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Remarks on arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$

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Abstract

In this paper some properties of the arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$ defined by Šalát in 1994 and Mycielski in 1951, respectively are investigated from the point of view of $\mathcal{I}$-convergence of sequences ($\mathcal{I}$-convergence was defined by Kostyrko, Šalát and Wilczynski in 2000).

1. Introduction

We shall study some properties of the $\mathcal{I}$–convergence of sequences of arithmetical functions $f\colon \mathbb{N} \to \mathbb{N}$, $a_p(n)$, $\gamma(n)$, $\tau(n)$. Elementary properties of the function $a_p(n)$ were studied in [6]. We shall extend these results with properties of $\mathcal{I}$–convergence of the sequence $(a_p(n))_{n=1}^{\infty}$.

We also want to investigate the asymptotic density of the sets $M_f = \{n : f(n) \mid n\}$ and the $\mathcal{I}$–convergence of arithmetical functions $\gamma(n)$, $\tau(n)$ defined by Mycielski in [4].

As usual we put for $A \subset \mathbb{N}$: $A(n) = |\{1, 2, \dots n\} \cap A|$,

$$\underline{d}(A) = \liminf \frac{A(n)}{n}, \overline{d}(A) = \limsup \frac{A(n)}{n}$$

the lower and upper density of A. If $\underline{d}(A) = \overline{d}(A)$, then we set

$$d(A) = \underline{d}(A) = \overline{d}(A), d(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

The system $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an admissible ideal if $\mathcal{I}$ is additive ($A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$), hereditary ($A \in \mathcal{I}$, $B \subseteq A \Rightarrow B \in \mathcal{I}$) and contains all finite sets. In this paper we are interested in ideals $\mathcal{I}_f = \{A \subseteq \mathbb{N}, |A| < +\infty\}$, $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum\limits_{a\in A} a^{-1} < +\infty\}$ and $\mathcal{I}_c^q = \{A \subseteq \mathbb{N} : \sum\limits_{a\in A} a^{-q} < +\infty\}$ for $q \in (0,1)$. It is easy to see that for $q \leqslant q' \in (0,1)$ the following inclusions hold:

$$\mathcal{I}_f \subseteq \mathcal{I}_c^q \subseteq \mathcal{I}_c^{q'} \subseteq \mathcal{I}_c \subseteq \mathcal{I}_d.$$

A given sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}$–convergent to $L \in R$, if for each $\varepsilon > 0$ we have $A_\varepsilon = \{n : \left|x_n - L\right| \geqslant \varepsilon\} \subseteq \mathcal{I}$ (shortly $\mathcal{I}$–$\lim x_n = L$). The cases of $\mathcal{I}_f$-convergence and $\mathcal{I}_d$-convergence coincide with the usual convergence and the statistical convergence (see [3], [7]), respectively. Therefore we will write $\lim x_n = L$ and $\lim\operatorname{stat} x_n = L$ instead of $\mathcal{I}_f$–$\lim x_n = L$ and $\mathcal{I}_d$–$\lim x_n = L$, respectively.

In [7, Lemma 2.2] it is shown that

$$\mathcal{I} \subseteq \mathcal{I}' \Rightarrow \mathcal{I} - \lim x_n = L \Rightarrow \mathcal{I}' - \lim x_n = L.$$

Using this result we completely determine for which q the sequences $a_p(n)$, $\gamma(n)$ and $\tau(n)$ are $\mathcal{I}_c^q$-convergent.

2. $\mathcal{I}$-convergence of $(a_p(n))_{n=1}^{\infty}$

Let p be a prime number. The function $a_p(n)$ is defined in the following way: $a_p(1) = 0$ and if $n > 1$, then $a_p(n)$ is the unique integer $j \geqslant 0$ satisfying $p^j | n$ but $p^{j+1} \nmid n$, i.e., $p^{a_p(n)} \parallel n$. At first we are going to generalize the result that the sequence $\left((\log p)\frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is statistically convergent to 0 [6, Th. 4.2].

Proposition 2.1. *Let $g(n) > 0\,(n = 1, 2 \dots)$ and $\lim\limits_{n\to\infty} g(n) = +\infty$. We have*

$$\lim\operatorname{stat}(\log p)\frac{a_p(n)}{g(n)} = 0.$$

Proof. Let $\varepsilon > 0$. Put $A_\varepsilon = \{n > 1 : (\log p)\frac{a_p(n)}{g(n)} \geqslant \varepsilon\}$. We will show that $d(A_\varepsilon) = 0$. Let $\eta > 0$. Choose $m \in N$ such that

$$p^{-m} < \eta. \tag{2.1}$$

By the conditions of the proposition there exists an n_0, such that for any $n > n_0$ we have

$$\frac{\varepsilon g(n)}{\log p} > m. \tag{2.2}$$

Let $n > n_0$ and $n \in A_\varepsilon$. It follows from (2.2) and the definition of A_ε that

$$(\log p)\frac{a_p(n)}{g(n)} \geqslant \varepsilon,$$

$$a_p(n) \geqslant \frac{\varepsilon g(n)}{\log p} > m.$$

Hence for the numbers $n > n_0, n \in A_\varepsilon$ implies $p^m | n$. This leads to the conclusion that $A_\varepsilon \subseteq \{1, 2, \dots, n_0\} \cup \{n > n_0 : p^m \mid n\}$ and considering (2.1) we get $\overline{d}(A_\varepsilon) \leqslant p^{-m} < \eta$. Since $\eta > 0$ is an arbitrary positive number, $d(A_\varepsilon) = 0$. $\square$

Remark 2.2. It is proved [6, Th. 4.1] that the sequence $\left((\log p)\frac{a_p(n)}{\log n}\right)_{n=2}^\infty$ is dense in interval $(0, 1)$. But $\left((\log p)\frac{a_p(n)}{g(n)}\right)_{n=2}^\infty$ which is statistically convergent to zero if $g(n) \to +\infty$, is not always dense in $(0, 1)$: For example if we define the function $g(n) = max\{1, \log^2 n\}$, then we have

$$\lim_{n\to\infty} (\log p)\frac{a_p(n)}{\log^2 n} = 0$$

and also

$$\lim \mathrm{stat}\, \frac{a_p(n)}{\log^2 n} = 0,$$

but this sequence is not dense in $(0, 1)$.

Theorem 2.3. *The sequence $(a_p(n))_{n=1}^\infty$ is I_c–convergent to 0 and $\mathcal{I}_c^q$–divergent for $q \in (0, 1)$.*

Proof. Let $\varepsilon > 0$ and denote

$$A_\varepsilon = \{n \in \mathbb{N} : (\log p)\frac{a_p(n)}{\log n} \geqslant \varepsilon\}.$$

Let $q \in (0, 1)$. We want to show that

$$\sum_{n\in A_\varepsilon} \frac{1}{n} < +\infty \tag{2.3}$$

and for $0 < \varepsilon < 1 - q$

$$\sum_{n\in A_\varepsilon} \frac{1}{n^q} = +\infty. \tag{2.4}$$

For nonnegative integer i denote $A_\varepsilon^i = \{n \in A_\varepsilon; n = p^i u, (u, p) = 1\}$. We have $A_\varepsilon^i \cap A_\varepsilon^j = \emptyset$ for $i \neq j$ and for any $t > 0$

$$\sum_{n\in A_\varepsilon} \frac{1}{n^t} = \sum_{i=0}^{\infty} \sum_{n\in A_\varepsilon^i} \frac{1}{n^t}. \tag{2.5}$$

a) Consider that $n \in A_\varepsilon^i$ if and only if $n = p^i u$ where $(u,p) = 1$ and also

$$(\log p)\frac{a_p(n)}{\log n} \geqslant \varepsilon.$$

Then

$$(\log p)\frac{i}{i\log p + \log u} \geqslant \varepsilon$$

from which we obtain $u \leqslant p^{i\delta}$, where $\delta = (1-\varepsilon)/\varepsilon$. Hence

$$\sum_{n\in A_\varepsilon^i}\frac{1}{n} \leqslant \frac{1}{p^i}\sum_{u\leqslant p^{i\delta}}\frac{1}{u} \leqslant \frac{1}{p^i}\left(1+\int_1^{p^{i\delta}} dt/t\right) = \frac{1}{p^i}(1+i\delta\log p) \leqslant A\delta\frac{i}{p^i}\log p$$

where $A > 0$ is only dependent on ε, p and not on i. The series $\sum\limits_{i=0}^{\infty}\frac{i}{p^i}$ converges, this proves (2.3).
b) We write

$$\sum_{n\in A_\varepsilon^i}\frac{1}{n^q} = \frac{1}{p^{iq}}\sum_{\substack{u\leqslant p^{i\delta}\\(u,p)=1}}\frac{1}{u^q}.$$

Then we have

$$\begin{aligned}
\sum_{\substack{u\leqslant p^{i\delta}\\(u,p)=1}}\frac{1}{u^q} &= \sum_{u\leqslant p^{i\delta}}\frac{1}{u^q} - \sum_{k\leqslant p^{i\delta-1}}\frac{1}{(kp)^q} = \sum_{u\leqslant p^{i\delta}}\frac{1}{u^q} - \frac{1}{p^q}\sum_{k\leqslant p^{i\delta-1}}\frac{1}{k^q}\\
&= \left(1-\frac{1}{p^q}\right)\sum_{v\leqslant p^{i\delta-1}}\frac{1}{v^q} + \sum_{p^{i\delta-1}<v\leqslant p^{i\delta}}\frac{1}{v^q}\\
&\geqslant \sum_{p^{i\delta-1}<v\leqslant p^{i\delta}}\frac{1}{v^q} \geqslant (p^{i\delta}-p^{i\delta-1})\frac{1}{p^{i\delta q}}\\
&= p^{i\delta}(1-\frac{1}{p})\frac{1}{p^{i\delta q}} = (1-\frac{1}{p})p^{i\delta(1-q)}.
\end{aligned}$$

Finally we obtain

$$\sum_{n\in A_\varepsilon}\frac{1}{n^q} = \sum_{i=0}^{\infty}\sum_{v\in A_\varepsilon^i}\frac{i}{v^q} \geqslant (1-\frac{1}{p})\sum_{i=0}^{\infty}\frac{1}{p^{i[q+(q-1)\delta]}}.$$

The series on the right-hand side diverges if $q+(q-1)\delta < 0$, i.e. $\varepsilon < 1-q$. This proves the I_c^q–divergence of $(a_p(n))_{n=1}^{\infty}$. $\square$

3. On the functions $\gamma(n)$ and $\tau(n)$

In [4] there were new arithmetical functions defined and investigated in connection with the representation of natural numbers of the form $n = a^b$, where a, b are positive integers. Let

$$n = a_1^{b_1} = a_2^{b_2} = \cdots = a_{\gamma(n)}^{b_{\gamma(n)}} \tag{3.1}$$

be all such representations of a given natural number n, where $a_i, b_i \in N$.

Denote by

$$\tau(n) = b_1 + \cdots + b_{\gamma(n)}, (n > 1).$$

It is clear that $\gamma(n) \geqslant 1$, because for any $n > 1$ there exists a representation in the form n^1.

We are going to study some new properties of the functions $\gamma(n)$ and $\tau(n)$.

Put $T(n) = \gamma(2) + \cdots + \gamma(n), (n \geqslant 2)$. It is proved in [4], that

$$T(n) = \sum_{s=1}^{[\log_2 n]} [\sqrt[s]{n}] - [\log_2 n] = n + \sum_{s=2}^{[\log_2 n]} [\sqrt[s]{n}] - [\log_2 n]. \tag{3.2}$$

Remark 3.1. It is easy to show that the average order of the function $\gamma(n)$ is 1, i.e.,

$$\lim_{n\to\infty} \frac{T(n)}{n} = 1.$$

It follows from (3.2) that

$$T(n) = n + T_1(n) - [\log_2 n],$$

where $T_1(n) = n + \sum\limits_{s=2}^{[\log_2 n]} [\sqrt[s]{n}]$. Then simple estimations give

$$([\log_2 n] - 1)[\sqrt[[\log_2 n]]{n}] \leqslant T_1(n) \leqslant ([\log_2 n] - 1)\sqrt{n}$$

from which we get $\lim\limits_{n\to\infty} \frac{T_1(n)}{n} = 0$.

In papers [1, 2] sets of the form $M_f = \{n \in \mathbb{N} : f(n) \mid n\}$, $f : \mathbb{N} \to \mathbb{N}$ are investigated. For some of the known arithmetical functions the sets M_f have zero asymptotic density: e.g. the functions $\omega(n)$ (the number of prime divisors of n), $s_g(n)$ (the digital sum of n in the representation with base g), $\pi(n)$ (the number of primes not exceeding n).

Proposition 3.2. *Put* $A_k = \{n > 1 : n = p_1^{\alpha_1} \dots p_n^{\alpha_n}, (\alpha_1, \dots, \alpha_n) = k\}$ $(k = 1, 2, \dots)$. *Then*

$$d(A_1) = 1. \tag{3.3}$$

Proof. Denote by $B = \cup_{k=2}^{\infty} A_k$, then $\mathbb{N}\setminus\{1\} = A_1 \cup B$, where $A_1 \cap B = \emptyset$. It can be easily shown that $d(B) = 0$, from which (3.3) follows immediately. The elements of the set B are only numbers of the form $t^s (t > 1,\ s > 1)$. Denote by H the set of all numbers $t^s (t > 1,\ s > 1)$. The series of reciprocal values of these numbers is equal to $\sum\limits_{t=2}^{\infty}\sum\limits_{s=2}^{\infty} \frac{1}{t^s}$ which is convergent to 1 (cf. [4]). Then we have $d(H) = 0$ and it implies that also $d(B) = 0$. □

Let us investigate the asymptotic density of $M_\gamma = \{n : \gamma(n) \mid n\}$ and $M_\tau = \{n : \tau(n) \mid n\}$.

Proposition 3.3. *We have*
(i) $d(M_\gamma) = 1$,
(ii) $d(M_\tau) = 1$.

Proof. (i) If $n \in A_1$, then evidently $\gamma(n) = 1$ and $n \in M_\gamma$. Thus $A_1 \subseteq M_\gamma$ and considering (3.3) we get $d(M_\gamma) = 1$.
(ii) Similarly. □

In [4, Th. 3, Th. 5] there are proofs of the following results:

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n} = 1, \sum_{n=2}^{\infty} \frac{\tau(n)-1}{n} = 1 + \frac{\pi^2}{6}.$$

In connection with these results we have investigated the convergence of series for any $\alpha \in (0,1)$

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha}, \sum_{n=2}^{\infty} \frac{\tau(n)-1}{n^\alpha}.$$

Theorem 3.4. *The series*

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha}$$

diverges for $0 < \alpha \leqslant \frac{1}{2}$ *and converges for* $\alpha > \frac{1}{2}$.

Proof. a) Let $0 < \alpha \leqslant \frac{1}{2}$. Put $K = \{k^2 : k > 1\}$. A simple estimation gives

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha} \geqslant \sum_{n \in K} \frac{\gamma(n)-1}{n^\alpha}.$$

Clearly $\gamma(n) \geqslant 2$ for $n \in K$. Therefore

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha} \geqslant \sum_{n \in K} \frac{1}{n^\alpha} = \sum_{k=2}^{\infty} \frac{1}{k^{2\alpha}} \geqslant \sum_{k=2}^{\infty} \frac{1}{k} = +\infty. \tag{3.4}$$

b) Let $\alpha > \frac{1}{2}$. We will use the formula

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha} = \sum_{k=2}^{\infty}\sum_{s=2}^{\infty} \frac{1}{k^{\alpha s}} = \sum_{k=2}^{\infty} \frac{1}{k^\alpha(k^\alpha-1)}. \tag{3.5}$$

For a sufficiently large number k $(k > k_0)$ we have $\frac{k^\alpha}{k^\alpha-1} < 2$. We can estimate the series on the right-hand side of (3.5) with

$$\sum_{k=2}^{\infty} \frac{1}{k^\alpha(k^\alpha-1)} < \sum_{k=2}^{k_0} \frac{1}{k^\alpha(k^\alpha-1)} + 2\sum_{k>k_0} \frac{1}{k^{2\alpha}}.$$

Since $2\alpha > 1$ we get

$$\sum_{n=2}^{\infty} \frac{\gamma(n)-1}{n^\alpha} < +\infty.$$

□

Corollary 3.5. *The sequence $\gamma(n)$ is*
(i) *$\mathcal{I}_c$-convergent to 1,*
(ii) *$\mathcal{I}_c^q$–divergent for $q \in (0, \frac{1}{2}]$ and $\mathcal{I}_c$–convergent to 1 for $q \in \left(\frac{1}{2}, 1\right)$.*

Proof. (i) Let $\varepsilon > 0$. The set of numbers $\{n > 1 : \left|\gamma(n) - 1\right| \geqslant \varepsilon\}$ is a subset of $H = \{t^s, t > 1, s > 1\}$ and $\sum\limits_{a\in H} \frac{1}{a} < +\infty$. From the definition of I_c–convergence (i) follows.

(ii) Let $\varepsilon > 0$ and denote $A_\varepsilon = \{n \in \mathbb{N} : \left|\gamma_n - 1\right| \geqslant \varepsilon\}$. When $0 < q \leqslant \frac{1}{2}$ then for the numbers $n \in K$, $K = \{k^2 : k > 1\}$ considering (3.4) holds

$$\sum_{n\in A_\varepsilon} \frac{1}{n^\alpha} \geqslant \sum_{n\in K} \frac{1}{n^\alpha} \geqslant +\infty.$$

Therefore $\gamma(n)$ is $\mathcal{I}_c^q$–divergent. When $\frac{1}{2} < q < 1$, then $A_\varepsilon \subset H$ and

$$\sum_{n=2}^{\infty} \frac{1}{n^\alpha} \leqslant \sum_{k=2}^{\infty}\sum_{s=2}^{\infty} \frac{1}{k^{\alpha s}}.$$

The convergence of the series on the right-hand side we proved previously in Theorem 3.4. Therefore $\gamma(n)$ is $\mathcal{I}_c$–convergent to 1 if $q \in (\frac{1}{2}, 1)$. □

Remark 3.6. We have $\lim\operatorname{stat}\gamma(n) = 1$.

Theorem 3.7. *The series*

$$\sum_{n=2}^{\infty} \frac{\tau(n)-1}{n^\alpha}$$

diverges for $0 < \alpha \leqslant \frac{1}{2}$ and converges for $\alpha > \frac{1}{2}$.

Proof. Let $0 < \alpha < 1$. We write the given series in the form

$$\sum_{n=2}^{\infty} \frac{\tau(n)-1}{n^{\alpha}} = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{s}{k^{\alpha s}}, \tag{3.6}$$

We shall try to use a similar method to Mycielski's proof of the convergence of $\sum\limits_{n=2}^{\infty} \frac{\tau(n)-1}{n^{\alpha}}$ to explain the equality (3.6). Since $\frac{s}{k^{\alpha s}} = -\frac{k}{\alpha}\frac{d}{dt}(\frac{1}{t^{\alpha s}})_{t=k}$ and $\sum\limits_{s=2}^{\infty} \frac{1}{t^{\alpha s}} = \frac{1}{t^{\alpha}(t^{\alpha}-1)}$ the right-hand side of (3.6) is equal to

$$\sum_{s=2}^{\infty} \frac{2k^{\alpha}-1}{k^{\alpha}(k^{\alpha}-1)^2} = \sum_{s=2}^{\infty} a_k.$$

For the k-th term of $\sum a_k$ we have

$$a_k = \frac{2-\frac{1}{k^{\alpha}}}{(1-\frac{1}{k^{\alpha}})^2} \cdot \frac{1}{k^{2\alpha}}.$$

Denote by $b_k = \frac{1}{k^{2\alpha}}$ and consider that $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = 2$. Hence the series $\sum\limits_{s=2}^{\infty} a_k$ converges (diverges) if and only if the series $\sum\limits_{s=2}^{\infty} b_k$ converges (diverges). Since $\sum b_k$ is convergent (divergent) for any $\alpha > \frac{1}{2}$ $(0 < \alpha \leqslant \frac{1}{2})$ so does the series $\sum a_k$ and therefore the series $\sum \frac{\tau(n)-1}{n^{\alpha}}$. □

Corollary 3.8. *The sequence* $\tau(n)$ *is*

(i) $\mathcal{I}_c$*–convergent to* 1*,*

(ii) $\mathcal{I}_c^q$*–divergent for* $q \in (0, \frac{1}{2}]$ *and* $\mathcal{I}_c$*–convergent to* 1 *for* $q \in \left(\frac{1}{2}, 1\right)$*.*

Proof. Similar to the proof of Corollary 3.5. □

Remark 3.9. We have $\lim \operatorname{stat} \tau(n) = 1$.

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