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Lebesgue constants in polynomial interpolation

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Abstract

Lagrange interpolation is a classical method for approximating a continuous function by a polynomial that agrees with the function at a number of chosen points (the "nodes"). However, the accuracy of the approximation is greatly influenced by the location of these nodes. Now, a useful way to measure a given set of nodes to determine whether its Lagrange polynomials are likely to provide good approximations is by means of the Lebesgue constant. In this paper a brief survey of methods and results for the calculation of Lebesgue constants for some particular node systems is presented. These ideas are then discussed in the context of Hermite–Fejér interpolation and a weighted interpolation method where the nodes are zeros of Chebyshev polynomials of the second kind.

Keywords: interpolation, Lagrange interpolation, Hermite–Fejér interpolation, Lebesgue constant, Lebesgue function

MSC: 41-02, 41A05, 41A10

1. Introduction

For each integer $n \geqslant 1$, consider n points (*nodes*) $x_{k,n}$ $(k = 1, 2, \ldots, n)$ in $[-1, 1]$ with

$$-1 \leqslant x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} \leqslant 1, \tag{1.1}$$

and let X be the infinite triangular matrix

$$X = \{x_{k,n} : k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots\}. \tag{1.2}$$

Given $f \in C[-1,1]$, the classical Lagrange interpolation polynomial $L_{n-1}(X,f)$ of degree $n-1$ (or less) for f, based on X, can be written as

$$L_{n-1}(X,f)(x) = \sum_{i=1}^{n} f(x_{i,n})\ell_{i,n}(X,x),$$

where the *fundamental polynomial* $\ell_{i,n}(X,x)$ is the unique polynomial of degree $n-1$ with $\ell_{i,n}(X,x_{k,n}) = \delta_{i,k}$, $1 \leqslant k \leqslant n$. (Here $\delta_{i,k}$ denotes the Kronecker delta.)

Let $\|f\|$ denote the uniform norm

$$\|f\| = \max_{-1 \leqslant x \leqslant 1} |f(x)|.$$

When studying the uniform convergence behaviour of the $L_{n-1}(X,f)$ as $n \to \infty$, a crucial role is played by the *Lebesgue function*

$$\lambda_n(X,x) = \max_{\|f\| \leqslant 1} |L_{n-1}(X,f)(x)| = \sum_{i=1}^{n} |\ell_{i,n}(X,x)|$$

and the *Lebesgue constant*

$$\Lambda_n(X) = \max_{\|f\| \leqslant 1} \|L_{n-1}(X,f)\| = \max_{-1 \leqslant x \leqslant 1} \lambda_n(X,x)$$

(see, for example, Rivlin [13, Chapter 4] or Szabados and Vértesi [19]).

Now, it is known that for *any* X, $\Lambda_n(X)$ is unbounded with respect to n. A consequence of this is (by the uniform boundedness theorem) Faber's 1914 result [6] that there exists $f \in C[-1,1]$ such that $L_n(X,f)$ does *not* converge uniformly to f. However, if f is not too badly behaved (as measured by the modulus of continuity, for instance) and the $\Lambda_n(X)$ are not too large, then uniform convergence *is* achieved (see, for example, Rivlin [13, Chapter 4]).

Figure 1 illustrates some basic properties of Lebesgue functions for Lagrange interpolation. For example, for any X and $n \geqslant 3$, $\lambda_n(X,x)$ is a piecewise polynomial that satisfies $\lambda_n(X,x) \geqslant 1$ with equality if and only if x is one of the nodes $x_{k,n}$. As well, on each interval $(x_{k+1,n}, x_{k,n})$ for $1 \leqslant k \leqslant n-1$, $\lambda_n(X,x)$ has precisely one local maximum, while $\lambda_n(X,x)$ is decreasing and concave upward on $(-1, x_{n,n})$ and is increasing and concave upward on $(x_{1,n}, 1)$. (For a discussion of these and other properties see, for example, Luttmann and Rivlin [11].)

2. The Lebesgue function for specific node systems

For some particular node systems, the Lebesgue function and constant have been studied in considerable detail. In this section, a summary of some of these results is given — for a more detailed account of many of the results, see the comprehensive survey paper by Brutman [4] and the references therein.

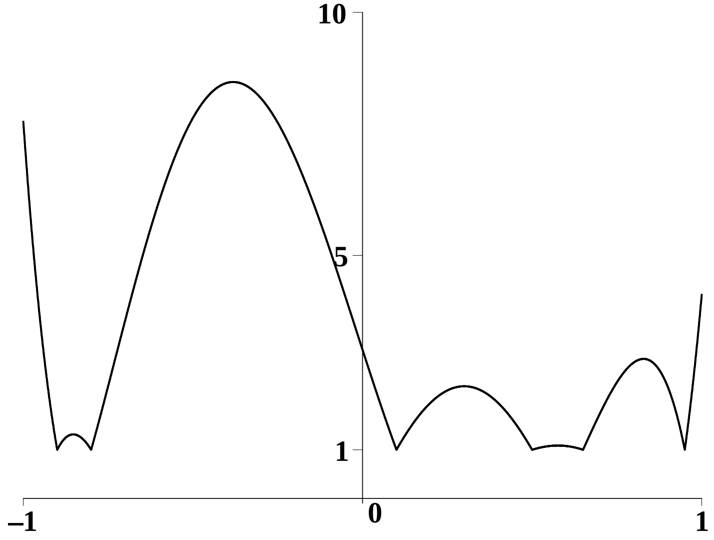

Figure 1: Lebesgue function for Lagrange interpolation based on the six nodes −0.9, −0.8, 0.1, 0.5, 0.65 and 0.95.

2.1. Equally-spaced nodes

Figure 2 illustrates a typical Lebesgue function for Lagrange interpolation based on the equally-spaced nodes

$$E = \{x_{k,n} = 1 - 2(k-1)/(n-1) : k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots\}.$$

As suggested by the graph, the local maxima of $\lambda_n(E, x)$ are strictly decreasing from the outside towards the middle of the interval $[-1, 1]$, a result that was established by Tietze [20]. Later Turetskii [21] showed that the Lebesgue constant $\Lambda_n(E)$ has the asymptotic expansion as $n \to \infty$,

$$\Lambda_n(E) \sim \frac{2^n}{en \log n}. \tag{2.1}$$

This result has been subsequently refined (to a small extent) by other authors.

2.2. Chebyshev nodes

Figure 3 shows a typical Lebesgue function for Lagrange interpolation based on the Chebyshev nodes

$$T = \{x_{k,n} = \cos(2k-1)\pi/(2n) : k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots\}.$$

(For each n these nodes are the zeros of the nth Chebyshev polynomial of the first kind.) The graph illustrates that the maximum of the Lebesgue function on $[-1, 1]$

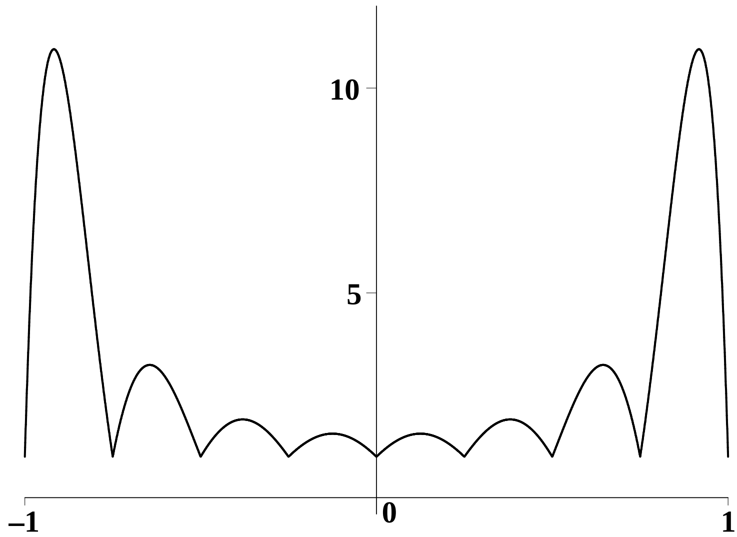

Figure 2: Lebesgue function for Lagrange interpolation on the equally-spaced nodes $x_{k,n} = 1 - 2(k-1)/(n-1)$ [with $n = 9$].

occurs at ± 1, a result due to Ehlich and Zeller [5]. From the representation

$$\Lambda_n(T) = \lambda_n(T, \pm 1) = \frac{1}{n} \sum_{i=1}^{n} \cot(2i-1)\pi/(4n),$$

asymptotic results such as

$$\Lambda_n(T) = \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\gamma + \log \frac{8}{\pi} \right) + O\left(\frac{1}{n^2} \right) \tag{2.2}$$

can be deduced, where γ denotes Euler's constant $0.577\ldots$ (see [4] for references and more precise results). On comparing (2.1) and (2.2) it can be seen that the Lebesgue constant for Chebyshev nodes is *much* smaller than for equally-spaced nodes. This confirms the "bad" status of equally-spaced nodes for Lagrange interpolation, a fact that has become well-known largely because of the example of Runge [15].

Figure 3 also suggests that, as with $\lambda_n(E, x)$, the local maxima of $\lambda_n(T, x)$ are strictly decreasing from the outside towards the middle of the interval $[-1, 1]$. This was proved by Brutman [3] (see also Günttner [8]).

2.3. Extended Chebyshev nodes

The *extended Chebyshev nodes* $\widehat{T}$ are defined by

$$\widehat{T} = \{x_{k,n} = \cos[(2k-1)\pi/(2n)]/\cos[\pi/(2n)] : k = 1, 2, \ldots, n;\ n = 2, 3, 4, \ldots\}.$$

That is, they are obtained by rescaling the Chebyshev nodes so that the nodes of

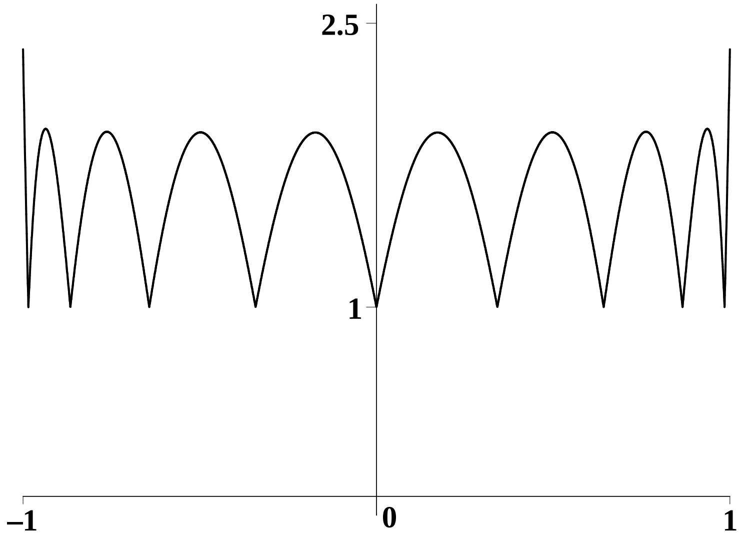

Figure 3: Lebesgue function for Lagrange interpolation on the Chebyshev nodes $x_{k,n} = \cos(2k-1)\pi/(2n)$ [with $n = 9$].

greatest magnitude for each n are at ± 1. Now, it is readily shown that

$$\lambda_n(\widehat{T}, x) = \lambda_n(T, x\cos[\pi/(2n)]).$$

Thus, by the monotonicity result for the local maxima of $\lambda_n(T, x)$, it follows that $\Lambda_n(\widehat{T})$ is strictly less than $\Lambda_n(T)$ and is equal to the maximum of $\lambda_n(T, x)$ on the interval $(\cos 3\pi/(2n), \cos\pi/(2n))$. This characterisation was used by Günttner [9] to obtain an asymptotic result for $\Lambda_n(\widehat{T})$, a simplified version of which is

$$\Lambda_n(\widehat{T}) = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi} - \frac{2}{3}\right) + O\left(\frac{1}{\log n}\right). \tag{2.3}$$

2.4. Augmented Chebyshev nodes

Another modification of T is to add ± 1 to each row of the matrix. These *augmented Chebyshev nodes* T_a are given by $x_{1,n+2} = 1$, $x_{n+2,n+2} = -1$ and $x_{k,n+2} = \cos(2k-3)\pi/(2n)$ for $k = 2, 3, \ldots, n+1$.

Now, interpolation polynomials on T and T_a are related by

$$\begin{aligned} L_{n+1}(T_a, f)(x) = L_{n-1}(T, f)(x) + \\ T_n(x) \times \{(1+x)[f(1) - L_{n-1}(T, f)(1)] \\ +(-1)^n(1-x)[f(-1) - L_{n-1}(T, f)(-1)]\}/2 \end{aligned} \tag{2.4}$$

where $T_n(x) = \cos(n \arccos x), -1 \leqslant x \leqslant 1$, is the nth Chebyshev polynomial of the first kind. (To verify (2.4), it is a simple matter to check that the RHS is a polynomial of degree no more than $n+1$ which agrees with f at the nodes $x_{k,n+2}$ for $1 \leqslant k \leqslant n+2$.) Thus if $L_n(T, f) \to f$ uniformly on $[-1, 1]$, then $L_n(T_a, f) \to f$ uniformly on $[-1, 1]$.

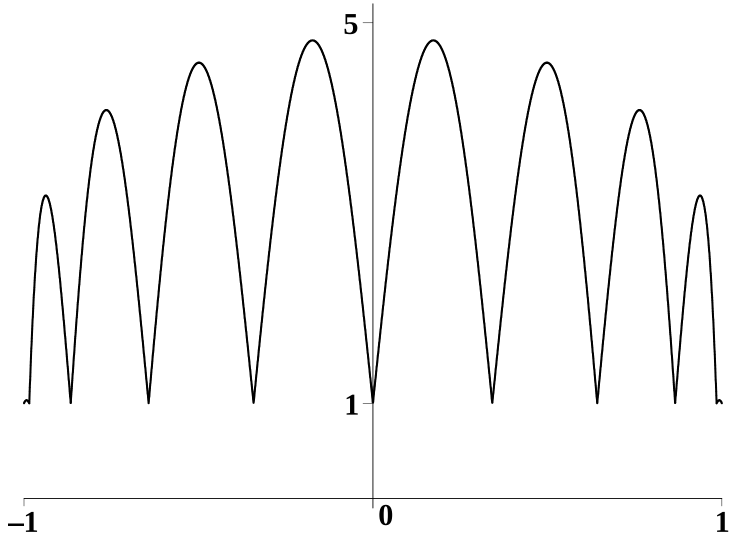

Figure 4: Lebesgue function for Lagrange interpolation on the augmented Chebyshev nodes $\{\cos(2k-3)\pi/(2n) : 2 \leqslant k \leqslant n+1\} \cup \{\pm 1\}$ [with $n = 9$].

Figure 4 appears to show that the local maximum values of $\lambda_{n+2}(T_a, x)$ *increase* from the outside towards the middle of $[-1, 1]$ (which is the reverse of the situation for T). This was proved by Smith [17], who used essentially the method that was employed by Brutman in [3] to establish the monotonic behaviour of the local maxima of $\lambda_n(T, x)$. Smith also obtained the asymptotic result

$$\Lambda_{n+2}(T_a) = \frac{4}{\pi} \log n + \frac{4}{\pi}\left(\gamma + \log \frac{4}{\pi}\right) + 1 + O\left(\frac{1}{n^2}\right) \tag{2.5}$$

which, when compared with (2.2), shows that $\Lambda_{n+2}(T_a)$ is effectively double $\Lambda_n(T)$.

2.5. Optimal nodes

The topic of the *optimal nodes* X^* for Lagrange interpolation, defined by

$$\Lambda_n(X^*) = \min_X \Lambda_n(X), \qquad n = 2, 3, 4, \ldots,$$

has been the subject of much research. Although no explicit formulation of X^* is known, Vértesi [22] showed that $\Lambda_n(X^*)$ has the asymptotic expansion

$$\Lambda_n(X^*) = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right). \tag{2.6}$$

A comparison of (2.3) and (2.6) suggests that $\widehat{T}$ is close to optimal. This point is discussed at some length (and made more precise) in Brutman [4, Section 3].

3. Hermite–Fejér interpolation

Given $f \in C[-1,1]$ and X defined by (1.2), the *Hermite–Fejér interpolation polynomial* $H_{2n-1}(X,f)$ of degree $2n-1$ (or less) for f, based on X, is the unique polynomial of degree no greater than $2n-1$ which interpolates f and has zero derivative at the nodes $x_{k,n}$ for $k = 1, 2, \ldots, n$. It can be written as

$$H_{2n-1}(X,f)(x) = \sum_{i=1}^{n} f(x_{i,n})A_{i,n}(X,x), \tag{3.1}$$

where the fundamental polynomial $A_{i,n}(X,x)$ is the unique polynomial of degree no greater than $2n-1$ such that $A_{i,n}(X,x_{k,n}) = \delta_{i,k}$ and $A'_{i,n}(X,x_{k,n}) = 0$ for $k = 1, 2, \ldots, n$.

The Lebesgue function for Hermite–Fejér interpolation on X is

$$\lambda_{1,n}(X,x) = \max_{\|f\|\leqslant 1} |H_{2n-1}(X,f)(x)| = \sum_{i=1}^{n} |A_{i,n}(X,x)|$$

and the Lebesgue constant is

$$\Lambda_{1,n}(X) = \max_{\|f\|\leqslant 1} \|H_{2n-1}(X,f)\| = \max_{-1\leqslant x\leqslant 1} \lambda_{1,n}(X,x).$$

For future reference, note that $H_{2n-1}(X,1)(x) = 1$ (from uniqueness considerations), so by (3.1),

$$\sum_{i=1}^{n} A_{i,n}(X,x) = 1. \tag{3.2}$$

3.1. Chebyshev nodes

Interest in Hermite–Fejér interpolation was sparked by Fejér's famous 1916 result (see [7]) that if $f \in C[-1,1]$, then $H_{2n-1}(T,f)$ converges uniformly to f. Thus there is a simple node system for which the Hermite–Fejér method succeeds for all $f \in C[-1,1]$, whereas no such system (simple or otherwise) exists for Lagrange interpolation.

A key point in Fejér's proof is that $A_{i,n}(T,x) \geqslant 0$ for $-1 \leqslant x \leqslant 1$ and $i = 1,2,\dots n$. Thus, by (3.2),

$$\lambda_{1,n}(T,x) = \sum_{i=1}^{n} |A_{i,n}(T,x)| = \sum_{i=1}^{n} A_{i,n}(T,x) = 1,$$

and so the Lebesgue constant $\Lambda_{1,n}(T)$ is simply 1.

3.2. A modified Hermite–Fejér method on the augmented Chebyshev nodes

As a "stepping stone" to the study of Hermite–Fejér interpolation on the augmented Chebyshev nodes, consider the following interpolation method.

For $n = 1,2,3,\dots$, write the Chebyshev nodes as

$$t_k = t_{k,n} = \cos(2k-1)\pi/(2n), \qquad k = 1,2,\dots,n,$$

and let $t_0 = 1$, $t_{n+1} = -1$. Given $f \in C[-1,1]$, define a polynomial $K_{2n+1}(f)$ of degree $2n+1$ (or less) by

$$\begin{cases} K_{2n+1}(f)(t_k) = f(t_k), & 0 \leqslant k \leqslant n+1, \\ K_{2n+1}(f)'(t_k) = 0, & 1 \leqslant k \leqslant n. \end{cases} \tag{3.3}$$

Thus $K_{2n+1}(f)$ interpolates f on the augmented Chebyshev nodes and has vanishing derivative at the Chebyshev nodes.

An explicit formula for $K_{2n+1}(f)$ in terms of the the Hermite–Fejér interpolation polynomial $H_{2n-1}(T,f)$ is

$$\begin{aligned} K_{2n+1}(f)(x) = H_{2n-1}(T,f)(x) + \\ T_n^2(x) \times \{(1+x)[f(1) - H_{2n-1}(T,f)(1)] \\ +(1-x)[f(-1) - H_{2n-1}(T,f)(-1)]\}/2. \end{aligned} \tag{3.4}$$

(Again, to verify (3.4), it is a simple matter to check that the RHS is a polynomial of degree no more than $2n+1$ that satisfies the conditions (3.3).) From (3.4) it follows immediately by Fejér's result that if $f \in C[-1,1]$, then $K_{2n+1}(f)$ converges uniformly to f.

Now, $K_{2n+1}(f)$ can also be written in terms of fundamental polynomials as

$$K_{2n+1}(f)(x) = \sum_{i=0}^{n+1} f(t_i) B_i(x),$$

where for each $i = 0,1,\dots,n+1$, $B_i(x) = B_{i,n}(x)$ is the unique polynomial of degree no greater than $2n+1$ so that $B_i(t_k) = \delta_{i,k}$ for $k = 0,1,\dots,n+1$ and $B_i'(t_k) = 0$ for $k = 1,2,\dots,n$. The Lebesgue function and constant are respectively

$$\lambda_n(x) = \sum_{i=0}^{n+1} |B_i(x)|, \quad \Lambda_n = \max_{-1 \leqslant x \leqslant 1} \lambda_n(x).$$

By using elementary properties of the Chebyshev polynomials $T_n(x)$ (see, for example, Rivlin [14, Chapter 1]), it is easy to verify that

$$B_0(x) = \frac{1+x}{2}\, T_n^2(x), \quad B_{n+1}(x) = \frac{1-x}{2}\, T_n^2(x) \tag{3.5}$$

and

$$B_k(x) = \frac{(1-x^2)(1+xt_k-2t_k^2)}{n^2(x-t_k)^2(1-t_k^2)}\, T_n^2(x), \quad 1 \leqslant k \leqslant n. \tag{3.6}$$

Observe that for $1 \leqslant k \leqslant n$, the sign of $B_k(x)$ is that of $1+xt_k-2t_k^2$. Thus if $n \geqslant 2$, then $B_1(x)$ (for example) is negative for all values of x in some interval in $[-1,1]$, and so, unlike the Hermite–Fejér method on T, the fundamental polynomials for the modified method are not all non-negative in $[-1,1]$. In terms of the Lebesgue constant, this means that $\Lambda_n > 1$ for all $n \geqslant 2$. On the other hand, since $K_{2n+1}(f)$ converges uniformly to f for all $f \in C[-1,1]$, it follows from the uniform boundedness theorem that the Λ_n are uniformly bounded. In the following theorem, the best possible bound for the Λ_n is derived.

Theorem 3.1. *The Lebesgue constant* Λ_n *satisfies*

$$\Lambda_n < 3, \qquad n = 1, 2, \ldots \tag{3.7}$$

and

$$\lim_{n\to\infty} \Lambda_n = 3. \tag{3.8}$$

Proof. By (3.5) and (3.6),

$$\lambda_n(x) = T_n^2(x)\left[1 + \frac{(1-x^2)}{n^2}\sum_{k=1}^{n} \frac{|1+xt_k-2t_k^2|}{(x-t_k)^2(1-t_k^2)}\right].$$

Observe that $1+xt_k-2t_k^2 > 0$ if and only if $p(x) < t_k < q(x)$, where

$$p(x) = \frac{x-\sqrt{x^2+8}}{4}, \quad q(x) = \frac{x+\sqrt{x^2+8}}{4}.$$

Let $J_n = \{1, 2, \ldots, n\}$ and to given $x \in [-1,1]$ define

$$\mathcal{R}(x) = \{k \in J_n : p(x) < t_k < q(x)\}, \quad \mathcal{S}(x) = \{k \in J_n : t_k \leqslant p(x) \text{ or } t_k \geqslant q(x)\}.$$

Therefore

$$\lambda_n(x) = T_n^2(x)\left[1 + \frac{(1-x^2)}{n^2}\, F(x)\right], \tag{3.9}$$

where

$$F(x) = \sum_{k\in\mathcal{R}(x)} \frac{1+xt_k-2t_k^2}{(x-t_k)^2(1-t_k^2)} - \sum_{k\in\mathcal{S}(x)} \frac{1+xt_k-2t_k^2}{(x-t_k)^2(1-t_k^2)}.$$

Next employ the partial fraction expansion

$$\frac{1+xt_k-2t_k^2}{(x-t_k)^2(1-t_k^2)} = \\ \frac{x}{(1-x^2)(x-t_k)} + \frac{1}{(x-t_k)^2} - \frac{1/2}{(1-x)(1-t_k)} - \frac{1/2}{(1+x)(1+t_k)}.$$

This leads to

$$\begin{aligned} F(x) = \sum_{k=1}^{n} &\left[\frac{x}{(1-x^2)(x-t_k)} + \frac{1}{(x-t_k)^2} + \frac{1/2}{(1-x)(1-t_k)} + \frac{1/2}{(1+x)(1+t_k)}\right] \\ &- \sum_{k\in\mathcal{R}(x)} \left[\frac{1}{(1-x)(1-t_k)} + \frac{1}{(1+x)(1+t_k)}\right] \\ &- 2\sum_{k\in\mathcal{S}(x)} \left[\frac{x}{(1-x^2)(x-t_k)} + \frac{1}{(x-t_k)^2}\right]. \end{aligned}$$

Now, from the identity

$$\frac{T_n'(x)}{T_n(x)} = \sum_{k=1}^{n} \frac{1}{x-t_k}$$

and elementary properties of the Chebyshev polynomials (see, for example, Rivlin [14, Chapter 1]), it follows that

$$\sum_{k=1}^{n} \frac{1}{1-t_k} = \sum_{k=1}^{n} \frac{1}{1+t_k} = n^2$$

and

$$\sum_{k=1}^{n} \frac{1}{(x-t_k)^2} = \frac{n^2 - xT_n(x)T_n'(x)}{(1-x^2)T_n^2(x)}.$$

Therefore (3.9) becomes

$$\lambda_n(x) = 1 + 2T_n^2(x) - \frac{2T_n^2(x)}{n^2}\left[\sum_{k\in\mathcal{R}(x)} \frac{1+xt_k}{1-t_k^2} + \sum_{k\in\mathcal{S}(x)} \frac{1-xt_k}{(x-t_k)^2}\right]. \tag{3.10}$$

If $x \in [-1, 1]$ the expression in square brackets is positive, so $\lambda_n(x) \leqslant 1 + 2T_n^2(x)$, with equality if and only if $T_n(x) = 0$. In particular, $\lambda_n(x) < 3$, from which (3.7) follows.

To establish (3.8), note that it follows from (3.10) that

$$\lambda_{2n}(0) = 3 - \frac{1}{2n^2}\left[\sum_{k\in\mathcal{R}(0)} \frac{1}{1-t_k^2} + \sum_{k\in\mathcal{S}(0)} \frac{1}{t_k^2}\right], \tag{3.11}$$

where $\mathcal{R}(0) = \{k \in J_{2n} : -1/\sqrt{2} < t_k < 1/\sqrt{2}\}$ and $\mathcal{S}(0) = \{k \in J_{2n} : t_k \leqslant -1/\sqrt{2} \text{ or } t_k \geqslant 1/\sqrt{2}\}$. The sums within the square brackets of (3.11) contain a total of $2n$ terms, each of which is no greater than 2, so

$$\Lambda_{2n} \geqslant \lambda_{2n}(0) \geqslant 3 - \frac{2}{n}.$$

By similar means it can be shown that there exists an absolute constant c so that

$$\Lambda_{2n+1} \geqslant \lambda_{2n+1}\left(\cos\left[n\pi/(2n+1)\right]\right) \geqslant 3 - \frac{c}{2n+1},$$

and hence (3.8) is proved. □

3.3. Hermite–Fejér interpolation on the augmented Chebyshev nodes

If $f \in C[-1,1]$, then by Fejér's result, the Hermite–Fejér interpolation polynomials $H_{2n-1}(T,f)$ converge uniformly to f. In Section 3.2 it was shown that if interpolation conditions at ± 1 are added to the Hermite–Fejér interpolation conditions at the Chebyshev nodes, the resulting interpolation polynomials will still converge uniformly to f. Thus it might be expected that if the full Hermite–Fejér interpolation conditions are applied at ± 1 as well as at the Chebyshev nodes, the resulting polynomials $H_{2n+3}(T_a,f)$ will converge uniformly to f. Perhaps surprisingly, this does not occur.

In fact, Hermite–Fejér interpolation on the augmented Chebyshev nodes exhibits some very bad properties! For example, Berman [1] showed that even for $f(x) = x^2$, $H_{2n+3}(T_a,f)(x)$ diverges as $n \to \infty$ for all $x \in (-1,1)$. (Note that this result doesn't extend to $[-1,1]$ because ± 1 are nodes for all n.) An explanation for "Berman's phenomenon" was provided by R. Bojanić [2], who showed that if $f \in C[-1,1]$ and the left and right derivatives $f'_L(1)$ and $f'_R(-1)$ exist, then $H_{2n-1}(T_a,f) \to f$ uniformly if and only if $f'_L(1) = f'_R(-1) = 0$.

Figure 5 shows a typical Lebesgue function $\lambda_{1,n+2}(T_a,x)$ for Hermite–Fejér interpolation on the augmented Chebyshev nodes T_a. On comparing Figures 4 and 5, it appears that the Lebesgue constant $\Lambda_{1,n+2}(T_a)$ for Hermite–Fejér interpolation is much larger than the Lebesgue constant $\Lambda_{n+2}(T_a)$ for Lagrange interpolation. This was confirmed by Smith [16], who used methods similar to those employed in the proof of Theorem 3.1 of this paper to show

$$\Lambda_{1,n+2}(T_a) = \begin{cases} 2n^2 + 3 + O(1/n), & \text{if } n \text{ is even,} \\ 2n^2 + 3 - \pi^2/2 + O(1/n), & \text{if } n \text{ is odd.} \end{cases}$$

4. A weighted interpolation method

In a paper in 1995, Mason and Elliott [12] studied certain weighted interpolation methods based on the zeros of the Chebyshev polynomials of the second, third and

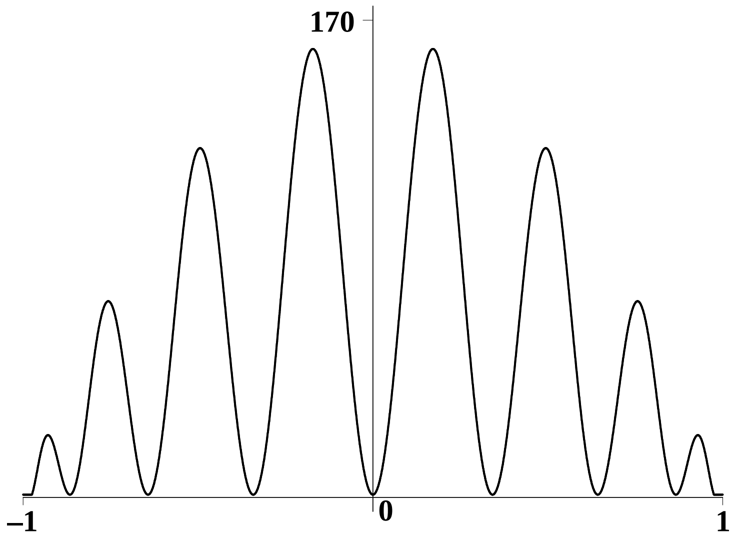

Figure 5: Lebesgue function for Hermite–Fejér interpolation on the augmented Chebyshev nodes $\{\cos(2k-3)\pi/(2n) : 2 \leqslant k \leqslant n+1\} \cup \{\pm 1\}$ [with $n = 9$].

fourth kinds. Although the resulting interpolating functions are not polynomials, there are many similarities between the study of these functions and the study of Lagrange interpolation polynomials. We illustrate Mason and Elliott's ideas by discussing their weighted interpolation method based on the zeros of Chebyshev polynomials of the second kind.

Denote the set of algebraic polynomials of degree at most n by Π_n, let $w(x)$ denote the weight function $w(x) = \sqrt{1-x^2}$, and let X and $x_{i,n}$ be given by (1.1) and (1.2) with $x_{i,n} \neq \pm 1$. We consider the interpolating projection $P_{n-1}(X)$ of $C[-1,1]$ on $w\Pi_{n-1}$ that is defined by

$$P_{n-1}(X)(f)(x) = w(x) \sum_{i=1}^{n} f(x_{i,n})\ell_{i,n}(X,x)/w(x_{i,n}). \tag{4.1}$$

Also define $\theta_k = \theta_{k,n} = k\pi/(n+1)$ and put

$$U = \{x_{k,n} = \cos\theta_{k,n} : k = 1,2,\ldots,n;\ n = 1,2,3,\ldots\}.$$

(Thus for fixed n, the $x_{k,n}$ are the zeros of the nth Chebyshev polynomial of the second kind.)

Mason and Elliott showed that the projection norm (or Lebesgue constant)

$$\|P_{n-1}(U)\| = \max_{\|f\| \leqslant 1} \|P_{n-1}(U)(f)\|$$

has the representation

$$\|P_{n-1}(U)\| = \max_{0 \leqslant \theta \leqslant \pi} F_n(\theta),$$

where

$$F_n(\theta) = \frac{|\sin(n+1)\theta|}{n+1} \sum_{i=1}^{n} \left| \frac{\sin\theta_i}{\cos\theta - \cos\theta_i} \right|.$$

Based on numerical computations, Mason and Elliott conjectured that the maximum of $F_n(\theta)$ occurs at $\pi/2$ for even n and asymptotically at $n\pi/(2n+2)$ (which is midway between the θ-nodes of $\theta_{(n-1)/2}$ and $\theta_{(n+1)/2} = \pi/2$) for odd n. This conjecture is supported by the graph of $F_7(\theta)$ in Figure 6.

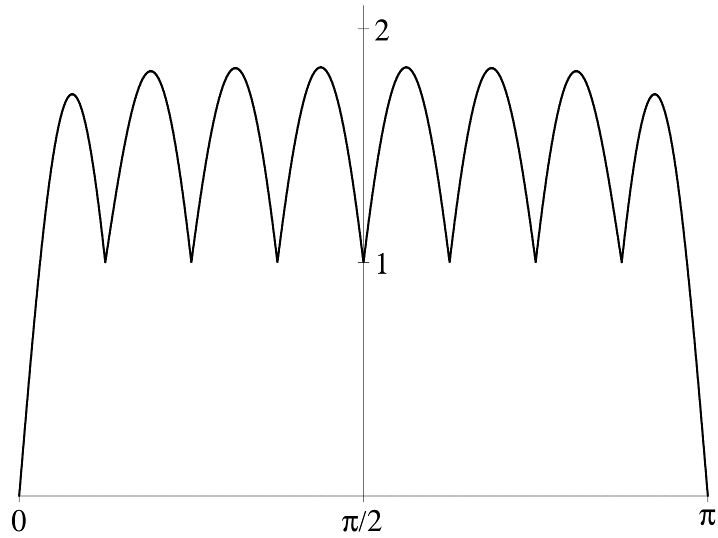

Figure 6: Plot of $F_7(\theta)$

Now, assuming that their conjecture about the maximum of $F_n(\theta)$ is true, Mason and Elliott showed that

$$\|P_{n-1}(U)\| = \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\gamma + \log \frac{4}{\pi} \right) + o(1). \tag{4.2}$$

Smith [18] later established the validity of (4.2), although the proof did not depend on Mason and Elliott's conjecture (which remains unresolved). The result (4.2) means that, to within $o(1)$ terms, $\|P_{n-1}(U)\|$ is equal to $\Lambda_n(X^*)$, the smallest possible Lebesgue constant for *unweighted* Lagrange interpolation (see Section 2.5). Furthermore, by a result of Kilgore [10], the minimum of $\|P_{n-1}(X)\|$ over all X is no smaller than $\Lambda_n(X^*)$. Thus

$$\min_X \|P_{n-1}(X)\| = \|P_{n-1}(U)\| + o(1),$$

which means that for the weighted interpolation method defined by (4.1), there is a simple description of nodes that are essentially optimal.

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