

In [3] we delt the zeros of the polynomials $q_n(x)$, where

$$q_n(x) = \sum_{k=0}^{n} R_{k+1}x^{n-k} = R_1x^n + R_2x^{n-1} + \cdots + R_nx + R_{n+1},$$

that is, our results concerned to a family of the linear recursive sequences of second order.

The aim of this revisit of the theme is to investigate the zeros of the much more general polynomials $q_n^{(i)}(x)$ and $q_n^{(i,t)}(x)$, where $i \geqslant 1$ and $t \geqslant 1$ are fixed integers:

$$q_n^{(i)}(x) = \sum_{k=0}^{n} R_{i+k}x^{n-k} = R_ix^n + R_{i+1}x^{n-1} + \cdots + R_{i+n-1}x + R_{i+n}, \tag{1.2}$$

$$q_n^{(i,t)}(x) = \sum_{k=0}^{n} R_{i+kt}x^{n-k} = R_ix^n + R_{i+t}x^{n-1} + R_{i+2t}x^{n-2} \cdots + R_{i+(n-1)t}x + R_{i+nt}.$$

2. Preliminary and known results

At first we mention that the polynomials $q_n^{(i)}(x)$ can easily be rewritten in a recursive manner. That is, if $q_0^{(i)}(x) = R_i$ then for $n \geqslant 1$

$$q_n^{(i)}(x) = xq_{n-1}^{(i)}(x) + R_{i+n}.$$

We need the following three lemmas:

Lemma 2.1. *For $n \geqslant 1$ let $g_n^{(i)}(x) = (x^2 - Ax - B)q_n^{(i)}(x)$. Then*

$$g_n^{(i)}(x) = R_ix^{n+2} + BR_{i-1}x^{n+1} - R_{i+n+1}x - BR_{i+n}.$$

Proof. Using (1.2) we get $q_1^{(i)}(x) = R_ix + R_{i+1}$ and by (1.1) $g_1^{(i)}(x) = (x^2 - Ax - B)q_1^{(i)}(x) = (x^2 - Ax - B)(R_ix + R_{i+1}) = \cdots = R_ix^3 + BR_{i-1}x^2 - R_{i+2}x - BR_{i+1}$.

Continuing the proof with induction on n, we suppose that the statement is true for $n-1$ and we prove it for n. Applying (1.2) and (1.1), after some numerical calculations one can get that

$$\begin{aligned} g_n^{(i)}(x) &= (x^2 - Ax - B)q_n^{(i)}(x) \\ &= xg_{n-1}^{(i)}(x) + (x^2 - Ax - B)R_{i+n} = \cdots \\ &= R_i x^{n+2} + BR_{i-1}x^{n+1} - R_{i+n+1}x - BR_{i+n}. \end{aligned}$$

□

Lemma 2.2. *If every coefficients of the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ are positive numbers and the roots of equation $f(x) = 0$ are denoted by $z_1, z_2, \ldots, z_n$, then*

$$\gamma \leqslant |z_i| \leqslant \delta$$

hold for every $1 \leqslant i \leqslant n$, where γ is the minimal, while δ is the maximal value in the sequence

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}.$$

Proof. Lemma 2.2 is known as theorem of S. Kakeya [4]. □

Lemma 2.3. *Let us consider the sequence R defined by* (1.1)*. The increasing order of the elements of the set*

$$\left\{ \frac{R_{j+1}}{R_j} \;:\; 1 \leqslant j \leqslant n \right\}$$

is

$$\frac{R_2}{R_1}, \frac{R_4}{R_3}, \frac{R_6}{R_5}, \ldots, \frac{R_7}{R_6}, \frac{R_5}{R_4}, \frac{R_3}{R_2}.$$

Proof. Lemma 2.3 can be found in [2]. □

3. Results and proofs

At first we deal with the number of the real zeros of the polynomial $q_n^{(i)}(x)$ defined in (1.2), that is

$$q_n(^{(i)}x) = \sum_{k=0}^{n} R_{i+k}x^{n-k} = R_ix^n + R_{i+1}x^{n-1} + \cdots + R_{i+n-1}x + R_{i+n}.$$

Theorem 3.1. a) *If $n \geqslant 2$ and even, then the polynomial $q_n^{(1)}(x)$ has not any real zero, while if $i \geqslant 2$ then $q_n^{(i)}(x)$ has no one or has two negative real zeros, that is, every zeros – except at most two – are non-real complex numbers.*

b) *If $n \geqslant 3$ and odd, then the polynomial $q_n^{(i)}(x)$ has only one real zero and this is negative. That is, every but one zeros are non-real complex numbers.*

Proof. Because of the definition (1.1) of the sequence R the coefficients of the polynomials $q_n^{(i)}(x)$ are positive ones, thus positive real root of the equation $q_n^{(i)}(x) = 0$ does not exist. That is, it is enough to deal with only the existence of negative roots of the equation $q_n^{(i)}(x) = 0$. a) Since n is even, the coefficients of the polynomial

$$\begin{aligned} g_n^{(i)}(-x) &= R_i(-x)^{n+2} + BR_{i-1}(-x)^{n+1} - R_{i+n-1}(-x) - BR_{i+n} \\ &= R_i x^{n+2} - BR_{i-1}x^{n+1} + R_{i+n-1}x - BR_{i+n} \end{aligned}$$

has only one change of sign if $i = 1$, thus according to the Descartes' rule of signs, the polynomial $g_n^{(i)}(x)$ has exactly one negative real zero. But $g_n^{(i)}(x) = (x^2 - Ax - B)q_n^{(i)}(x)$ implies that $g_n^{(i)}(\beta) = 0$, where $\beta < 0$, and so the polynomial $q_n^{(i)}(x)$ can not have any negative real zero if $i = 1$. But in the case $i \geqslant 2$ the polynomial $g_n^{(i)}(-x)$ has 3 changes of sign, that is, $q_n^{(i)}(x) = 0$ has no one or 2 negative roots.

b) Since $n \geqslant 3$ is odd, thus the existence of at least one negative real zero is obvious. We have only to prove that exactly one negative real zero exists. The polynomial

$$\begin{aligned} g_n^{(i)}(-x) &= R_i(-x)^{n+2} + BR_{i-1}(-x)^{n+1} - R_{i+n-1}(-x) - BR_{i+n} \\ &= -R_i x^{n+2} + BR_{i-1}x^{n+1} + R_{i+n-1}x - BR_{i+n} \end{aligned}$$

shows that among its coefficients there are two changes of signs, thus according to the Descartes' rule of signs, the polynomial $g_n^{(i)}(x)$ has either two negative real zeros or no one. But $g_n^{(i)}(x) = (x^2 - Ax - B)q_n^{(i)}(x)$ implies that for $\beta < 0$ $g_n^{(i)}(\beta) = 0$. Although, $g_n^{(i)}(\alpha) = 0$ also holds, but $\alpha > 0$. That is, an other negative real zero of $g_n^{(i)}(x)$ must exist. Because of $g_n^{(i)}(x) = (x^2 - Ax - B)q_n^{(i)}(x)$ this zero must be the zero of the polynomial $q_n^{(i)}(x)$.

This terminated the proof of the theorem. □

Remark 3.2. Some numerical examples imply the conjection that if n is even and $i \geqslant 2$ then $q_n^{(i)}(x)$ has no negative real root.

In the following part of this note we deal with the localization of the zeros of the polynomials

$$q_n^{(i)}(x) = \sum_{k=0}^{n} R_{i+k}x^{n-k} = R_i x^n + R_{i+1}x^{n-1} + \dots + R_{i+n-1}x + R_{i+n}.$$

Theorem 3.3. *Let $z \in \mathbb{C}$ denote an arbitrary zero of the polynomial $q_n^{(i)}(x)$ if $n \geqslant 1$. Then*

$$\frac{R_{i+1}}{R_i} \leqslant |z| \leqslant \frac{R_{i+2}}{R_{i+1}},$$

if i is odd, while

$$\frac{R_{i+2}}{R_{i+1}} \leqslant |z| \leqslant \frac{R_{i+1}}{R_i},$$

if i is even.

Proof. To apply Lemma 2.2 for the polynomial $q_n^{(i)}(x)$ we have to determine the minimal and maximal values in the sequence

$$\frac{R_{i+n}}{R_{i+n-1}}, \frac{R_{i+n-1}}{R_{i+n-2}}, \dots, \frac{R_{i+1}}{R_i}.$$

Applying Lemma 2.3, one can get the above stated bounds. □

Remark 3.4. Even more there is an other possibility for further generalization. Let $i \geqslant 1$ and $t \geqslant 1$ be fixed integers.

$$q_n^{(i,t)}(x) := \sum_{k=0}^{n} R_{i+kt}x^{n-k} = R_i x^n + R_{i+t}x^{n-1} + R_{i+2t}x^{n-2} \cdots + R_{i+(n-1)t}x + R_{i+nt}.$$

The following recursive relation also holds if $q_0^{(i,t)}(x) = R_i$ then for $n \geqslant 1$

$$q_n^{(i,t)}(x) = xq_{n-1}^{(i,t)}(x) + R_{i+nt}.$$

Using similar methods for the set

$$\left\{ \frac{R_{i+jt}}{R_{i+(j-1)t}} \ : \ 1 \leqslant j \leqslant n \right\}$$

it can be proven that for any zero z of $q_n^{(i,t)}(x) = 0$:
if i and t are odd then:

$$\frac{R_{i+t}}{R_i} \leqslant |z| \leqslant \frac{R_{i+2t}}{R_{i+t}},$$

if i is even and t is odd then:

$$\frac{R_{i+2t}}{R_{i+t}} \leqslant |z| \leqslant \frac{R_{i+t}}{R_i},$$

if i and t are even then:

$$\frac{R_{i+nt}}{R_{i+(n-1)t}} \leqslant |z| \leqslant \frac{R_{i+t}}{R_i},$$

if i is odd and t is even then:

$$\frac{R_{i+t}}{R_i} \leqslant |z| \leqslant \frac{R_{i+nt}}{R_{i+(n-1)t}}.$$

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