Annales Mathematicae et Informaticae
33 *(2006) pp. 15–21*
http://www.ektf.hu/tanszek/matematika/ami

On Heron triangles

Andrew Bremner

Department of Mathematics and Statistics, Arizona State University
e-mail: bremner@asu.edu

Submitted 30 August 2006; Accepted 15 November 2006

Abstract

There has previously been given a one-parameter family of pairs of Heron triangles with equal perimeter and area. In this note, we find two two-parameter families of such triangle pairs, one of which contains the known one-parameter family as a special case. Second, for an arbitrary integer $n \geqslant 2$ we show how to find a set of n Heron triangles in two parameters such that all triangles have equal perimeter and area.

MSC: 11D72, 14G05.

1. Introduction

A.-V. Kramer and F. Luca [3] investigate several problems related to Heron triangles (triangles with integral sides and integral area; *rational* triangles are those with rational sides and rational area, which by scaling thus become Heron triangles). They give a one-parameter family of pairs of such triangles having equal perimeter and area (curiously, Aassila [1] in a paper that gives the appearance of plagiarism produces exactly the same parametric family). This note shows first in completely elementary manner how to construct a doubly infinite family of such triangle pairs. In fact we produce two such parametrizations, in three (homogeneous) parameters, containing the Kramer-Luca family as a special case.

Recently, van Luijk [4] answers a question posed by Kramer and Luca by showing that there exist arbitarily many Heron triangles having equal perimeter and area, and gives a method whereby a one-parameter family may be written down for n such triangles for a given integer n. We use the same ideas in showing how to produce a set of n Heron triangles in two parameters with the property of equal perimeter and area.

2. Pairs of Heron triangles

Brahmagupta gave a parametrization for all Heron triangles, with sides proportional to

$$(v+w)(u^2-vw), \quad v(u^2+w^2), \quad w(u^2+v^2),$$

where the semi-perimeter is equal to $u^2(v+w)$, and the area is equal to $uvw(v+w)(u^2-vw)$. Thus to find a pair of Heron triangles with equal perimeter and area, we take the two triangles with independent parameters u, v, w and r, s, t and demand solutions of the system

$$u^2(v+w) = mr^2(s+t), \qquad uvw(v+w)(u^2-vw) = m^2rst(s+t)(r^2-st),$$

for a scaling factor m. The general solution will be difficult to obtain. However, we focus on the situation $u = r$ and consider two cases.
First, $m = 1$. Then $w = s+t-v$ and equality of the area demands

$$(s+t)u(s-v)(t-v)(st-u^2+sv+tv-v^2) = 0.$$

For non-trivial solutions, we thus have $st-u^2+sv+tv-v^2 = 0$, and this quadric surface is birationally equivalent to the projective plane under the mapping

$$s:t:u:v = b(b+c):(a^2-bc+c^2):a(b+c):c(b+c),$$

(with inverse $a:b:c = u:s:v$). Accordingly,

$$r:s:t:u:v:w = a(b+c):b(b+c):a^2-bc+c^2:a(b+c):c(b+c):a^2+b^2-bc,$$

leading to the triangle-pair

$$\begin{aligned} &b(a^2-bc+c^2)(a^2+b^2+c^2),\\ &c(a^4+3a^2b^2+b^4-2b^3c+a^2c^2+b^2c^2),\\ &(b+c)(a^2+b^2-bc)(a^2+c^2), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &c(a^2+b^2-bc)(a^2+b^2+c^2),\\ &b(a^4+a^2b^2+3a^2c^2+b^2c^2-2bc^3+c^4),\\ &(a^2+b^2)(b+c)(a^2-bc+c^2) \end{aligned} \tag{2.2}$$

with the common semi-perimeter $a^2(b+c)(a^2+b^2+c^2)$ and the common area $abc(b+c)(a^2+b^2-bc)(a^2-bc+c^2)(a^2+b^2+c^2)$. As an example, at $(a, b, c) = (2, 3, 4)$, we obtain the triangles $(174, 197, 35)$ and $(29, 195, 182)$, both with perimeter 406 and area 2436.

Second, we assume $m \neq 1$ and restrict to $v = ms$, $w = mt$, when the perimeters become equal. Equality of the area demands

$$-r^2 + st + mst + m^2st = 0,$$

and considered as a quadric curve over $\mathbf{Q}(m)$ we have a birational correspondence with the projective line given by

$$r : s : t = (1 + m + m^2)\pi\rho : (1 + m + m^2)\rho^2 : \pi^2, \qquad \pi : \rho = r : s.$$

Thus

$$\begin{aligned} &r : s : t : u : v : w = \\ &(1+m+m^2)\pi\rho : (1+m+m^2)\rho^2 : \pi^2 : (1+m+m^2)\pi\rho : m(1+m+m^2)\rho^2 : m\pi^2 = \\ &P(Q^2+QR+R^2) : R(Q^2+QR+R^2) : P^2R : P(Q^2+QR+R^2) : Q(Q^2+QR+R^2) : P^2Q, \end{aligned}$$

on setting $\pi/\rho = P/R$, $m = Q/R$ (so P, Q, R independent parameters). This leads to the triangle pair

$$\begin{aligned} &Q(Q+R)(P^2+Q^2+QR+R^2), \\ &Q^4+2Q^3R+P^2R^2+3Q^2R^2+2QR^3+R^4, \\ &(P^2+R^2)(Q^2+QR+R^2) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &R(Q+R)(P^2+Q^2+QR+R^2), \\ &P^2Q^2+Q^4+2Q^3R+3Q^2R^2+2QR^3+R^4, \\ &(P^2+Q^2)(Q^2+QR+R^2). \end{aligned} \tag{2.4}$$

If we put $(P, Q, R) = (t(3 + 3t^2 + t^4), 1, 1 + t^2)$, then the resulting triangle pair is the one-parameter family of Kramer and Luca [3].

3. Sets of Heron triangles

Kramer and Luca [3] essentially ask whether one can find sets of k Heron triangles with equal perimeter and area, for a given positive integer k. Relatedly, for a given triangle with rational sides a_0, b_0, c_0 of perimeter $2s$ and area A, we can ask to find other triangles with the same perimeter and area. If such a triangle has sides a, b, c then

$$a + b + c = 2s, \qquad s(s-a)(s-b)(s-c) = A^2.$$

Equivalently,

$$C : s(s-a)(s-b)(a+b-s) = A^2, \tag{3.1}$$

the equation of a cubic curve in the a, b-plane. Certainly C contains the points at infinity $(0, 1, 0), (1, 0, 0), (-1, 1, 0)$, so is an elliptic curve. Fixing one of these points as the zero of the group law, then the other two points become torsion points of order 3. Moreover, C contains the rational points at $(a,b) = (a_0,b_0)$, (b_0,a_0), (b_0,c_0), (c_0,b_0), (c_0,a_0), (a_0,c_0), the sextet comprising the points $\pm(a_0,b_0)+3$-torsion in the group $C(\mathbf{Q})$. In general, the point (a_0,b_0) will be of infinite order, allowing arbitrarily large sets of rational points (a,b) to be determined, each in turn defining a triangle with sides $(a,b,2s-a-b)$, having the perimeter $2s$ and area A. The triangle may of course not be geometrically realisable if $a<0$, $b<0$, or $2s<a+b$, or if the triangle inequality is violated; but since (a_0,b_0) corresponds to a genuine triangle, a density argument of points on the elliptic curve (dating back to Hurwitz: see Theorem 13 of [2]) guarantees the existence of arbitrarily many (a,b) corresponding to genuine triangles. (van Luijk [4] makes this argument explicit: if points P_i correspond to real triangles, then $\sum_{i=1}^{i=k} n_i P_i$ corresponds to a real triangle if and only if $\sum_{i=1}^{i=k} n_i$ is odd). Scaling will now produce arbitrarily large sets of Heron triangles with equal perimeter and area.

Remark 3.1. The isosceles triangle (a_0,a_0,c_0) with $b_0 = a_0$ has corresponding curve C with (homogeneous) equation

$$2ab(a+b)-(2a_0+c_0)(a^2+3ab+b^2)d+(2a_0+c_0)^2(a+b)d^2-a_0(2a_0^2+3a_0c_0+2c_0^2)d^3 = 0,$$

and the points (a_0,a_0), (a_0,c_0), and (c_0,a_0) have the property that doubling them results in a torsion point at infinity: so the points are either of order 2 or of order 6. The point (a_0,b_0) may also be of finite order for a non-isosceles triangle, for example the triangle $(a_0,b_0,c_0) = (13,27,34)$, where (a_0,b_0) has order 12. If the rational rank of C is 0 (as is the case for example with the (non-Heron) triangles given by $(a_0,b_0,c_0) = (1,1,1)$ or $(13,27,34)$) then there are at most finitely many rational-sided triangles with same perimeter and area, arising from the torsion points on C. When (a_0,b_0) is a torsion point therefore, to determine arbitrarily many triangles with equal perimeter and area we require C to have an additional rational non-torsion point (corresponding to a real triangle) in order to start the above construction. For instance, the Heron triangle $(14,25,25)$ has (a_0,b_0) a torsion point, but the respective curve C exhibits the additional non-torsion point $\left(\frac{39}{2},\frac{136}{5}\right)$, leading to the triangle $\left(\frac{39}{2},\frac{136}{5},\frac{173}{10}\right)$, with same perimeter and area.

As illustration of the above construction of sets of points, take as example the Heron triangle $(3,4,5)$, with semi-perimeter 6 and area 6. The construction of taking multiples of the point $(3,4)$ on C provides the triangles

$$\left(\frac{156}{35},\frac{41}{15},\frac{101}{21}\right), \quad \left(\frac{81831}{16159},\frac{27689}{8023},\frac{35380}{10153}\right), \quad \left(\frac{678541575}{151345267},\frac{683550052}{142637329},\frac{221167193}{81180907}\right), \quad \ldots$$

with perimeter 6 and area 6. The numbers grow rapidly because the underlying elliptic curve here has rank 1, and the heights on an elliptic curve of multiples of a fixed point are rapidly increasing. When the underlying curve has higher rank, then by taking linear combinations of the generators there is expectation of a greater supply of rational points with relatively small height, and accordingly an expectation of a more plentiful supply of triangles, as for example in the table of van Luijk [4], where 20 triangles are generated with same area and perimeter as the triangle $(75, 146, 169)$; in this instance, the underlying elliptic curve has rank 4 (independent points on C are $(111, 104)$, $(125, 91)$, $(146, 75)$, and $(265, 203)$).

Of course, we can use as our initial triangle one given by a one- or two-parameter family, and construct arbitrarily many triangles in the corresponding number of parameters, all having the same perimeter and area. The formulae rapidly become lengthy, and we give as example a three (homogeneous) parameter family of only four such triangles, arising from the parametrizations at (2.3), (2.4). Denote the points on C corresponding to the parametrizations (2.3), (2.4), by S and T respectively. Then the parametrizations corresponding to the points S, T, $2S+T$, $S+2T$ are given by:

$$\begin{aligned}
&Q(Q+R)(2Q+R)(Q+2R)(P^2+Q^2+QR+R^2)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+\\
&\quad Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3),\\
&(2Q+R)(Q+2R)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+Q^2R+QR^2+R^3)(P^2Q+Q^3+\\
&\quad 2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3)(Q^4+2Q^3R+P^2R^2+3Q^2R^2+2QR^3+R^4),\\
&(2Q+R)(Q+2R)(P^2+R^2)(Q^2+QR+R^2)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+\\
&\quad Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3),
\end{aligned}$$

$$\begin{aligned}
&R(Q+R)(2Q+R)(Q+2R)(P^2+Q^2+QR+R^2)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+\\
&\quad Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3),\\
&(2Q+R)(Q+2R)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+Q^2R+QR^2+R^3)(P^2Q+Q^3+\\
&\quad 2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3)(P^2Q^2+Q^4+2Q^3R+3Q^2R^2+2QR^3+R^4),\\
&(P^2+Q^2)(2Q+R)(Q+2R)(Q^2+QR+R^2)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+\\
&\quad Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+2QR^2+R^3),
\end{aligned}$$

$$\begin{aligned}
&(2Q+R)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+\\
&\quad 2QR^2+R^3)(P^4Q^4+P^2Q^6+4P^4Q^3R+6P^2Q^5R+6P^4Q^2R^2+13P^2Q^4R^2+4P^4QR^3+\\
&\quad 16P^2Q^3R^3+P^4R^4+13P^2Q^2R^4+Q^4R^4+6P^2QR^5+2Q^3R^5+2P^2R^6+3Q^2R^6+2QR^7+R^8),\\
&(2Q+R)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+\\
&\quad 2QR^2+R^3)(P^2Q^6+Q^8+6P^2Q^5R+6Q^7R+13P^2Q^4R^2+17Q^6R^2+16P^2Q^3R^3+30Q^5R^3+\\
&\quad P^4R^4+13P^2Q^2R^4+36Q^4R^4+6P^2QR^5+30Q^3R^5+2P^2R^6+17Q^2R^6+6QR^7+R^8),\\
&R(Q+R)(2Q+R)(Q+2R)(Q^2+QR+R^2)(P^2+Q^2+QR+R^2)(P^2Q+Q^3+P^2R+Q^2R+QR^2)\\
&\quad (P^2Q+Q^3+2Q^2R+2QR^2+R^3)(P^4-P^2Q^2+Q^4+2P^2QR+2Q^3R+2P^2R^2+3Q^2R^2+2QR^3+R^4),
\end{aligned}$$

$$\begin{aligned}
&(Q+2R)(P^2Q+P^2R+Q^2R+QR^2+R^3)(P^2Q+Q^3+2Q^2R+2QR^2+R^3)(Q^3+P^2R+2Q^2R+\\
&\quad 2QR^2+R^3)(P^4Q^4+2P^2Q^6+Q^8+4P^4Q^3R+6P^2Q^5R+2Q^7R+6P^4Q^2R^2+13P^2Q^4R^2+3Q^6R^2+\\
&\quad 4P^4QR^3+16P^2Q^3R^3+2Q^5R^3+P^4R^4+13P^2Q^2R^4+Q^4R^4+6P^2QR^5+P^2R^6),\\
&(Q+2R)(P^2Q+Q^3+P^2R+Q^2R+QR^2)(P^2Q+P^2R+Q^2R+QR^2+R^3)(Q^3+P^2R+2Q^2R+\\
&\quad 2QR^2+R^3)(P^4Q^4+2P^2Q^6+Q^8+6P^2Q^5R+6Q^7R+13P^2Q^4R^2+17Q^6R^2+16P^2Q^3R^3+\\
&\quad 30Q^5R^3+13P^2Q^2R^4+36Q^4R^4+6P^2QR^5+30Q^3R^5+P^2R^6+17Q^2R^6+6QR^7+R^8),\\
&Q(Q+R)(2Q+R)(Q+2R)(Q^2+QR+R^2)(P^2+Q^2+QR+R^2)(P^2Q+P^2R+Q^2R+QR^2+R^3)\\
&\quad (Q^3+P^2R+2Q^2R+2QR^2+R^3)(P^4+2P^2Q^2+Q^4+2P^2QR+2Q^3R-P^2R^2+3Q^2R^2+2QR^3+R^4).
\end{aligned}$$

Remark 3.2. The family of elliptic curves at (3.1) is actually one-dimensional parameterized by $t = A/s^2$, namely

$$(1-x)(1-y)(x+y-1) = t^2, \tag{3.2}$$

where $(x, y) = (a/s, b/s)$, $t = A/s^2$. For the triangles at (2.1), (2.2), we have

$$t = \frac{bc(a^2+b^2-bc)(a^2+c^2-bc)}{a^3(b+c)(a^2+b^2+c^2)}, \tag{3.3}$$

which for general t defines a curve in the (a, b, c)-plane of genus 5. Thus by Falting's proof of the Mordell Conjecture, only finitely many a, b, c give rise to the same t. Specialization of a, b, c therefore in general produces n-tuples of triangles each corresponding to a different elliptic curve. A similar remark holds for the triangles at (2.3), (2.4), where

$$t = \frac{PQR(Q+R)}{(Q^2+QR+R^2)(P^2+Q^2+QR+R^2)}$$

defines for general t a curve of genus 2 in the (P, Q, R)-plane.

Remark 3.3. The curve (3.2) comprises a bounded component lying within the region $0 < x < 1$, $0 < y < 1$, $x + y > 1$, and an unbounded component in the region $x > 1$. Real triangles correspond to points on the bounded component, and it is immediately apparent from the geometrical definition of addition on the curve (and straightforward to prove) that if points P_i lie on the bounded component, then $\sum_{i=1}^{i=k} n_i P_i$ lies on the bounded component if and only $\sum_{i=1}^{i=k} n_i$ is odd, recovering the density argument mentioned above.

Remark 3.4. The triangles at (2.1), (2.2) give rise to points S' and T' on the elliptic curve (3.2), with t given by (3.3); and by specialization, S', T' are seen to be generically linearly independent in the Mordell-Weil group. Similarly the two points S and T arising from triangles (2.3), (2.4) are independent in the corresponding Mordell-Weil group. It may well be possible to specialize to polynomials in one variable so that the Mordell-Weil group acquires further independent points, so will have rank at least 3. As remarked previously, the larger the rank, the greater the expectation of a supply of points of small height, and hence the expectation of providing parametrizations of smaller degree.

We refine the question of Kramer and Luca by asking how many distinct *primitive* Heron triangles may be found (those with sides having no non-trivial common divisor), with equal perimeter and area. It is straightforward to find pairs with this property, and there is the triple

$$(75, 146, 169), \qquad (91, 125, 174), \qquad (104, 111, 175)$$

(implicit in the table of van Luijk) with perimeter 390 and area 5460; but I am not aware of a quadruple of such triangles.

Acknowledgements. My thanks to the referee for helpful criticism of the first draft of this paper.

References

[1] M. Aassila, Some results on Heron triangles, *Elem. Math.*, 56 (2001), 143-146.

[2] A. Hurwitz, Über ternäre diophantische Gleichungen dritten Grades, *Viertel Jahrschrift Naturforsch. Gesellsch.* Zürich 62 (1917) 207-229.

[3] A.-V. Kramer and F. Luca, Some results on Heron triangles, *Acta Acad. Paed. Agriensis, Sectio Math.* 26 (2000), 1-10.

[4] R. van Luijk, An elliptic K3 surface associated to Heron triangles, (see `http://arxiv.org/abs/math/0411606`) *J. Number Theory*, to appear.

Andrew Bremner
Department of Mathematics and Statistics
Arizona State University
Tempe
AZ 85287-1804
USA