Annales Mathematicae et Informaticae
33 (2006) pp. 69–75
http://www.ektf.hu/tanszek/matematika/ami

Ljunggren's Diophantine problem connected with virus structure

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Submitted 5 March 2006; Accepted 22 December 2006

Abstract

In this paper we give an effective method for determination of all solutions of the Ljunggren's Diophantine equation

$$x^2 + 3y^2 + 12z = 4M, \tag{L}$$

in odd positive integers x, y and non-negative integers z, where $M = a^2 + ab + b^2$, $N = 10M + 2$ and a, b are given non-negative integers. Equation (L) is strictly connected with virus structure.

1. Introduction

In virology are known (see [3, pp. 171–200]) different groups of viruses. One of such groups has been found by Stoltz [5], [6] and by Wrigley [7], [8] and is called as symmetrons. Virus particles are invariably enclosed by shells of protein subunits and these are packed geometrically according to symmetry rules. More of known examples are close packed with each subunit surrounded by six neighbours, except the twelve vertices which have five neighbours. In the paper [1], Goldberg indicated that total number of nearly identical subunits which may be regularly packed on the closed icosahedral surface is given by the following formula:

$$N = 10\left(a^2 + ab + b^2\right) + 2, \tag{G}$$

where a, b are given non-negative integers.

Stoltz ([5], [6]) and Wrigley ([7], [8]) discovered that the symmetrons have the construction of linear, triangular and pentagonal and are called: disymmetrons, trisymmetrons and pentasymmetrons, respectively. Moreover, it is known [7], that an icosahedron has 30 axes of twofold symmetry, 20 of threefold symmetry and 12 of fivefold symmetry. Hence, the subunits on the surface of an icosahedral virus may be divided into 30, 20 or 12 corresponding previously listed groups symmetry. Let the 30 disymmetrons contain d_u subunits, the 20 trisymmetrons contain t_v subunits and the 12 pentasymmetrons contain p_w subunits, then we have

$$N = 30d_u + 20t_v + 12p_w, \tag{S-W}$$

where

$$d_u = u - 1, \quad t_v = \frac{(v-1)\,v}{2}, \quad p_w = \frac{5w\,(w-1)}{2} + 1 \tag{1.1}$$

and u, v, w are positive integers.

Now, we remark that for each value of N given by the equation (G) the number $f(N)$ of the solutions of the equation (S-W) corresponds to the number theoretically possible ways of making a virus with N subunits, but with different combinations of symmetrons.

Putting

$$x = 2v - 1, \quad y = 2w - 1, \quad z = u - 1, \quad N = 10M + 2, \quad M = a^2 + ab + b^2$$

and using (1.1) Ljunggren [2] transformed the equation (S-W) to the following form:

$$x^2 + 3y^2 + 12z = 4M. \tag{L}$$

Moreover, he proved that total number $f(N)$ of solutions of the Diophantine equation (L) is equal to

$$f(N) = \frac{\pi\sqrt{3}}{180}N + k_1\sqrt{N}, \tag{L$_1$}$$

where k_1 is bounded and is independent of N. From (L_1) immediately follows that

$$\lim_{N\to\infty} \frac{f(N)}{N} = \frac{\pi\sqrt{3}}{180} \approx 0.03.$$

Geometrically, the formula (L_1) denote that the points (x, y) satisfying of the equation (L) all lie in the neighbourhood of the two lines:

$$y = 0.03x, \quad y = 0.015x.$$

On page 54 of the paper [2] Ljunggren remarked (see [2, p. 54]) that the following problem is important for applications in virology:

Ljunggren's Problem. *Find all odd, positive integers x, y and all non-negative integers z satisfying the equation* (L) *for given non-negative integers values of a and b.*

In this paper we give an effective method for the solution of this Ljunggren's Problem.

2. Solution of the Ljunggren's Problem

The Diophantine equation (L) we can present in the following form

$$x^2 + 3y^2 = 4(M - 3z), \tag{2.1}$$

where $M = a^2 + ab + b^2$ and a, b are given non-negative integers. Since $x^2 + 3y^2 \geqslant 0$ and $z \geqslant 0$, then by (2.1) it follows that

$$0 \leqslant z \leqslant \frac{1}{3}M. \tag{2.2}$$

From (2.2) follows that there is only finite number of integers z satisfying (2.2), because for given non-negative a, b the number $M = a^2 + ab + b^2$ is fixed.

Now, let $z = z_0 \in \left[0, \frac{1}{3}M\right]$ and let

$$M_0 = M - 3z_0. \tag{2.3}$$

From (2.1) and (2.3) we have

$$x^2 + 3y^2 = 4M_0. \tag{2.4}$$

Since M_0 is non-negative integer then we can present this number in the form

$$M_0 = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \tag{2.5}$$

where $\alpha \geqslant 0$, $\alpha_j \geqslant 1$ are integers for $j = 1, 2, \ldots, r$ and p_j are odd distinct primes. Substituting (2.5) to (2.4) we obtain

$$x^2 + 3y^2 = 2^{\alpha+2} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \tag{2.6}$$

From (2.6) and well-known properties of the divisibility and congruence relations we get

$$p_j \mid x^2 + 3y^2 \;\Leftrightarrow\; x^2 \equiv -3y^2 \pmod{p_j}, \tag{2.7}$$

for each $j = 1, 2, \ldots, r$.

By (2.7) and the properties of the Legendre's symbol it follows that

$$\left(\frac{-3y^2}{p_j}\right) = \left(\frac{-3}{p_j}\right)\left(\frac{y^2}{p_j}\right) = \left(\frac{-3}{p_j}\right)\left(\frac{y}{p_j}\right)^2 = \left(\frac{-3}{p_j}\right) = +1.$$

It is to observe that the equality $\left(\frac{-3}{p_j}\right) = +1$, imply that for each $j = 1, 2, \ldots, r$ the prime p_j is the form $p_j = 6k_j + 1$.

Indeed, suppose that for some $j = 1, 2, \ldots, r$ the equality $\left(\frac{-3}{p_j}\right) = +1$, imply that $p_j \neq 6k_j + 1$. Since p_j is prime, then $p_j = 6m_j + 5$.

Hence, by well-known properties of Legendre's symbol it follows that

$$\left(\frac{-3}{p_j}\right)=\left(\frac{-1}{p_j}\right)\left(\frac{3}{p_j}\right)=(-1)^{\frac{p_j-1}{2}}\left(\frac{3}{p_j}\right). \tag{2.8}$$

On the other hand from the Gauss reciprocity law we have

$$\left(\frac{3}{p_j}\right)\left(\frac{p_j}{3}\right)=(-1)^{\frac{(3-1)(p_j-1)}{2}}=(-1)^{\frac{p_j-1}{2}}. \tag{2.9}$$

Since $p_j=6m_j+5$, then we have

$$\left(\frac{p_j}{3}\right)=\left(\frac{6m_j+5}{3}\right)=\left(\frac{2}{3}\right)=-1. \tag{2.10}$$

By (2.9) and (2.10) it follows that

$$\left(\frac{3}{p_j}\right)=(-1)^{\frac{p_j-1}{2}+1}. \tag{2.11}$$

From (2.11) and (2.8) follows that

$$\left(\frac{-3}{p_j}\right)=(-1)^{p_j}=(-1)^{6m_j+5}=-1, \tag{2.12}$$

so proves our assertion. This fact implies that every odd prime p_j of the right hand of (2.6) is the form

$$p_j=6k_j+1, \quad j=1,2,\ldots,r.$$

By the Theorem 5 of the monograph [4, p. 349] it follows that every prime p which is of the form $p=6k+1$ is of the form $p=m^2+3n^2$, where m,n are positive integers. Therefore, we have

$$p_j=x_j^2+3y_j^2, \quad \text{for every} \quad j=1,2,\ldots,r.$$

Now, we note that if the equation (2.6) has a solution in odd positive integers x,y then we have

$$2^{\alpha+2} \mid x^2+3y^2. \tag{2.13}$$

Since $x=2v-1$ and $y=2w-1$ then

$$x^2+3y^2=(2v-1)^2+3(2w-1)^2=4[v(v-1)+3w(w-1)+1]. \tag{2.14}$$

By (2.13) and (2.14) it follows that

$$2^{\alpha} \mid v\,(v-1)+3w\,(w-1)+1. \tag{2.15}$$

It is easy to see that the sum $v\,(v-1)+3w\,(w-1)+1$ is odd positive integer for any positive integers v,w and consequently the relation (2.15) is impossible for any positive integers $\alpha \geqslant 1$. Since $\alpha \geqslant 0$, then we have $\alpha = 0$ and the equation (2.6) reduces to the form:

$$x^2 + 3y^2 = 4p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \tag{2.16}$$

where

$$p_j = x_j^2 + 3y_j^2, \quad \alpha_j \geqslant 1, \quad j = 1, 2, \ldots, r. \tag{2.17}$$

The representation p_j in the form (2.17) is unique. This fact follows by the Theorem 10 on page 221 of [4].

Further, we note that the following identity is true,

$$\left(u^2 + 3v^2\right)\left(r^2 + 3s^2\right) = (ur - 3vs)^2 + 3\,(us + vr)^2. \tag{2.18}$$

From (2.18) and by uniqueness representation the prime number p_j in the form (2.17) it follows that

$$p_j^{\alpha_j} = \left(x_j^2 + 3y_j^2\right)^{\alpha_j} = R_j^2 + 3S_j^2, \quad j = 1, 2, \ldots, r, \tag{2.19}$$

where R_j, S_j are positive integers of different parity and representation (2.19) is also unique.

Moreover, we remark that we can determine R_j and S_j in explicit form. Namely, we have

$$R_j = \frac{\left(x_j + i\sqrt{3}y_j\right)^{\alpha_j} + \left(x_j - i\sqrt{3}y_j\right)^{\alpha_j}}{2}, \quad S_j = \frac{\left(x_j + i\sqrt{3}y_j\right)^{\alpha_j} - \left(x_j - i\sqrt{3}y_j\right)^{\alpha_j}}{i\sqrt{3}}, \tag{2.20}$$

for $j = 1, 2, \ldots, r$. By (2.19), (2.16) and (2.18) it follows that

$$x^2 + 3y^2 = 4\prod_{j=1}^{r}\left(R_j^2 + 3S_j^2\right) = 4\left(R^2 + 3S^2\right), \tag{2.21}$$

where R, S are positive integers of different parity and are effectively determined by (2.18), (2.20) and (2.21).

Now, we observe that

$$4 = 1^2 + 3 \times 1^2. \tag{2.22}$$

From (2.22) and (2.18) we get

$$4\left(R^2 + 3S^2\right) = \left(1^2 + 3 \times 1^2\right)\left(R^2 + 3S^2\right) = (R - 3S)^2 + 3\,(R + S)^2. \tag{2.23}$$

By (2.21) and (2.23) it follows that odd positive integers satisfy the following equation:

$$x^2 + 3y^2 = (R - 3S)^2 + 3\,(R + S)^2. \tag{2.24}$$

Immediately, from (2.24) we get that

$$x = |R - 3S|, \quad y = R + S, \tag{2.25}$$

and the positive integers x, y determined by the formula (2.25) are both odd and satisfy the virulogical Ljunggren's Diophantine equation (L).

On the other hand we note that the representation of (2.24) can be nonuniquenes and for determined eventuelle other solutions of (2.24) we can applied the following estimate, whose immediately follows from (2.21);

$$x < 4\max\{R, S\}, \quad y < 3\max\{R, S\}. \tag{2.26}$$

In this way we determine all odd positive integers solutions of the Ljunggren's Diophantine equation (L).

3. Remark and an example

We note that if the equation (2.6) has a solution in odd positive integers x, y then the number $M - 3z$ on the right hand of (2.6) must be odd non-negative integer. Therefore, if M is odd then it suffices consider only even non-negative integers $z \in \left[0, \frac{1}{3}M\right]$.

The following example is illustration of our method for this case:

Let $a = 5, b = 3$. Then $M = a^2+ab+b^2 = 5^2+5\times3+3^2 = 49$ and consequently the equation (2.6) has the form:

$$x^2 + 3y^2 = 4(49 - 3z), \tag{3.1}$$

where $0 \leqslant z \leqslant 49\frac{1}{3}$. Since z must be even integer then we can consider only the case when $z = 0, 2, 4, 6, 8, 10, 12, 14$ and 16.

If $z = 0$ then the equation (3.1) has the form:

$$x^2 + 3y^2 = 4 \times 7^2.$$

Since $7 = 6 \times 1 + 1 = 2^2 + 3 \times 1^2$, then by (2.18) it follows that $7^2 = 1^2 + 3 \times 4^2$ and we have $R = 1, S = 4$, so

$$x = |R - 3S| = |1 - 12| = 11, \quad y = R + S = 1 + 4 = 5.$$

Moreover, using (2.26) we obtain second solution, $x = y = 7$.

If $z = 2$, then $M - 6 = 49 - 6 = 43 = 6 \times 7 + 1 = 4^2 + 3 \times 3^2$, so $R = 4$, $S = 3$ and $x = 5$, $y = 7$ or $x = 13$, $y = 1$.

If $z = 4$, then $M - 12 = 37 = 6 \times 6 + 1 = 5^2 + 3 \times 2^2$, so $R = 5$, $S = 2$ and we have $x = 1$, $y = 7$ or $x = 11$, $y = 3$.

If $z = 6$, then we obtain $M - 18 = 31 = 6 \times 5 + 1 = 2^2 + 3 \times 3^2$, so $R = 2$, $S = 3$, and $x = 7$, $y = 5$ or $x = 11$, $y = 1$.

If $z = 8$, then $M - 24 = 25 = 5^2$ and $5 \neq 6k + 1$, so the equation (2.6) has no solutions.

If $z = 10$, then $M - 30 = 19 = 6 \times 3 + 1 = 4^2 + 3 \times 1^2$, so $R = 4$, $S = 1$ and $x = 1$, $y = 5$ or $x = 7$, $y = 3$.

If $z = 12$, then $M - 36 = 13 = 6 \times 2 + 1 = 1^2 + 3 \times 2^2$, so $R = 1$, $S = 2$ and $x = 5$, $y = 3$ or $x = 7$, $y = 1$.

If $z = 14$, then $M - 42 = 7 = 6 \times 1 + 1 = 2^2 + 3 \times 1^2$, so $R = 2$, $S = 1$ and $x = 1$, $y = 3$ or $x = 5$, $y = 1$.

If $z = 16$, then $M - 48 = 1$ and we have $x^2 + 3y^2 = 4$, so there is only one trivial solution in odd positive integers, namely $x = y = 1$.

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