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On positive integers with a certain nondivisibility property*

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Abstract

For a positive integer $k \geqslant 3$ let $(u_m^{(k)})_{m \geqslant 0}$ be the Lucas sequence given by $u_0^{(k)} = 0$, $u_1^{(k)} = 1$ and $u_{m+2}^{(k)} = k u_{m+1}^{(k)} - u_m^{(k)}$ for all $m \geqslant 0$. In this paper, we study the positive integers n such that

$$\frac{n-k}{1+(k-2)(u_m^{(k)})^2} \notin \mathbb{Z} \qquad \text{for any } 3 \leqslant k < n \text{ and } m \geqslant 1.$$

Keywords: Diophantine Equations, Primes, Euler Function, Fibonacci Numbers

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1. Introduction

For a positive integer $k \geqslant 3$ let $(u_m^{(k)})_{m \geqslant 0}$ be the Lucas sequence given by $u_0^{(k)} = 0$, $u_1^{(k)} = 1$ and $u_{m+2}^{(k)} = k u_{m+1}^{(k)} - u_m^{(k)}$ for all $m \geqslant 0$. In this paper, we study the positive integers n such that

$$\frac{n-k}{1+(k-2)(u_m^{(k)})^2} \notin \mathbb{Z} \qquad \text{for any } 3 \leqslant k < n \text{ and } m \geqslant 1. \tag{1.1}$$

Let $\mathcal{N}$ be the set of positive integers satisfying property (1.1). The study of this set of integers is motivated by the study of the solutions of the Diophantine equation

$$x_1^2 + \cdots + x_n^2 = y x_1 \cdots x_n, \quad n \geqslant 3, \tag{1.2}$$

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in positive integers $x_1, \dots, x_n, y$. Hurwitz [5], proved that the Diophantine equation (1.2) has no solutions with $y > n$ and has infinitely many solutions with $y = n$. Herzberg [4], showed that there are only 15 values of $n \leqslant 301020$ for which (1.2) has no solutions with $y < n$. In particular, for any $2688 < n \leqslant 301020$, equation (1.2) has solutions with $y < n$. Using Herzberg's algorithm, we checked all $n \leqslant 10^8$ and didn't find any other exceptional values. It is conjectured that for a sufficiently large n, equation (1.2) has a solution with $y < n$. Let us remark that Hurwitz's results yield that $(u_{m+1}^{(k)} - u_m^{(k)}, u_m^{(k)} - u_{m-1}^{(k)}, \underbrace{1, \dots, 1}_{k-2}, k)$ is a solution of the equation

$$y_1^2 + \cdots + y_k^2 = z y_1 \cdots y_k$$

for any $k \geqslant 3$ and $m \geqslant 1$. It is easy to check that

$$(u_{m+1}^{(k)} - u_m^{(k)})(u_m^{(k)} - u_{m-1}^{(k)}) = 1 + (k-2)(u_m^{(k)})^2.$$

Hence, if for a given n there exist $3 \leqslant k < n$ and $m \geqslant 1$ such that $\dfrac{n-k}{1+(k-2)(u_m^{(k)})^2}$ is an integer, then $(u_{m+1}^{(k)} - u_m^{(k)}, u_m^{(k)} - u_{m-1}^{(k)}, \underbrace{1, \dots, 1}_{n-2}, y)$ is a solution of (1.2), where

$$\begin{aligned} y &= \frac{(u_{m+1}^{(k)} - u_m^{(k)})^2 + (u_m^{(k)} - u_{m-1}^{(k)})^2 + k - 2}{(u_{m+1}^{(k)} - u_m^{(k)})(u_m^{(k)} - u_{m-1}^{(k)})} + \frac{n-k}{1+(k-2)(u_m^{(k)})^2} \\ &= k + \frac{n-k}{1+(k-2)(u_m^{(k)})^2} < n. \end{aligned}$$

In particular, if for any sufficiently large n we could find such values of k and m, then the conjecture would follow. Unfortunately, there are infinitely many values of n which are in the set $\mathcal{N}$, and this is the content of our paper.

2. Result

Our precise result is the following. For a set $\mathcal{A}$ of positive integers and a positive real number x let $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$.

Theorem 2.1. *There exists x_0 such that $\#\mathcal{N}(x) > 0.09x/\log x$ for $x > x_0$.*

For the proof, we will need the following lemma. For a positive integer m let $\phi(m)$ denote the Euler function of m.

Lemma 2.2. *We have the estimate*

$$S = \sum_{k \geqslant 3} \sum_{m \geqslant 2} \frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)} < 0.91. \tag{2.1}$$

Proof. Let $\omega(m)$ be the number of distinct prime factors of the positive integer m. Thus, if $p_1 < p_2 < \cdots < p_{\omega(m)}$ denote all the prime factors of $m > 1$, then

$$\frac{\phi(m)}{m} = \prod_{i=1}^{\omega(m)} \left(1 - \frac{1}{p_i}\right) \geqslant \prod_{i=1}^{\omega(m)} \left(1 - \frac{1}{i+1}\right) = \frac{1}{\omega(m)+1}.$$

From here, we can deduce various things. For example, since $m \geqslant 2^{\omega(m)}$, we get that $\omega(m) \leqslant (\log m)/(\log 2)$, therefore the above inequality gives

$$\frac{\phi(m)}{m} \geqslant \frac{1}{(\log m)/(\log 2)+1} = \frac{\log 2}{\log(2m)}. \tag{2.2}$$

Then

$$\frac{1}{\phi(m)} \leqslant \frac{\log(2m)}{m \log 2}.$$

Applying this to $1+(k-2)(u_m^{(k)})^2$, we get

$$\frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)} \leqslant \frac{\log(2(1+(k-2)(u_m^{(k)})^2))}{(\log 2)(1+(k-2)(u_m^{(k)})^2)}.$$

For $m \geqslant 2$ and $k \geqslant 3$ we have that

$$1+(k-2)(u_m^{(k)})^2 \geqslant 1+(k-2)(u_2^{(k)})^2 \geqslant 1+(k-2)k^2 \geqslant 10,$$

and the function $\log(2t)/t$ is decreasing for $t \geqslant 2$. So, we need a lower bound on $1+(k-2)(u_m^{(k)})^2$.

It is well-known and easy to prove that if we write

$$\alpha_k = \frac{k+\sqrt{k^2-4}}{2} \qquad \text{and} \qquad \beta_k = \frac{k-\sqrt{k^2-4}}{2}$$

for the two roots of the quadratic equation $x^2 - kx + 1 = 0$, then

$$u_m^{(k)} = \frac{\alpha_k^m - \beta_k^m}{\alpha_k - \beta_k}.$$

Note that $\alpha_k - \beta_k = \sqrt{k^2-4}$ and $\alpha_k \beta_k = 1$. Hence,

$$\begin{aligned} 1+(k-2)(u_m^{(k)})^2 &= 1 + \frac{k-2}{(\alpha_k - \beta_k)^2}\left(\alpha_k^{2m} + \beta_k^{2m} - 2\right) \\ &> 1 + \frac{1}{k+2}\left(\alpha_k^{2m} - 2\right) = \frac{\alpha_k^{2m}+k}{k+2} > \frac{\alpha_k^{2m}}{k+2} \\ &> \frac{(k^2-4)^m}{k+2} = (k-2)^m (k+2)^{m-1}. \end{aligned} \tag{2.3}$$

Note that for $k \geqslant 3$ and $m \geqslant 2$ we have that $(k-2)^m(k+2)^{m-1} \geqslant 5$. Thus,

$$\begin{aligned}
\frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)} &< \frac{\log(2(k-2)^m(k+2)^{m-1})}{(\log 2)(k-2)^m(k+2)^{m-1}} \\
&= \frac{1}{(k-2)^m(k+2)^{m-1}} \\
&+ \frac{m\log(k-2)}{(\log 2)(k-2)^m(k+2)^{m-1}} \\
&+ \frac{(m-1)\log(k+2)}{(\log 2)(k-2)^m(k+2)^{m-1}}.
\end{aligned}$$

We shall apply the above inequality for all $k \geqslant 4$. The case $k=3$ is special since in this case $u_m^{(2)} = F_{2m}$ for all $n \geqslant 1$, where $(F_m)_{m\geqslant 0}$ denotes the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geqslant 0$. Thus,

$$1 + (u_m^{(2)})^2 = 1 + F_{2m}^2 = F_{2m+1}F_{2m-1},$$

therefore

$$\phi(1+(u_m^{(2)})^2) = \phi(F_{2m+1}F_{2m-1}) = \phi(F_{2m+1})\phi(F_{2m-1}),$$

where the last relation holds because F_{2m+1} and F_{2m-1} are coprime. Summing up over all $m \geqslant 2$ and $k \geqslant 3$, we find that

$$S < S_0 + S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
S_0 &= \sum_{m\geqslant 2} \frac{1}{\phi(F_{2m+1})\phi(F_{2m-1})}, \\
S_1 &= \sum_{k\geqslant 4}\sum_{m\geqslant 2} \frac{1}{(k-2)^m(k+2)^{m-1}}, \\
S_2 &= \sum_{k\geqslant 4}\sum_{m\geqslant 2} \frac{m\log(k-2)}{(\log 2)(k-2)^m(k+2)^{m-1}}, \\
S_3 &= \sum_{k\geqslant 4}\sum_{m\geqslant 2} \frac{(m-1)\log(k+2)}{(\log 2)(k-2)^m(k+2)^{m-1}}.
\end{aligned}$$

We now compute the four sums above. We computed,

$$S_0 < 0.277.$$

To deduce this inequality, we first computed the first 100 terms in S_0 getting an answer < 0.2769. For $n \geqslant 199$, we have $\phi(n) \geqslant 48$. Indeed, to see this note first that

$$\phi(n) \geqslant \frac{n\log 2}{\log(2n)} > 48,$$

where the inequality on the left holds always by inequality (2.2) and the inequality on the right holds for all $n \geqslant 500$. For $n \in [199, 500]$, we checked that the minimal value of the Euler function is 48. Next recall that a result of Luca [6] says that

$$\phi(F_n) \geqslant F_{\phi(n)}$$

holds for all n. In particular,

$$\frac{1}{\phi(F_{2n+1})\phi(F_{2n-1})} \leqslant \frac{1}{F_{\phi(2n+1)}F_{\phi(2n-1)}} \leqslant \frac{1}{\alpha^{\phi(2n+1)+\phi(2n-1)-4}},$$

where we use $\alpha = (1+\sqrt{5})/2$ together with the fact that the inequality $F_n \geqslant \alpha^{n-2}$ holds for all $n \geqslant 2$. Let $m = \phi(2n+1) + \phi(2n-1) - 4$. Since $n \geqslant 100$, we have that $2n - 1 \geqslant 199$, and so $m \geqslant 92$. Clearly,

$$4n - 4 > m > \frac{(2n+1)\log 2}{\log(4n+2)} + \frac{(2n-1)\log 2}{\log(4n-2)} - 4.$$

We checked that the square of the above lower bound is larger than the upper bound for all $n \geqslant 21$, which is our case. This implies that the number of n such that $\phi(2n+1) + \phi(2n-1) - 4 = m$ does not exceed m^2 for n in our range. Note that m is even. To summarize,

$$S_0 \leqslant \sum_{n=1}^{100} \frac{1}{\phi(F_{2n-1})\phi(F_{2n+1})} + \sum_{\ell \geqslant 46} \frac{4\ell^2}{\alpha^{2\ell}}.$$

For $\ell \geqslant 12$, we have that $\alpha^{\ell} \geqslant 4\ell^2$. Thus,

$$S_0 < \sum_{n=1}^{100} \frac{1}{\phi(F_{2n-1})\phi(F_{2n+1})} + \sum_{\ell \geqslant 46} \frac{1}{\alpha^{\ell}}$$

Thus, the error in approximating S_0 by its first 100 terms is

$$< \sum_{\ell \geqslant 46} \frac{1}{\alpha^{\ell}} = \frac{1}{\alpha^{45}(\alpha - 1)} < 10^{-9}.$$

So, indeed $S_0 < 0.277$. Next,

$$S_1 = \sum_{k \geqslant 4} \frac{1}{(k-2)} \sum_{m \geqslant 1} \frac{1}{(k^2-4)^m} = \sum_{k \geqslant 4} \frac{1}{(k-2)(k^2-5)} < 0.0861.$$

Further,

$$S_2 = \sum_{k \geqslant 4} \frac{\log(k-2)}{(\log 2)(k-2)} \sum_{m \geqslant 1} \frac{m+1}{(k^2-4)^m} < \sum_{k \geqslant 4} \frac{2(k+2)\log(k-2)}{(\log 2)(k^2-5)^2} < 0.2845.$$

Finally,

$$S_3 = \sum_{k\geqslant 4} \frac{\log(k+2)}{(\log 2)(k-2)} \sum_{m\geqslant 1} \frac{m}{(k^2-4)^m} = \sum_{k\geqslant 4} \frac{(k+2)\log(k+2)}{(\log 2)(k^2-5)^2} < 0.2607.$$

The upper bounds on S_1, S_2, S_3 were computed with Mathematica. We shall justify only S_1. Clearly,

$$\sum_{m\geqslant 1} \frac{1}{(k^2-4)^m} = \frac{1}{(k^2-4)} \cdot \frac{1}{1-\frac{1}{(k^2-4)}} = \frac{1}{(k^2-5)}.$$

With Mathematica, we obtained that

$$\sum_{k=4}^{1003} \frac{1}{(k-2)(k^2-5)} < 0.08607,$$

while certainly

$$\begin{aligned}\sum_{k>1003} \frac{1}{(k-2)(k^2-5)} &< \sum_{k>1003} \frac{1}{(k-2)^3} = \sum_{k>1001} \frac{1}{k^3} < \int_{1000}^{\infty} \frac{dt}{t^3} \\ &= -\frac{1}{2t^2}\Big|_{t=1000}^{t=\infty} = \frac{1}{2\cdot 10^6} < 0.00001,\end{aligned}$$

which together imply that $S_1 < 0.0861$, as claimed. A similar argument can be used to justify the bounds on S_2 and S_3. Hence,

$$S < 0.277 + 0.0861 + 0.2845 + 0.2607 = 0.9083 < 0.91,$$

which completes the proof of the lemma. □

Proof of Theorem 2.1. Assume that relation (1.1) does not hold with $m = 1$. Then we get that $(n-1)/(k-1)$ is an integer for some $3 \leqslant k < n$, and this certainly is the case for some k if $n-1$ is not a prime. From now on, we fix a large positive real number x and we look only at numbers $n \leqslant x$ such that $n-1$ is prime and relation (1.1) is not satisfied for some $3 \leqslant k < n$ and $m \geqslant 2$. Then

$$n - 1 \equiv k - 1 \pmod{1 + (k-2)(u_m^{(k)})^2}.$$

Since $k < n$, it follows that $n-1 = (k-1) + \ell(1+(k-2)(u_m^{(k)})^2)$ for some positive integer ℓ, therefore $1 + (k-2)(u_m^{(k)})^2 < x$. Since $m \geqslant 2$, it follows that

$$x > 1 + (k-2)(u_m^{(k)})^2 \geqslant (k-2)^m(k+2)^{m-1} \geqslant \max\{(k-2)^2(k+2), 5^{m-1}\}$$

(see estimate (2.3)), leading to $k = O(x^{1/3})$ and $m = O(\log x)$. So, there are only $O(x^{1/3}\log x)$ such pairs (k, m). We may further assume that $k-1$ is coprime to $1 + (k-2)(u_m^{(k)})^2$, for if not any common prime factor q of these two integers will be $\leqslant k-1 < n-1$ and will divide $n-1$, which is impossible. For positive coprime integers a and b we write $\pi(x; a, b)$ for the number of primes $p \leqslant x$ which are congruent to $a \pmod b$ and we write $\pi(x)$ for the total number of prime numbers $p \leqslant x$. It then follows that the number of positive integers $n \leqslant x$ satisfying (1.1) for any $k \geqslant 3$ and $m \geqslant 1$ is

$$\#\mathcal{N}(x) \geqslant \pi(x-1) - \sum_{\substack{(k,m) \\ 1+(k-2)(u_m^{(k)})^2 < x}} \pi(x; k-1, 1+(k-2)(u_m^{(k)})^2). \qquad (2.4)$$

Thus, it suffices to show that the above expression exceeds $0.09x/\log x$ for all sufficiently large x.

Let x be large. We split the set of pairs (k, m) with $1 + (k-2)(u_m^{(k)})^2 < x$ in three subsets as follows:

(i) $\mathcal{S}_1 = \{(k, m) : 1 + (k-2)(u_m^{(k)})^2 < (\log x)^{10}\}$;

(ii) $\mathcal{S}_2 = \{(k, m) : (\log x)^{10} \leqslant 1 + (k-2)(u_m^{(k)})^2 < x^{1/2}\}$;

(iii) $\mathcal{S}_3 = \{(k, m) : x^{1/2} \leqslant 1 + (k-2)(u_m^{(k)})^2 < x\}$.

If $(k, m) \in \mathcal{S}_1$, then, by the Siegel-Walfiz theorem (see, for example, page 133 in [1]), we have that

$$\pi(x; k-1, 1+(k-2)(u_m^{(k)})^2) = \frac{\pi(x)}{\phi(1+(k-2)(u_m^{(k)})^2)} + O\left(\frac{x}{\exp(A\sqrt{\log x})}\right)$$

for some positive constant A. Note further that since for $(k, m) \in \mathcal{S}_1$ we have that

$$(\log x)^{10} \geqslant 1 + (k-2)(u_m^{(k)})^2 \geqslant \max\{(k-2)^2(k+2), 5^{m-1}\},$$

we get $k \ll (\log x)^{10/3}$ and $m \ll \log\log x \ll (\log x)^{2/3}$, therefore

$$\#\mathcal{S}_1 \ll (\log x)^4.$$

If $(k, m) \in \mathcal{S}_2$, then by the Brun-Titchmarsh theorem (see, for example, [2, Section 2.3.1, Theorem 1] or [3, Chapter 3, Theorem 3.7]), we have that

$$\begin{aligned}\pi(x; k-1, 1+(k-2)(u_m^{(k)})^2) &\ll \frac{x}{\phi(1+(k-2)(u_m^{(k)})^2)\log\left(\frac{x}{1+(k-2)(u_m^{(k)})^2}\right)} \\ &\ll \frac{\pi(x)}{\phi(1+(k-2)(u_m^{(k)})^2)},\end{aligned}$$

where we used the fact that

$$\log\left(\frac{x}{1+(k-2)(u_m^{(k)})^2}\right) \geqslant \log(x^{1/2}) = \frac{\log x}{2},$$

as well as the Prime Number Theorem.

Finally, if $(k,m) \in \mathcal{S}_3$, then

$$\pi(x; k-1, 1+(k-2)(u_m^{(k)})^2) \leqslant \frac{x}{1+(k-2)(u_m^{(k)})^2} + 1 \ll x^{1/2}.$$

Putting everything together, we get that

$$\sum_{\substack{(k,m)\\ 1+(k-2)(u_m^{(k)})^2<x}} \pi(x; k-1, 1+(k-2)(u_m^{(k)})^2) \leqslant \pi(x) \sum_{(k,m)\in\mathcal{S}_1} \frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)}$$
$$+ O\left(\frac{x(\log x)^4}{\exp(A\sqrt{\log x})} + \sum_{(k,m)\in\mathcal{S}_2} \frac{\pi(x)}{\phi(1+(k-2)(u_m^{(k)})^2)} + x^{1/2}\#\mathcal{S}_3\right).$$

Note that $\#\mathcal{S}_3 \ll x^{1/3}\log x$, and by the Prime Number Theorem, we have

$$\frac{x(\log x)^4}{\exp(A\sqrt{\log x})} = o(\pi(x))$$

as $x \to \infty$. Since the series (2.1) sums to S, it follows that both estimates

$$\pi(x) \sum_{(k,m)\in\mathcal{S}_2} \frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)} = o(\pi(x))$$
$$\pi(x) \sum_{(k,m)\in\mathcal{S}_1} \frac{1}{\phi(1+(k-2)(u_m^{(k)})^2)} = S\pi(x) + o(\pi(x))$$

hold as $x \to \infty$. Thus,

$$\sum_{\substack{(k,m)\\ 1+(k-2)(u_m^{(k)})^2<x}} \pi(x; k-1, 1+(k-2)(u_m^{(k)})^2) \leqslant \pi(x)(S+o(1)),$$

which together with estimate (2.4) and Lemma 2.2 implies the conclusion of the theorem. □

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