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Quadrature rules for periodic integrands. Bi-orthogonality and para-orthogonality*

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Abstract

In this paper, the algebraic construction of quadrature formulas for weighted periodic integrals is revised. For this purpose, two classical papers ([10] and [14]) in the literature are revisited and certain relations and connections are brought to light. In this respect, the concepts of "bi-orthogonality" and "para-orthogonality" are shown to play a fundamental role.

Key Words: Trigonometric polynomials, Szegő polynomials, quadratures, bi-orthogonality, para-orthogonality.

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1. Introduction

Let the integral $I_n(f) = \int_\Gamma f(z)d\mu(z)$ be given with Γ a certain curve in the complex plane and $d\mu$ a positive measure on Γ. By an n-point quadrature rule for this integral we mean an expression like $I_n(f) = \sum_{j=1}^n A_j f(z_j)$ with $z_j \neq z_k$ if $j \neq k$ and $\{z_j\}_{j=1}^n \subset \Gamma$ so that the weights or coefficients $\{A_j\}_{j=1}^n$ and nodes $\{z_j\}_{j=1}^n$ are to be determined by imposing that $I_n(f)$ exactly integrates i.e. $I_n(f)$ coincides with $I_\mu(f)$ for as many basis functions as possible in an appropriate function space S where the above integral exists. Two situations have been most widely considered in the literature. Namely, on the one hand, the case when Γ coincides with a subinterval of the real line, that is, $\Gamma = [a,b]$, $-\infty \leqslant a < b \leqslant \infty$ and on the other hand when Γ is the unit circle, i.e. $\Gamma = \mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Observe that the second case is equivalent to dealing with real integrals of the form $\int_{-\pi}^{\pi} f(\theta)d\mu(\theta)$, f being a 2π-periodic function (here by a slight abuse of notation

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we write $f(z) = f(\theta)$, $d\mu(z) = d\mu(\theta)$ for $z = e^{i\theta}$). As for the first case, it is well known that the construction of quadrature formulas to approximate integrals like $\int_a^b f(x)d\mu(x)$ represents an interesting research topic which has been exhaustively considered in the last decades and where orthogonal polynomials find one of their most direct and natural applications. Indeed, if $\{Q_k\}_{k=0}^{\infty}$ denotes the sequence of orthonormal polynomials for the measure μ, then $I_n(f) = \sum_{j=1}^{n} A_j f(x_j)$ with $\{x_j\}_{j=1}^{n}$ the zeros of $Q_n(x)$ and $\lambda_j = \left(\sum_{k=0}^{n} Q_k^2(x_j)\right)^{-1}$ for $j = 1, \ldots, n$ (Christoffel numbers) satisfies $I_n(P) = \int_a^b P(x)d\mu(x)$ for any polynomial P of degree $2n-1$. In this case, $\{I_n(f) \ : \ n = 1, 2, \ldots\}$ represent the well known sequence of Gaussian or Gauss–Christoffel quadrature formulas (see e.g. [8] for a survey). On the other hand, although quadratures on the unit circle and other related topics such as Szegő polynomials and the trigonometric moment problem have been receiving much recent attention because of their applications in rapidly growing fields of pure and applied mathematics (Digital Signal Processing, Operator Theory, Probability Theory, ...), there do not exist so many results about quadratures on the unit circle as in the real case. In this respect, the main aim of this paper is to emphasize the role played by certain sequences of orthogonal trigonometric polynomials introduced by Szegő [14] in the construction of quadrature rules on the unit circle by carrying out a comparision with the approach given by Jones et. al in [10]. In both approaches, a fundamental tool will be the so-called Szegő polynomials or polynomials which are orthogonal on the unit circle in the following sense: given $n \geqslant 1$, it is known (see e.g. [13]) that a unique monic polynomial $\rho_n(z)$ exists such that $\int_{-\pi}^{\pi} \rho_n(e^{i\theta})e^{-ik\theta}d\mu(\theta) = 0$ for $k = 0, 1, \ldots, n-1$. Furthermore, if we assume that the support of μ has infinitely many points, then $\int_{-\pi}^{\pi} \rho_n^2(e^{i\theta})d\mu(\theta) = \| \rho_n \|_{\mu}^2 > 0$. Setting $\rho_0 \equiv 1$, then $\{\rho_n\}_{n=0}^{\infty}$ is called the orthogonal sequence of monic Szegő polynomials. On the other hand, the sequence $\{\varphi_n\}_{n=0}^{\infty}$ with $\varphi_n(z) = \frac{\rho_n(z)}{\|\rho_n\|_{\mu}}$ represents an orthonormal sequence of Szegő polynomials (observe that such a sequence is uniquely determined by assuming that the leading coefficient of $\varphi_n(z)$ for $n = 0, 1, \ldots$ is positive). Setting $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ (sometimes we will use $\mathbb{E} = \{z \in \mathbb{C} \ : \ |z| > 1\}$, $\mathbb{C} = \mathbb{T} \cup \mathbb{D} \cup \mathbb{E}$) a fundamental property concerning the zeros of $\rho_n(z)$ for $n \geqslant 1$ (and apparently rather negative for our purposes) is the following (see e.g. [1]): "For each $n \geqslant 1$, all the zeros of $\rho_n(z)$ lie in $\mathbb{D}$". Thus, unlike the Gauss–Christoffel formulas, now the zeros of Szegő polynomials can not be directly used as nodes in our quadratures. Following two initially different paths, throughout the paper we will see how this drawback can be overcome. The paper is organized as follows. In Section 2, some preliminary results concerning trigonometric polynomials, Laurent polynomials and algebraic polynomials are presented. Then, in Section 3 the problem of the interpolation by trigonometric polynomials is analyzed whereas in Section 4 the so-called bi-orthogonal systems of trigonometric polynomials are introduced and their most relevant properties studied. The construction of quadrature rules exactly integrating trigonometric polynomials with degree as large as possible is considered in Section 5 and a connection with the unit circle presented in Section 6. Finally some illustrative numerical experiments are shown in Section 7.

2. Preliminary results

We will start by fixing some definitions and notations. Thus, for a nonnegative integer n, Π_n will denote the space of (in general complex) algebraic polynomials of degree n at most and Π the space of all polynomials. On the other hand, a real trigonometric polynomial of degree n is a function of the form

$$T_n(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta), \quad a_k, b_k \in \mathbb{R}, \quad |a_n| + |b_n| > 0.$$

Clearly, when a_0, a_k and b_k are in general complex numbers for $k = 1, \ldots, n$, we shall be dealing with trigonometric polynomials with complex coefficients. Thus, when we refer to a trigonometric polynomial we are implicitly meaning with real coefficients. We also denote by $\mathcal{T}_n$ the space of trigonometric polynomials of degree n at most, i.e.

$$\mathcal{T}_n = span\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$$

and hence, $dim\,(\mathcal{T}_n) = 2n + 1$. We occasionally deal with complex trigonometric polynomials, where a_0, a_k and b_k are arbitrary complex numbers. By using the transformation $z = e^{i\theta}$ and Euler's formulas, for any complex trigonometric polynomial one can write $T_n(\theta) = L_n(e^{i\theta})$ where

$$L_n(z) = \sum_{k=-n}^{n} c_k z^k. \tag{2.1}$$

Then

$$c_0 = a_0, \quad c_k = \frac{1}{2}(a_k - ib_k), \quad k = 1, \ldots, n,$$

and when the trigonometric polynomial T_n is real, a_0, a_k, b_k are real and $c_{-k} = \overline{c_k}$. Functions $L_n(z)$ as given above are called Laurent polynomials, or more generally, given p and q integers such that $p \leqslant q$, a Laurent polynomial is a function of the form

$$L_n(z) = \sum_{j=p}^{q} \alpha_j z^j, \quad \alpha_j \in \mathbb{C}. \tag{2.2}$$

We also denote by $\Lambda_{p,q}$ the space of Laurent polynomials (2.2). Observe that

$$\Lambda_{p,q} = span\left\{z^k : p \leqslant k \leqslant q\right\}.$$

Hence, $dim\,(\Lambda_{p,q}) = q - p + 1$. Thus, $L_n(z)$ given by (2.1) belongs to $\Lambda_{-n,n}$.

Now, by recalling that a double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of complex numbers is said to be "Hermitian" if $\mu_{-k} = \overline{\mu_k}$, a Laurent polynomial $L \in \Lambda_{-n,n}$ is called Hermitian if the sequence of its coefficients is Hermitian. That is, with $L_n(z)$ in (2.1) we have $c_k = \overline{c_k}$ for $k = 0, 1, \ldots, n$ and the following trivially holds,

Theorem 2.1. *Let $T_n(\theta)$ be a complex trigonometric polynomial, and set $L_n(e^{i\theta}) = T_n(\theta)$. Then T_n is real if and only if L_n is Hermitian.*

Remark 2.2. If we define $\Lambda_n^H = \{L \in \Lambda_{-n,n} \ : \ L\ Hermitian\}$ then Λ_n^H is a real vector space of dimension $2n+1$ and one can write

$$\mathcal{T}_n = \left\{T(\theta) \ : \ T(\theta) = L(e^{i\theta}) \text{ with } L \in \Lambda_n^H\right\}.$$

Let us next consider the connection between trigonometric polynomials and certain algebraic polynomials. For this purpose, let $P(z)$ be an algebraic polynomial of degree n, i.e.,

$$P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \quad a_n \neq 0.$$

Then, the reciprocal $P^*(z)$ of $P(z)$ is a polynomial defined by $P^*(z) = z^n P_*(z)$ where $P_*(z)$ represents the "sub-star" conjugate of $P(z)$, i.e., $P_*(z) = \overline{P\left(1/\bar{z}\right)}$. Thus,

$$P^*(z) = z^n \overline{P\left(1/\bar{z}\right)} = z^n \overline{P}\left(1/\bar{z}\right) = \sum_{j=0}^{n} \overline{a_{n-j}} z^j$$

where $\overline{P}(z) = \sum_{j=0}^{n} \overline{a_j} z^j$. Now, a usefull property of the polynomials that we shall work with is the following: for $k \in \mathbb{C}\backslash\{0\}$, a polynomial $P(z)$ is called "invariant" or more precisely, "k-invariant" if

$$P^*(z) = kP(z) \ \ \forall z \in \mathbb{C}.$$

Some direct consequences of this definition are:

1. If $P(z)$ is invariant, then $P(0) \neq 0$.

2. Let α be a zero of the invariant polynomial $P(z)$. Then, $1/\bar{\alpha}$ is also a zero of $P(z)$.

3. Let $P(z)$ be an invariant polynomial of odd degree n. Then, the number of zeros of $P(z)$ on $\mathbb{T}$ (counting multiplicities) is also odd. On the other hand, if $P(z)$ is an invariant polynomial with even degree n, it has an even number of zeros on $\mathbb{T}$.

4. Let $P(z)$ be invariant and set $P(z) = \sum_{j=0}^{n} c_j z^j = c_n \prod_{k=1}^{n}(z - z_k)$, then $|P(0)| = |c_0| = |c_n| \prod_{k=1}^{n} |z_k|$ and taking into account that $\prod_{k=1}^{n} |z_k| = 1$ it follows that $|c_0| = |c_n|$. Consequently, $c_n = k\overline{c_0}$ with $|k| = 1$. Set $k = e^{i\omega}$, $\omega \in \mathbb{R}$, and define $Q(z) = \lambda P(z)$, $\lambda \neq 0$. Then, $Q^*(z) = \overline{\lambda} k P(z) = \frac{\overline{\lambda}}{\lambda} k Q(z)$, that is, $Q(z)$ is $\frac{\overline{\lambda}}{\lambda}k$-invariant. Set now $\lambda = Re^{i\gamma}$, then $\frac{\overline{\lambda}}{\lambda}k = e^{i(\omega - 2\gamma)}$. Thus, by taking $\gamma \in \mathbb{R}$ such that $\gamma = \frac{\omega}{2} + m\pi$, with $m \in \mathbb{Z}$, then $\frac{\overline{\lambda}}{\lambda}k = 1$ and $Q(z)$ is 1-invariant.

Remark 2.3. The term "k-invariant" was introduced by Jones et. al. in [10], whereas Szegő in [14] says that a polynomial $P(z)$ is "autoreciprocal" if $P^*(z) = P(z)$ (1-invariant). Hence, we see that "invariant" polynomials are essentially "autoreciprocal".

Let $P_{2n}(z)$ be an invariant polynomial of degree $2n$. Then, there exists $\lambda_{2n} \in \mathbb{C}\backslash\{0\}$ such that $Q_{2n}(z) = \lambda_{2n} P_{2n}(z)$ is 1-invariant and we can write:

$$L_n(z) = \frac{Q_{2n}(z)}{z^n} = \sum_{j=-n}^{n} c_j z^j, \quad c_{-j} = \overline{c_j}, \; j = 0, 1, \ldots, n$$

that is, $L_n \in \Lambda_n^H$ and by Theorem 2.1, $L_n(e^{i\theta}) = T_n(\theta)$ with $T_n \in \mathcal{T}_n$. Thus, we have

$$e^{-in\theta} P_{2n}(e^{i\theta}) = \lambda_{2n}^{-1} T_n(\theta).$$

Conversely, let $T_n \in \mathcal{T}_n$. Then

$$T_n(\theta) = L_n(e^{i\theta}), \; L_n \in \Lambda_n^H.$$

Again, $L_n(z) = \frac{P_{2n}(z)}{z^n}$, where $P_{2n}(z) \in \Pi_{2n}$ and 1-invariant. Indeed, $P_{2n}(z) = z^n L_n(z)$. Hence,

$$\begin{aligned} P_{2n}^*(z) &= z^{2n}\overline{P_{2n}\left(1/\bar{z}\right)} = z^{2n} z^{-n} \overline{L_n\left(1/\bar{z}\right)} \\ &= z^n \textstyle\sum_{j=-n}^{n} \overline{c_j} z^{-j} = z^n \sum_{j=-n}^{n} c_{-j} z^{-j} = z^n L_n(z) = P_{2n}(z). \end{aligned}$$

Next, we will see how the connection between trigonometric polynomials and invariant algebraic polynomials allows us to recover some classical results about zeros of trigonometric polynomials. Thus, let α and β be arbitrary constants, then $\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)$ represents a trigonometric polynomial of degree one. Furthermore, it can be easily proved by induction that the function

$$T(\theta) = C \prod_{j=1}^{n} \sin\left(\frac{\theta - \theta_{2j-1}}{2}\right) \sin\left(\frac{\theta - \theta_{2j}}{2}\right), \quad C \neq 0 \tag{2.3}$$

where $\{\theta_j\}_{j=1}^{2n}$ are given constants, represents a trigonometric polynomial of degree n. We will now show that a converse result also holds, i.e. any trigonometric polynomial can be factorized as (2.3). Indeed, let $T_n \in \mathcal{T}_n$, then $T_n(\theta) = L_n(e^{i\theta})$, $L_n \in \Delta_n^H$ and one can write $L_n(z) = \frac{P_{2n}(z)}{z^n}$ with $P_{2n}(z)$ an 1-invariant polynomial of degree $2n$. Therefore, $P_{2n}(z) = c_n \prod_{k=1}^{2n}(z - z_k)$, $c_n \neq 0$ (counting multiplicities) with $z_j \neq 0$ and if $z_j \notin \mathbb{T}$, then $1/\bar{z}_j$ is also a root of $P_{2n}(z)$. Let $2m$ denote the number of zeros of $P_{2n}(z)$ on $\mathbb{T}$ ($0 \leqslant m \leqslant n$). Then

$$P_{2n}(z) = c_n \prod_{j=1}^{2m}(z - z_j) \prod_{k=1}^{n-m}(z - \tilde{z}_k)\left(z - \frac{1}{\overline{\tilde{z}_k}}\right), \; c_n \neq 0 \tag{2.4}$$

where $z_j = e^{i\theta_j}$, $\theta_j \in \mathbb{R}$ for $1 \leqslant j \leqslant 2m$, are the zeros of $P_{2m}(z)$ on $\mathbb{T}$ and $\tilde{z}_k$ and $1/\overline{\tilde{z}_k}$ for $1 \leqslant k \leqslant n-m$ are the zeros not on $\mathbb{T}$, so that $\tilde{z}_k = e^{i\omega_k}$ with $\omega_k \in \mathbb{C}$, wich implies that $1/\overline{\tilde{z}_k} = e^{i\overline{\omega_k}}$. Furthermore, it can be easily checked that $e^{i\theta} - e^{i\omega} = 2i\sin\left(\frac{\theta-\omega}{2}\right)e^{i\left(\frac{\theta+\omega}{2}\right)}$. Therefore,

$$\begin{aligned} P_{2n}(e^{i\theta}) &= c_n \prod_{j=1}^{2m}\left(e^{i\theta}-e^{i\theta_j}\right)\prod_{k=1}^{n-m}\left(e^{i\theta}-e^{i\omega_k}\right)\left(e^{i\theta}-e^{i\overline{\omega_k}}\right) \\ &= c_n(-1)^n 2^{2n}\prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)e^{i\left[\frac{\theta-\theta_j}{2}\right]}\times \\ &\times \prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right)e^{i\left[\theta+\frac{\omega_k+\overline{\omega_k}}{2}\right]}. \end{aligned}$$

Then, it follows that,

$$P_{2n}(e^{i\theta}) = \lambda_n e^{in\theta}\prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)\prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right), \quad \lambda_n \neq 0.$$

Consequently,

$$\begin{aligned} T_n(\theta) &= L_n(e^{i\theta}) = \frac{P_{2n}(e^{i\theta})}{e^{in\theta}} \\ &= \lambda_n \prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)\prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right), \end{aligned} \tag{2.5}$$

where $\lambda_n \neq 0$, $\theta_j \in \mathbb{R}$ and $\omega_k \in \mathbb{C}$ such that $\Re(\omega_k) = \psi_k + 2t\pi$, $\psi_k \in (-\pi, \pi]$, $t \in \mathbb{Z}$ and $k = 1, \ldots, n-m$. Then, we have proved the following

Theorem 2.4. *A real trigonometric polynomial $T_n(\theta)$ of the precise degree n has exactly $2n$ real or complex zeros provided that we count them as usual with their multiplicity and we restrict ourselves to the strip $-\pi < \Re(\theta) \leqslant \pi$. Furthermore, the non-real zeros appear in conjugate pairs.*

Remark 2.5. The representation (2.3) is of course not unique.

Furthermore, from (2.3) and (2.5) it can be also proved

Theorem 2.6 (L.Fejér and F.Riesz). *A real trigonometric polynomial $T(\theta)$ is nonnegative for all real θ, if and only if, it can be written in the form*

$$T(\theta) = |g(z)|^2, \quad z = e^{i\theta}$$

where $g(z)$ is an algebraic polynomial of the same degree as $T(\theta)$.

Proof. Assume $T(\theta)$ a trigonometric polynomial of degree n such that $T(\theta) = \frac{P(e^{i\theta})}{e^{in\theta}}$ with $P(z)$ a polynomial of degree $2n$. Since $T(\theta) \geqslant 0$ for all $\theta \in \mathbb{R}$, then possible real zeros of $T(\theta)$ must have even multiplicity. Furthermore, if $\theta = \alpha$ is a real zero of $T(\theta)$ then $z = e^{i\alpha}$ is a zero of $P(z)$ on $\mathbb{T}$. Hence, from (2.4), $P(z)$ can be expressed as:

$$P(z) = \lambda_n p_m^2(z) q_{n-m}(z) q_{n-m}^*(z), \quad \lambda_n \neq 0$$

where $p_m(z) \in \Pi_m$ for $0 \leqslant m \leqslant n$ and $q_{n-m}(z) \in \Pi_{n-m}$. Since $T(\theta) \geqslant 0$, for any $\theta \in \mathbb{R}$,

$$\begin{aligned} T(\theta) &= |T(\theta)| = \left|\frac{P(e^{i\theta})}{e^{in\theta}}\right| = |\lambda_n| \left|p_m^2(e^{i\theta})\right| \left|q_{n-m}(e^{i\theta})\right| \left|\overline{q_{n-m}(e^{i\theta})}\right| \\ &= |\lambda_n| \left|p_m^2(e^{i\theta})\right| \left|q_{n-m}(e^{i\theta})\right|^2 = \left|g(e^{i\theta})\right|^2 \end{aligned}$$

where $g(z) = \sqrt{|\lambda_n|} p_m(z) q_{n-m}(z) \in \Pi_n$.

Conversely, let $g(z)$ be an algebraic polynomial of degree n, then by setting $z = e^{i\theta}$ it follows that

$$|g(z)|^2 = g(z)\overline{g(z)} = g(z) g_*(z) = \frac{g(z) g^*(z)}{z^n} = \frac{P_{2n}(z)}{z^n}$$

where $P_{2n}(z) = g(z) g^*(z)$ is clearly an 1-invariant polynomial of degree $2n$ so that $|g(z)|^2 = L_n(z) \in \Lambda_n^H$, and by Theorem 2.1, $|g(z)|^2$ represents a trigonometric polynomial of degree n which is clearly nonnegative for any $\theta \in \mathbb{R}$. □

3. Interpolation by Trigonometric Polynomials

As it is well known, polynomial interpolation finds in the construction of quadrature formulas one of its most immediate applications. On the other hand, when considering quadrature rules based on trigonometric polynomials, similar results on interpolation will be needed. In this respect, some of the already known results will now be proved by means of the close connection between trigonometric polynomials and Hermitian Laurent polynomials shown in the preceding section. First we have,

Theorem 3.1. *Given $(2n+1)$ distinct nodes $\{\theta_j\} \subset (-\pi, \pi]$, there exists a unique $T_n \in \mathcal{T}_n$ such that*

$$T_n(\theta_j) = y_j, \quad j = 1, \ldots, 2n+1, \tag{3.1}$$

$\{y_j\}_{j=1}^{2n+1}$ being a given set of real numbers.

Proof. Set $T(\theta) = a_0 + \sum_{k=1}^{n} a_k \cos k\theta + b_k \sin k\theta$. We first show that the constants $\{a_k\}_{k=0}^{n} \cup \{b_k\}_{k=1}^{n}$ are uniquely determined from conditions (3.1). Now, $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{-n,n}$ so that (3.1) is equivalent to

$$L(z_j) = y_j, \quad z_j = e^{i\theta_k}, \quad j = 1, \ldots, 2n+1. \tag{3.2}$$

Now $L(z) \in \Lambda_{-n,n}$ implies that $L(z) = \frac{P(z)}{z^n}$, $P(z) \in \Pi_{2n}$ so that (3.2) yields

$$P(z_j) = z_j^n y_j, \quad j = 1, \ldots, 2n+1. \tag{3.3}$$

Since $z_j \neq z_k$, $P(z)$ is uniquely determined by (3.3) and hence $T(\theta)$ has the desired interpolation properties. It remains to show that $T(\theta)$ has real coefficients. This will be proved by showing that $P(z)$ is 1-invariant. To see this we will show that also $P^*(z)$ satisfies the interpolation conditions (3.3). Indeed,

$$P^*(z_j) = z_j^{2n}\overline{P\left(1/\bar{z}_j\right)} = z_j^{2n}\overline{P(z_j)} = z_j^{2n}\overline{z_j^n y_j} = z_j^n y_j, \quad y_j \in \mathbb{R}.$$

Hence, by virtue of the uniqueness of polynomial $P(z)$, it follows that $P^*(z) = P(z)$ and the proof is completed. □

As for an explicit representation of $T_n \in \mathcal{T}_n$ satisfying (3.1), because of uniqueness, one can write

$$T_n(\theta) = \sum_{j=1}^{2n+1} l_j(\theta) y_j \tag{3.4}$$

where $l_j(\theta) = l_{j,n}(\theta) \in \mathcal{T}_n$ such that $l_j(\theta_k) = \delta_{j,k} = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$. Since $l_j(\theta_k) = 0$ for $k = 1, \ldots, 2n+1$, $k \neq j$, clearly by (2.5),

$$l_j(\theta) = \lambda_j \prod_{k=1,k\neq j}^{2n+1} \sin\left(\frac{\theta - \theta_k}{2}\right), \quad \lambda_j \neq 0,$$

λ_j being a normalization constant such that $l_j(\theta_j) = 1$. More precisely, setting

$$W_n(\theta) = \prod_{k=1}^{2n+1} \sin\left(\frac{\theta - \theta_k}{2}\right)$$

then, it follows that

$$l_j(\theta) = \lambda_j \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n+1.$$

Thus,

$$l_j(\theta_j) = \lambda_j \lim_{\theta\to\theta_j} \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)} = \lambda_j \lim_{\theta\to\theta_j} \frac{W_n(\theta)}{\frac{\theta-\theta_j}{2}} = 2\lambda_j W_n'(\theta_j).$$

Hence, taking $\lambda_j = \frac{1}{2W_n'(\theta_j)}$ one has $l_j(\theta_j) = 1$ and we can write

$$l_j(\theta) = \frac{W_n(\theta)}{2W_n'(\theta_j)\sin\left(\frac{\theta-\theta_j}{2}\right)} \quad , \quad j = 1, \ldots, 2n+1.$$

Furthermore, when dealing with the construction of certain quadrature formulas exactly integrating trigonometric polynomials of degree as high as possible, the following result will be required:

Theorem 3.2. *Let $\theta_1 \dots \theta_{n+1}$ be $(n+1)$ distinct nodes on $(-\pi, \pi]$. Then there exists a unique trigonometric polynomial $H_n \in \mathcal{T}_n$ satisfying*

$$\left.\begin{array}{ll} H_n(\theta_j) = H_n^{(k)}(\theta_j) = y_j & j = 1, \dots, n+1 \\ \\ H_n^{'}(\theta_j) = H_n^{(k)\,'}(\theta_j) = y_j^{'} & j = 1, \dots, n+1, \ j \neq k \end{array}\right\} \quad (3.5)$$

where $k \in \{1, \dots, n+1\}$ is previously fixed and $\{y_j\}_{j=1}^{n+1} \cup \{y_j^{'}\}_{j=1, j \neq k}^{n+1}$ is a set of $(2n+1)$ real numbers.

Proof. Set $H_n(\theta) = L_n(e^{i\theta}) \in \Lambda_{-n,n}$. Then (3.5) becomes $H_n(\theta_j) = L_n(e^{i\theta_j}) = L_n(z_j) = y_j$ with $z_j = e^{i\theta_j} \in \mathbb{T}$ for all $j = 1, \dots, n+1$ and $z_j \neq z_k$ if $j \neq k$. On the other hand, $H_n^{'}(\theta) = L_n^{'}(e^{i\theta}) i e^{i\theta}$. Hence, $L_n^{'}(z_j) = -i\overline{z_j} H_n^{'}(\theta_j) = -i\overline{z_j} y_j^{'}$ for $j = 1, \dots, n+1$ and $j \neq k$. Since $L_n \in \Lambda_{-n,n}$, then $L_n(z) = \frac{P_{2n}(z)}{z^n}$ with $P_{2n}(z) \in \Pi_{2n}$ such that $P_{2n}(z_j) = z_j^n L_n(z_j) = z_j^n y_j$, $y_j \in \mathbb{R}$ and $z_j \in \mathbb{T}$. Furthermore, $P_{2n}^{'}(z) = n z^{n-1} L_n(z) + z^n L_n^{'}(z)$, hence

$$P_{2n}^{'}(z_j) = n z_j^{n-1} L_n(z_j) + z_j^n L_n^{'}(z_j) = z_j^{n-1}\left(n y_j - i y_j^{'}\right), \ j = 1, \dots, n+1, \ j \neq k.$$

Thus our Hermite-type trigonometric interpolation problem reduces to finding $P_{2n}(z) \in \Pi_{2n}$ such that

$$\left.\begin{array}{ll} P_{2n}(z_j) = z_j^n y_j & j = 1, \dots, n+1 \\ \\ P_{2n}^{'}(z_j) = z_j^{n-1}\left(n y_j - i y_j^{'}\right) & j = 1, \dots, n+1, \ j \neq k \end{array}\right\} \quad (3.6)$$

Now, since $z_j \neq z_l$ for $j \neq l$, it is known that the interpolation problem (3.6) has a unique solution $P_{2n}(z)$ and $T_n(\theta) = L_n(e^{i\theta}) = \frac{P_{2n}(e^{i\theta})}{e^{in\theta}}$ will be the unique solution to (3.5). As in Theorem 3.1, it remains to prove that $T_n(\theta)$ is a real trigonometric polynomial. To do this, we will show that $P_n^*(z)$ is also a solution to (3.6), hence because of uniqueness we have $P_{2n}(z) = P_{2n}^*(z)$ and the conclusion follows. Indeed,

$$\begin{aligned} P_{2n}^*(z_j) &= z_j^{2n} \overline{P_{2n}\left(1/\overline{z_j}\right)} = z_j^{2n} \overline{P_{2n}(z_j)} \\ &= z_j^{2n} \overline{z_j^n y_j} = z_j^n y_j = P_{2n}(z_j), \ j = 1, \dots, n+1. \end{aligned}$$

Furthermore, $\left(P_{2n}^*\right)^{'}(z) = 2n z^{2n-1} \overline{P_{2n}}(1/z) + z^{2n} \left(\overline{P_{2n}}\right)^{'}(1/z)\left(\frac{-1}{z^2}\right)$, yielding:

$$\left(P_{2n}^*\right)^{'}(z_j) = z_j^{2n-2}\left[2n z_j \overline{P_{2n}(z_j)} - \overline{P_{2n}^{'}(z_j)}\right].$$

(Here we are making use of the fact $\left(\overline{P}\right)^{'}(z) = \overline{(P^{\,'})}(z)$). Therefore, for $j = 1, \dots, n+1, \ j \neq k$:

$$\begin{aligned} \left(P_{2n}^*\right)^{'}(z_j) &= z_j^{2n-2}\left[2n z_j \overline{z_j^n y_j} - z_j^{-(n-1)}(n y_j + i y_j^{'})\right] \\ &= z_j^{n-1}\left[2n y_j - n y_j - i y_j^{'}\right] = z_j^{n-1}[n y_j - i y_j^{'}] = P_{2n}^{'}(z_j). \end{aligned}$$

□

As for an explicit representation of the interpolating trigonometric polynomial $H_n(\theta)$ satisfying (3.5), by virtue of uniqueness we can write for any $k \in \{1, \dots, n+1\}$,

$$H_n(\theta) = H_n^{(k)}(\theta) = t_k^{(k)}(\theta) y_k + \sum_{j=1, j \neq k}^{n+1} \left[t_j^{(k)}(\theta) y_j + s_j^{(k)}(\theta) y_j^{'} \right] \tag{3.7}$$

where $t_j^{(k)}(\theta)$ and $s_j^{(k)}(\theta)$ are trigonometric polynomials in $\mathcal{T}_n$, such that

$$\begin{aligned}
& t_j^{(k)}(\theta_r) = \delta_{j,r} && 1 \leqslant j, r \leqslant n+1 \\
& \left(t_j^{(k)}\right)^{'}(\theta_r) = 0 && 1 \leqslant j, r \leqslant n+1,\ r \neq k \\
& s_j^{(k)}(\theta_r) = 0 && 1 \leqslant r \leqslant n+1,\ j \neq k \\
& \left(s_j^{(k)}\right)^{'}(\theta_r) = \delta_{j,r} && 1 \leqslant j, r \leqslant n+1,\ r \neq k,\ j \neq k.
\end{aligned} \tag{3.8}$$

Define now $W_n(\theta) = \prod_{j=1}^{n+1} \sin\left(\frac{\theta - \theta_j}{2}\right)$. If we proceed as in the previous case, after some elementary calculations we deduce the following expressions for such trigonometric polynomials for $1 \leqslant j \leqslant n+1,\ j \neq k$:

$$s_j^{(k)}(\theta) = \frac{W_n^2(\theta) \sin\left(\frac{\theta_j - \theta_k}{2}\right)}{2 \sin\left(\frac{\theta - \theta_j}{2}\right) \sin\left(\frac{\theta - \theta_k}{2}\right) \left[W_n^{'}(\theta_j)\right]^2} \in \mathcal{T}_n, \tag{3.9}$$

$$\begin{aligned}
t_j^{(k)}(\theta) &= \frac{W_n^2(\theta)}{\sin^2 \frac{\theta - \theta_j}{2} \ \sin \frac{\theta - \theta_k}{2} \ \left[2 W_n^{'}(\theta_j)\right]^2} \times \\
&\times \left[\sin\left(\frac{\theta_j - \theta_k}{2}\right) + \cos\left(\frac{\theta_j - \theta_k}{2}\right) \sin\left(\frac{\theta - \theta_j}{2}\right) \right] \in \mathcal{T}_n
\end{aligned} \tag{3.10}$$

and

$$t_k^{(k)}(\theta) = \left[\frac{W_n(\theta)}{2 W_n^{'}(\theta_k) \sin\left(\frac{\theta - \theta_k}{2}\right)} \right]^2 \in \mathcal{T}_n. \tag{3.11}$$

In the rest of the section we shall be concerned with certain interpolation problems using an even number of nodes, say $2n$, in subspaces $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ of dimension $2n$. For instance, $\tilde{\mathcal{T}}_n = \mathcal{T}_n \backslash span\{\cos n\theta\}$ or $\tilde{\mathcal{T}}_n = \mathcal{T}_n \backslash span\{\sin n\theta\}$. In this respect, it should be recalled that a system of continuous functions $\{f_0, \dots, f_m\}$ on an interval $[a, b]$ represents a Haar system on $[a, b]$ if and only if for any k, $1 \leqslant k \leqslant m$, $\{f_0, \dots, f_k\}$ is a Chebyshev system on $[a, b]$. Clearly,

$$\{1, \cos\theta, \sin\theta, \dots, \cos n\theta, \sin n\theta\},$$

can not be a Haar system on $[-\pi, \pi]$ (check simply that $\{1, \cos\theta\}$ is not a Chebyshev system). Hence, we can not initially assume that given $2n$ nodes $\{\theta_j\}_{j=1}^{2n}$ on $(-\pi, \pi]$ there exists $T_n \in \mathcal{T}_n \backslash span\{\cos n\theta\}$ or in $\mathcal{T}_n \backslash span\{\sin n\theta\}$ such that $T_n(\theta_j) = y_j$ for all $j = 1, \dots, 2n$. However, we can prove the following

Theorem 3.3. *Let $\{\theta_j\}_{j=1}^{2n} \subset (-\pi, \pi]$ be $2n$ distinct nodes, let $\{y_j\}_{j=1}^{2n}$ be arbitrary real numbers, and consider the interpolation problem:*

$$\tilde{T}_n(\theta_j) = y_j, \quad j = 1, \ldots, 2n. \tag{3.12}$$

Then the following hold:

1. *If $\sum_{j=1}^{2n} \theta_j \neq k\pi$ for all $k \in \mathbb{Z}$, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\cos n\theta\}$ and a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\sin n\theta\}$.*

2. *If $\sum_{j=1}^{2n} \theta_j = k\pi$ for an odd integer k, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\cos n\theta\}$.*

3. *If $\sum_{j=1}^{2n} \theta_j = k\pi$ for an even integer k, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\sin n\theta\}$.*

Proof. Assume first that we are trying to find $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ satisfying (3.12). Thus, we can write:

$$\tilde{T}_n(\theta) = a_0 + \sum_{j=1}^{n-1} (a_j \cos j\theta + b_j \sin j\theta) + a_n \cos n\theta = L_n(e^{i\theta}) \in \Lambda_{-n,n}$$

with $L_n(z) = \sum_{j=-n}^{n} c_j z^j$, where

$$c_j = \frac{a_j - ib_j}{2}, \; c_{-j} = \overline{c_j}, \; 1 \leqslant j \leqslant n-1, \; c_0 = a_0.$$

Thus, $c_{-j} = \overline{c_j}$ for all $0 \leqslant j \leqslant n$. Setting as usual $z_j = e^{i\theta_j}$ for all $j = 1, \ldots, 2n$, ($z_j \neq z_k$ if $j \neq k$), (3.12) becomes

$$\tilde{T}_n(\theta_j) = L_n(e^{i\theta_j}) = L_n(z_j) = y_j, \; j = 1, \ldots, 2n$$

giving rise to the linear system

$$\sum_{k=-(n-1)}^{n-1} c_k z_j^k + c_n(z_j^n + z_j^{-n}) = y_j, \; j = 1, \ldots, 2n. \tag{3.13}$$

Now, the system (3.13) has a unique solution if and only if $\Delta_n \neq 0$, where

$$\Delta_n = \begin{vmatrix} z_1^{-(n-1)} & z_1^{-(n-2)} & \cdots & 1 & \cdots & z_1^{n-1} & \left(z_1^n + z_1^{-n}\right) \\ z_2^{-(n-1)} & z_2^{-(n-2)} & \cdots & 1 & \cdots & z_2^{n-1} & \left(z_2^n + z_2^{-n}\right) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ z_{2n}^{-(n-1)} & z_{2n}^{-(n-2)} & \cdots & 1 & \cdots & z_{2n}^{n-1} & \left(z_{2n}^n + z_{2n}^{-n}\right) \end{vmatrix}.$$

By introducing the Vandermonde determinant associated with $z_1, \ldots, z_{2n}$, i.e.,

$$\gamma_n = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{2n-1} \\ 1 & z_2 & \cdots & z_2^{2n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_{2n} & \cdots & z_{2n}^{2n-1} \end{vmatrix}$$

it can be easily checked that

$$\Delta_n = (z_1 \cdots z_{2n})^{n-1}(1 - z_1 \cdots z_{2n})\gamma_n. \tag{3.14}$$

On the other hand, if we consider our interpolation problem in $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\cos n\theta\}$, the associated determinant $\tilde{\Delta}_n$ of the corresponding system satisfies

$$\tilde{\Delta}_n = (z_1 \cdots z_{2n})^{n-1}(1 + z_1 \cdots z_{2n})\gamma_n. \tag{3.15}$$

Since $z_j = e^{i\theta_j}$, then $z_1 \cdots z_{2n} = e^{i\lambda_n}$ with $\lambda_n = \sum_{j=1}^{2n} \theta_j$. If $\lambda_n \neq k\pi$ for any integer k, then clearly $z_1 \cdots z_{2n} \neq \pm 1$ and from (3.14) and (3.15), both determinants Δ_n and $\tilde{\Delta}_n$ are nonzero since $\gamma_n \neq 0$, which means that the interpolation problem (3.12) has a unique solution both in $\mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\mathcal{T}_n \backslash span\{\cos n\theta\}$. Next, assume that $\lambda_n = k\pi$ for some integer k. Thus, if k is even, then $e^{i\lambda_n} = 1$ and (3.15) is different from zero, whereas if k is odd, then $e^{i\lambda_n} = -1$ and (3.14) does not vanish. Thus, for instance, if $\Delta_n \neq 0$, we have found a unique $L_n \in \Lambda_{-n,n}$, $L_n(z) = \sum_{j=-n}^{n} c_j z^j$ such that $c_{-n} = c_n$ and satisfying $L_n(z_j) = y_j$ for $j = 1, \ldots, 2n$. Therefore, $\tilde{T}_n(\theta) = L_n(e^{i\theta}) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\tilde{T}_n(\theta_j) = y_j$ for $j = 1, \ldots, 2n$. To check that $\tilde{T}_n(\theta)$ is actually a real trigonometric polynomial we proceed as in Theorem 3.1. □

Next, a Lagrange-type representation for the trigonometric polynomial $\tilde{T}_n(\theta)$ satisfying the conditions of Theorem 3.3 will be given. Indeed, set

$$\eta_n = \frac{1}{2}\sum_{j=1}^{2n} \theta_j = \frac{1}{2}\lambda_n$$

and assume that $\eta_n \neq k\pi$ for any integer k so that $\Delta_n \neq 0$. Thus, $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and by virtue of uniqueness, one has $\tilde{T}_n(\theta) = \sum_{j=1}^{2n} \tilde{t}_j(\theta) y_j$ where $\tilde{t}_j \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\tilde{t}_j(\theta_k) = \delta_{j,k}$ for $1 \leqslant j,k \leqslant 2n$. Fix $j \in \{1, \ldots, 2n\}$ and define $\alpha_j = \sum_{k=1,k\neq j}^{2n} \theta_j$. Now, we can write $\tilde{s}_j(\theta) = \frac{\tilde{l}_j(e^{i\theta})}{e^{in\theta}}$ where $\tilde{l}_j(z) \in \Pi_{2n}$ such that $\tilde{l}_j(z_k) = z_j^n \delta_{j,k}$ where, as usual, $z_k = e^{i\theta_k}$ for $k = 1, \ldots, 2n$. Since $\tilde{t}_j \in \mathcal{T}_n \backslash span\{\sin n\theta\}$, the leading coefficient of $\tilde{l}_j(z)$ must coincide with $\tilde{l}_j(0)$, and one has $\tilde{l}_j(z) = c_j(z - w_j)\prod_{k=1,k\neq j}^{2n}(z - z_j) = c_j z^{2n} + \cdots + \tilde{l}_j(0)$. But $\tilde{l}_j(0) = c_j w_j \prod_{j=1,j\neq k}^{2n} z_j$, hence

$$w_j = \frac{1}{\prod_{j=1,j\neq k}^{2n} z_j} = \prod_{j=1}^{2n} \overline{z_j} = e^{-\sum_{j=1,j\neq k}^{2n} \theta_j} = e^{-i\alpha_j}.$$

Therefore, by (2.5) it follows that

$$\tilde{s}_j(\theta) = \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \prod_{j=1, j\neq k}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right)$$

where $\tilde{c}_j$ is to be determined such that $\tilde{s}_j(\theta_j) = 1$. Setting

$$W_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right) \in \mathcal{T}_n,$$

we have

$$\tilde{s}_j(\theta) = \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}.$$

Now,

$$1 = \lim_{\theta\to\theta_j} \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)} = 2\tilde{c}_j \sin\left(\frac{\theta_j+\alpha_j}{2}\right) W_n^{'}(\theta_j).$$

Observe that $\frac{1}{2}(\theta_j + \alpha_j) = \frac{1}{2}\sum_{j=1}^{2n}\theta_j = \eta_n \neq k\pi$ for any integer k, so that $\sin\left(\frac{\theta_j+\alpha_j}{2}\right) = \sin\eta_n \neq 0$ and hence

$$\tilde{s}_j(\theta) = \frac{1}{2W_n^{'}(\theta_j)\sin\eta_n} \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n. \tag{3.16}$$

When dealing with the interpolant $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\cos n\theta\}$ it can be easily verified that the fundamental Lagrange-type trigonometric polynomials $\tilde{s}_j(\theta)$ are now given by

$$\tilde{s}_j(\theta) = \frac{1}{2W_n^{'}(\theta_j)\cos\eta_n} \cos\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n. \tag{3.17}$$

with α_j and η_n as previously given.

4. Bi-orthogonal systems

Let $\omega(\theta)$ be a weight function on $(-\pi,\pi]$, i.e., $\omega(\theta) \geqslant 0$ on $(-\pi,\pi]$ and $0 < \int_{-\pi}^{\pi}\omega(\theta)d\theta < \infty$. The main aim of this section is briefly collecting some results by Szegő (see [14]) concerning properties of an orthogonal basis for the space $\mathcal{T}$ of real trigonometric polynomials with respect to the inner product on $\mathcal{T}$ induced by $\omega(\theta)$, namely,

$$\langle f,g\rangle_\omega = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\omega(\theta)d\theta, \;\; \forall\, f,g \in \mathcal{T} \tag{4.1}$$

As indicated in [14], we might consider an arbitrary measure $d\mu(\theta)$ on the unit circle; in what follows, however we restrict ourselves for the sake of simplicity, to the previously defined case, i.e. to the case when $\mu(\theta)$ is absolutely continuous. Furthermore, when only real functions are considered, complex conjugation in (4.1) can be omitted. For this purpose, let us first consider the basis of $\mathcal{T}_n$ given by $\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$ which is clearly orthogonal for $\omega(\theta) \equiv 1$ on $[-\pi, \pi]$ and let us see how this property can be extended to an arbitrary weight function $\omega(\theta)$. Certainly, this can be done by orthogonalizing the elementary functions

$$1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta$$

arranged in a linear order, according to Gram-Schmidt process. Thus, a set

$$\{f_0, f_1, g_1, \ldots, f_n, g_n\}$$

of trigonometric polynomials is generated such that f_0 is a nonzero constant,

$$f_1 \in span\{1, \cos\theta\},\ g_1 \in span\{1, \cos\theta, \sin\theta\},\ f_2 \in span\{1, \cos\theta, \sin\theta, \cos 2\theta\}$$

$$g_2 \in span\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta\}\ \ \ldots\ \ f_n \in \mathcal{T}_n \backslash span\{\sin n\theta\},\ g_n \in \mathcal{T}_n$$

and it holds that

$$\begin{array}{lll} \langle f_j, f_k\rangle_\omega = \kappa_j \delta_{j,k} & , & \kappa_j > 0 \\ \langle g_j, g_k\rangle_\omega = \kappa_j^{'} \delta_{j,k} & , & \kappa_j^{'} > 0 \\ \langle f_j, g_k\rangle_\omega = 0,\ j = 0, 1, \ldots, n & , & k = 1, \ldots, n. \end{array} \tag{4.2}$$

When the process is repeated for each $n \in \mathbb{N}$, then $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ represents an orthogonal basis for $\mathcal{T}$ with respect to $\omega(\theta)$. Now, if we set

$$\begin{aligned} f_0 &= a_{0,0} \neq 0 \\ f_j &= a_{j,0} + \textstyle\sum_{k=1}^{j} \left(a_{j,k} \cos k\theta + b_{j,k} \sin k\theta\right) \\ g_j &= c_{j,0} + \textstyle\sum_{k=1}^{j} \left(c_{j,k} \cos k\theta + d_{j,k} \sin k\theta\right) \end{aligned} \tag{4.3}$$

then, because of the linear independence it clearly follows that

$$\begin{vmatrix} a_{n,n} & b_{n,n} \\ c_{n,n} & d_{n,n} \end{vmatrix} \neq 0, \quad n \geqslant 1.$$

Conversely, we also have (see [14])

Theorem 4.1. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a system of trigonometric polynomials such that $f_0(\theta) \equiv c \neq 0$ and for $n \geqslant 1$:*

$$\begin{aligned} f_n(\theta) &= a_{n,0} + \textstyle\sum_{k=1}^{n} \left(a_{n,k} \cos k\theta + b_{n,k} \sin k\theta\right), \\ g_n(\theta) &= c_{n,0} + \textstyle\sum_{k=1}^{n} \left(c_{n,k} \cos k\theta + d_{n,k} \sin k\theta\right). \end{aligned}$$

Assume that for $n \geqslant 1$,

$$\begin{vmatrix} a_{n,n} & b_{n,n} \\ c_{n,n} & d_{n,n} \end{vmatrix} \neq 0.$$

Then, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{T}$.

Now, according to Szegő (see [14]) we are in a position to state the following definitions:

Definition 4.2. Two trigonometric polynomials of degree n, of the form

$$f(\theta) = a\cos n\theta + b\sin n\theta + \cdots\,,\quad g(\theta) = c\cos n\theta + d\sin n\theta + \cdots$$

are said to be linearly independent if and only if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Definition 4.3. Given the weight function $\omega(\theta)$ on $[\pi, \pi]$, a system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ of real trigonometric polynomials with f_0 a nonzero constant will be called a bi-orthogonal system for $\omega(\theta)$ if the following holds:

1. For each $n \geqslant 1$, $f_n(\theta)$ and $g_n(\theta)$ are linearly independent.
2. The system is orthogonal with respect to the inner produc generated by $\omega(\theta)$, i.e., (4.2) is satisfied.

Next, let us see how a bi-orthogonal system can be constructed from a sequence of orthogonal polynomials on the unit circle (Szegő polynomials) for $\omega(\theta)$. To fix ideas, let $\{\rho_n(z)\}_{n=0}^{\infty}$ be the sequence of monic Szegő polynomials: $\rho_n(z) = z^n + \cdots + \delta_n$ for $n = 0, 1, \ldots$. Here, $\delta_n = \rho_n(0)$ ($\delta_0 \neq 0$; $|\delta_n| < 1$ for $n = 1, 2, \ldots$) represents the n-th reflection coefficient or Schur parameter. Let $\{\omega_n\}_{n=0}^{\infty}$ be a given sequence of nonzero complex numbers and consider $\frac{\omega_n \rho_{2n+1}(z)}{z^n} \in \Lambda_{-(n+1),n+1}$. Here, one can write

$$\omega_n e^{-in\theta} \rho_{2n+1}(e^{i\theta}) = f_{n+1}(\theta) + i g_{n+1}(\theta) \tag{4.4}$$

where $f_{n+1}(\theta)$ and $g_{n+1}(\theta)$ are real trigonometric polynomials of degree $n+1$ ($n = 0, 1, \ldots$), and we have (see [3])

Theorem 4.4. *Let $\{\omega_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that for any $n \geqslant 0$, $\omega_n \neq 0$ and $\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta} \rho_{2n+1}(e^{i\theta}) \omega(\theta) d\theta$ is a real number. Then the real trigonometric polynomials $f_0 \cup \{f_{n+1}, g_{n+1}\}_{n=0}^{\infty}$ given by (4.4) with $f_0(\theta) = f_0 \neq 0$ is a bi-orthogonal system for $\omega(\theta)$.*

Remark 4.5. For an alternative construction of a bi-orthogonal system making use of orthonormal Szegő polynomials of even instead of odd degree, see [14].

Example 4.6. Take $\omega(\theta) \equiv 1$ on $[-\pi, \pi]$ (Lebesgue measure). It is known that $\rho_n(z) = z^n$ for $n = 0, 1, \ldots$ so that, for any $\omega_n \in \mathbb{C}\backslash\{0\}$:

$$\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta} \rho_{2n+1}(e^{i\theta}) \omega(\theta) d\theta = \omega_n^2 \int_{-\pi}^{\pi} e^{i(2n+2)\theta} d\theta = 0.$$

Hence, we can take any nonzero complex number ω_n. Set $\omega_n = \alpha_n + i\beta_n$, $\alpha_n, \beta_n \in \mathbb{R}$ and $|\alpha_n| + |\beta_n| > 0$. Then,

$$\begin{aligned} f_{n+1}(\theta) &= \alpha_n \cos(n+1)\theta - \beta_n \sin(n+1)\theta \\ g_{n+1}(\theta) &= \beta_n \cos(n+1)\theta + \alpha_n \sin(n+1)\theta. \end{aligned} \tag{4.5}$$

Furthermore, by taking $\omega_n = 1$, for $n = 0, 1, \ldots$, we obtain

$$\tilde{f}_{n+1}(\theta) = \cos(n+1)\theta, \quad \tilde{g}_{n+1}(\theta) = \sin(n+1)\theta \tag{4.6}$$

and the well known orthogonal properties of the functions

$$\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta, \ldots\}$$

with respect to the weight function $\omega(\theta) \equiv 1$ are now recovered.

Remark 4.7. It should be noted that the relations (4.5) and (4.6) between two bi-orthogonal systems for $\omega(\theta) \equiv 1$ hold for any arbitrary $\omega(\theta)$. Indeed, let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ and $\tilde{f}_0 \cup \{\tilde{f}_k, \tilde{g}_k\}_{k=1}^{\infty}$ be two bi-orthogonal systems for a given weight function $\omega(\theta)$. Since $\tilde{f}_n \in \mathcal{T}_n$ and $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{T}_n$, one has

$$\tilde{f}_n(\theta) = \alpha_0 f_0 + \sum_{j=1}^{n} \left(\alpha_j f_j(\theta) + \beta_j g_j(\theta)\right).$$

On the other hand, because of the bi-orthogonality, $\langle \tilde{f}, T\rangle_\omega = 0$ for all $T \in \mathcal{T}_{n-1}$, yielding $\tilde{f}_n(\theta) = \alpha_n f_n(\theta) + \beta g_n(\theta)$. Similarly, $\tilde{g}_n(\theta) = \gamma_n f_n(\theta) + \delta g_n(\theta)$. Both relations can be expressed in a matrix form as,

$$\begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} = M_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad M_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

with

$$\begin{aligned} \alpha_n &= \tfrac{\langle \tilde{f}_n, f_n\rangle_\omega}{\|f_n\|_\omega^2}, \quad & \beta_n &= \tfrac{\langle \tilde{f}_n, g_n\rangle_\omega}{\|g_n\|_\omega^2} \\ \gamma_n &= \tfrac{\langle \tilde{g}_n, f_n\rangle_\omega}{\|\tilde{f}_n\|_\omega^2}, \quad & \delta_n &= \tfrac{\langle \tilde{g}_n, g_n\rangle_\omega}{\|\tilde{g}_n\|_\omega^2}. \end{aligned}$$

By changing the roles of both systems, it follows that

$$\begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} = \tilde{M}_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad \tilde{M}_n = M_n^{-1}.$$

Furthermore, when dealing with bi-orthonormal systems i.e., $\| f_n \|_\omega = \| g_n \|_\omega = \| \tilde{f}_n \|_\omega = \| \tilde{g}_n \|_\omega = 1$, then it can be verified that $M_n = M_n^T$ i.e., M_n is an orthogonal matrix, as remarked in [14].

Example 4.8. Consider the weight function $\omega(\theta) = \frac{1}{T(\theta)}$, $\theta \in [-\pi, \pi]$, $T(\theta)$ being a positive trigonometric polynomial of degree m (i.e., a rational modification of the Lebesgue measure). From 2.6 we can write $T(\theta) = |h(z)|^2$, $z = e^{i\theta}$, where

$h(z) \in \Pi_m$ without zeros on $\mathbb{T}$. Moreover, we can assume without loss of generality that $h(z)$ is a monic polynomial. Hence, from [15] the monic Szegő polynomials are given by $\rho_n(z) = z^{n-m}h(z)$ for $n \geqslant m$. Hence, as in Example 4.6 it holds that $\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta}\rho_{2n+1}(e^{i\theta})\omega(\theta)d\theta = 0$, and any nonzero complex number ω_n can be used, provided that $n \geqslant E\left[\frac{m-1}{2}\right] + 1$ where $E[x]$ denotes as usual the integer part of x. Thus, if we set $h(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ and take $\omega_n = 1$, then

$$\begin{aligned} \omega_n e^{-in\theta}\rho_{2n+1}(e^{i\theta}) &= e^{i(n+1-m)\theta}h(\theta) = e^{i(n+1-m)\theta}\left(e^{im\theta} + \cdots + a_0\right) \\ &= e^{i(n+1)\theta} + \cdots + a_0 e^{i(n+1-m)\theta} = f_{n+1}(\theta) + ig_{n+1}(\theta). \end{aligned}$$

Thus, for $n \geqslant E\left[\frac{m-1}{2}\right] + 1$ a bi-orthogonal system is given by

$$\begin{aligned} f_{n+1}(\theta) &= \cos(n+1)\theta + \cdots + a_0\cos(n+1-m)\theta, \\ g_{n+1}(\theta) &= \sin(n+1)\theta + \cdots + a_0\sin(n+1-m)\theta. \end{aligned}$$

Certainly, to have a bi-orthogonal system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ completely constructed, we must compute the Szegő polynomials $\rho_{2k+1}(z)$, $0 \leqslant k \leqslant E\left[\frac{m-1}{2}\right]$ which can be recursively done by Levinson's algorithm (see [7] or [12]).

In the rest of the section we shall be concerned with the zeros of a given bi-orthogonal system. We observe from Example 1 that $f_0 \equiv c \neq 0$, $f_n(\theta) = \cos n\theta$, $g_n(\theta) = \sin n\theta$, $n = 1, 2, \ldots$ represent a bi-orthogonal system for $\omega(\theta) \equiv 1$. Now, $f_n(\theta) = 0$ means $\theta = \frac{(2k+1)\pi}{2n}$, $k \in \mathbb{Z}$. Thus, taking $-(n-1) \leqslant k \leqslant n-1$ we see that $f_n(\theta)$ has exactly $2n$ distinct zeros on $(-\pi, \pi]$. Similarly, if a and b are two real numbers, not both zero, it can be seen that $af_n(\theta) + bg_n(\theta)$ has also $2n$ distinct zeros on $(-\pi, \pi]$. This property can be generalized to any arbitrary weight function $\omega(\theta)$.

Theorem 4.9. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system for $\omega(\theta)$ and let a and b be real numbers not both zero. Then the trigonometric polynomial $T(\theta) = af(\theta) + bg(\theta)$ has $2n$ real and distinct zeros on any interval of length 2π.*

Proof. To fix ideas we shall restrict ourselves to $(-\pi, \pi]$. By Theorem (2.4) we know that $T_n(\theta)$ has $2n$ real or complex zeros in the strip $-\pi < \Re(\theta) \leqslant \pi$. Furthermore, the non-real zeros appear in conjugate pairs. Le p be the number of zeros of $T_n(\theta)$ on $(-\pi, \pi]$ with odd multiplicity ($0 \leqslant p \leqslant 2n$). Since p should be even we can set $p = 2k$, $0 \leqslant k \leqslant n$. Assume that $k < n$ and define

$$U_k(\theta) = \prod_{j=1}^{k} \sin\left(\frac{\theta - \theta_{2j}}{2}\right) \sin\left(\frac{\theta - \theta_{2j-1}}{2}\right),$$

$\{\theta_j\}_{j=1}^{2k}$ being the zeros of $T_n(\theta)$ on $(-\pi, \pi]$ with odd multiplicity (obviously, if $k = 0$ we take $U_k(\theta) \equiv 1$). Then we can write $T_n(\theta) = af_n(\theta) + bg_n(\theta) = U_k(\theta)V_{n-k}(\theta)$, where $V_{n-k}(\theta) \in \mathcal{T}_{n-k}$ and $V_{n-k}(\theta)$ has a constant sign on $(-\pi, \pi]$. Since $k < n$, by virtue of orthogonality it follows on the one hand that

$$\begin{aligned} I &= \int_{-\pi}^{\pi} T_n(\theta) U_k(\theta) \omega(\theta) d\theta \\ &= a \int_{-\pi}^{\pi} f_n(\theta) U_k(\theta) \omega(\theta) d\theta + b \int_{-\pi}^{\pi} g_n(\theta) U_k(\theta) \omega(\theta) d\theta = 0, \end{aligned}$$

whereas on the other hand

$$I = \int_{-\pi}^{\pi} U_k^2(\theta) V_{n-k}(\theta) \omega(\theta) d\theta \neq 0$$

because $\omega(\theta)$ is a weight function on $(-\pi, \pi]$. >From this contradiction it follows that $k = n$. □

Furthermore, the following interlacing property of zeros holds:

Theorem 4.10. *Under the same assumptions as in Theorem 4.9, the zeros of* $af_n(\theta) + bg_n(\theta)$ *and* $-bf_n(\theta) + ag_n(\theta)$ *interlace.*

Proof. Since we are dealing with properties of zeros, we can assume, without loss of generality that the system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is bi-orthonormal. We introduce the function

$$\mathcal{K}_n(\alpha, \theta) = f_0(\alpha) f_0(\theta) + \sum_{k=1}^{n} \left(f_k(\alpha) f_k(\theta) + g_k(\alpha) g_k(\theta) \right)$$

which satisfies the following reproducing property:

$$T(\alpha) = \int_{-\pi}^{\pi} \mathcal{K}_n(\alpha, \theta) T(\theta) \omega(\theta) d\theta, \quad \forall T \in \mathcal{T}_n.$$

On the other hand, from the paper by Szegő [14], the following Christoffel-Darboux identity can be established,

$$\begin{aligned} \mathcal{K}_{n-1}(\alpha, \theta) &= \tfrac{1}{2} \tfrac{k_{2n-1}}{k_{2n}} \cot\left(\tfrac{\theta - \alpha}{2}\right) \left(f_n(\alpha) g_n(\theta) - f_n(\theta) g_n(\alpha) \right) - \\ &\quad - \left(r_n f_n(\alpha) f_n(\theta) + s_n g_n(\alpha) g_n(\theta) \right) \end{aligned} \tag{4.7}$$

where the coefficients k_n, r_n and s_n are related to the orthonormal sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ of Szegő polynomials as follows: Set $\varphi_n(z) = k_n z^n + \cdots + l_n$ $(k_n > 0)$, then $2s_n = 1 + \frac{|l_{2n}|}{k_{2n}} > 0$ and $2r_n = 1 - \frac{|l_{2n}|}{k_{2n}}$. Furthermore, since $\rho_n(z) = \frac{\varphi_n(z)}{k_n} = z^n + \cdots + \frac{l_n}{k_n}$, then $\frac{|l_{2n}|}{k_{2n}} < 1$ and r_n is also positive. Thus

$$\begin{aligned} \mathcal{K}_{n-1}(\alpha, \alpha) &= lim_{\theta \to \alpha} \mathcal{K}_{n-1}(\alpha, \theta) \\ &= \tfrac{k_{2n-1}}{k_{2n}} \left(f_n(\alpha) g_n^{'}(\alpha) - f_n^{'}(\alpha) g_n(\alpha) \right) - \left(r_n f_n^2(\alpha) + s_n g_n^2(\alpha) \right). \end{aligned}$$

Setting $M_n(\alpha) = \left(r_n f_n^2(\alpha) + s_n g_n^2(\alpha) \right)$ we obtain for all $\alpha \in \mathbb{R}$:

$$f_n(\alpha) g_n^{'}(\alpha) - f_n^{'}(\alpha) g_n(\alpha) = \frac{k_{2n-1}}{k_{2n}} \left(M_n(\alpha) + \mathcal{K}_{n-1}(\alpha, \alpha) \right) > 0,$$

since clearly $M_n(\alpha) > 0$ and $\mathcal{K}_{n-1}(\alpha, \alpha) > 0$. >From here it can be easily seen that the zeros of $f_n(\theta)$ and $g_n(\theta)$ interlace. Finally, let us consider

$$C_n(\theta) = af_n(\theta) + bg_n(\theta), \quad D_n(\theta) = -bf_n(\theta) + ag_n(\theta), \quad |a| + |b| > 0.$$

Then

$$C_n(\alpha)D_n^{'}(\alpha) - C_n^{'}(\alpha)D_n(\alpha) = (a^2 + b^2)\left(f_n(\alpha)g_n^{'}(\alpha) - f_n^{'}(\alpha)g_n(\alpha)\right) > 0$$

and the proof follows. □

Remark 4.11. The two previous theorems were earlier proved by Szegő in [14] making use of the fundamental property that the zeros of any Szegő polynomial $\rho_n(z)$ lie in $\mathbb{D}$. Here, we have given alternative proofs involving only bi-orthogonality properties.

As an immediate consequence of Theorems 4.9 and 4.10, we have

Corollary 4.12. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be an orthogonal system for $\omega(\theta)$. Then,*

1. *Both f_n and g_n have $2n$ distinct zeros on any interval of length 2π.*
2. *On any interval of length 2π, the zeros of f_n and g_n interlace.*

5. Quadratures

In this section we start to properly deal with the main topic of the paper, i.e., the approximate calculation of integrals

$$I_\omega(f) = \int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta \tag{5.1}$$

with $\omega(\theta)$ a weight function on $(-\pi, \pi]$ and f a 2π-periodic function such that $f\omega \in L_1(-\pi, \pi]$. $I_\omega(f)$ is going to be approximated by means of an n-point quadrature rule like:

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(\theta_j), \ \theta_j \neq \theta_k, \ \theta_j \in (-\pi, \pi]. \tag{5.2}$$

Here, the nodes $\{\theta_j\}_{j=1}^n$ and weights $\{\lambda_j\}_{j=1}^n$ are to be determined so that $I_n(f)$ is exact in certain subspaces of $\mathcal{T}$ with dimension as large as possible, i.e. it should hold that $I_\omega(T) = I_n(T)$ for any $T \in \mathcal{T}_{m(n)} \subset \mathcal{T}$ with $m(n)$ as large as possible. For this purpose the following results should first be taken into account:

Theorem 5.1. *There can not exist an n-point quadrature rule $I_n(f)$ like (5.2) which is exact in $\mathcal{T}_n$, i.e., $m(n) < n$.*

Proof. Proceed as in [11, pp. 73-74] for the case $\omega(\theta) \equiv 1$. □

Now, making use of the interpolation results in Section 3 the following can be proved:

Theorem 5.2. *Given n distinct nodes $\{\theta_j\}_{j=1}^n \subset (-\pi,\pi]$, there exists a certain subspace $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ with dimension n such that weights $\{\lambda_j\}_{j=1}^n$ satisfying*

$$I_n(T) = \sum_{j=1}^n \lambda_j T(\theta_j) I_\omega(T), \ \ \forall\, T \in \tilde{\mathcal{T}}_n.$$

are uniquely determined.

Theorem 5.3. *If there exists an n-point quadrature rule $I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j)$ which is exact in $\mathcal{T}_{n-1}$, then $\lambda_j > 0$ for all $j = 1,\dots,n$ (see [11]).*

Proof. Take $t_j(\theta) = \prod_{k=1,k\neq j}^n \sin^2\left(\frac{\theta-\theta_k}{2}\right)$. Thus, $t_j(\theta) \in \mathcal{T}_{n-1}$ and $t_j(\theta) \geqslant 0$. Hence, $0 < I_\omega(t_j) = I_n(t_j) = \lambda_j t_j(\theta_j)$. Since $t_j(\theta_j) > 0$, the proof follows. □

After these preliminary considerations, we are now in a position to investigate the following problem, namely: "For $n \in \mathbb{N}$, $n \geqslant 1$, find $\theta_1,\dots,\theta_n$ with $\theta_j \neq \theta_k$ if $j \neq k$ on $(-\pi,\pi]$ and real numbers $\lambda_1,\dots,\lambda_n$ such that

$$I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j) = I_\omega(f), \ \ \forall f \in \mathcal{T}_{n-1}.'' \tag{5.3}$$

Since $dim\,(\mathcal{T}_{n-1}) = 2n-1$, (5.3) leads to a nonlinear system with $2n-1$ equations and $2n$ unknowns: $\theta_1,\dots,\theta_n;\lambda_1,\dots,\lambda_n$. Now, proceeding as in the polynomial situation (see e.g. [6]), instead of directly attacking the system coming from (5.3) we will try to analyze the properties of the real trigonometric polynomial whose zeros are the nodes of $I_n(f)$. For this reason we are forced to assume that the number of nodes in our quadrature rules should be even. To fix ideas, assume that this number is $2n$. Then, in the sequel our rule will be of the form

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j), \ \ \{\theta_j\}_{j=1}^{2n} \subset (-\pi,\pi].$$

Set $T_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right) \in \mathcal{T}_n$. Then the following holds:

Theorem 5.4. *Let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ be a quadrature rule such that $I_{2n}(T) = I_\omega(T)$ for all $T \in \mathcal{T}_{2n-1}$ and let $f_0 \cup \{f_k, g_k\}_{k=1}^\infty$ be a bi-orthogonal system for $\omega(\theta)$. Set $T_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right)$. Then there exist real numbers a_n and b_n not both zero such that $T_n(\theta) = a_n f_n(\theta) + b_n g_n(\theta)$.*

Proof. Set $S \in \mathcal{T}_{n-1}$, then $T_n(\theta)S(\theta) \in \mathcal{T}_{2n-1}$. Hence

$$\begin{aligned} \langle T_n, S\rangle_\omega &= I_\omega(T_n\cdot S) = \textstyle\int_{-\pi}^{\pi} T_n(\theta)S(\theta)\omega(\theta)d\theta \\ &= I_n(T_n\cdot S) = \textstyle\sum_{j=1}^{2n} \lambda_j T_n(\theta_j)S(\theta_j) = 0. \end{aligned} \tag{5.4}$$

On the other hand, since $f_0 \cup \{f_k, g_k\}_{k=1}^{n}$ is a basis for $\mathcal{T}_n$, one can write

$$T_n(\theta) = a_0 + \sum_{k=1}^{n} (a_k f_k(\theta) + b_k g_k(\theta)), \quad a_k = \frac{\langle T_n, f_k \rangle_\omega}{\| f_k \|_\omega^2}, \quad b_k = \frac{\langle T_n, g_k \rangle_\omega}{\| f_k \|_\omega^2}.$$

By (5.4), $a_k = 0$ for $k = 0, 1, \ldots, n-1$ and $b_k = 0$ for $k = 1, \ldots, n-1$ and the proof follows. □

Conversely, we can prove the following

Theorem 5.5. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system for the weight function $\omega(\theta)$. Let a and b be real numbers not both zero and let $\{\theta_j\}_{j=1}^{2n}$ be the $2n$ zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$ on $(-\pi, \pi]$. Then, there exist positive numbers $\lambda_1, \ldots, \lambda_{2n}$ such that*

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j) = I_\omega(f), \quad \forall\, f \in \mathcal{T}_{2n-1}.$$

Proof. Throughout the proof, $\tilde{\mathcal{T}_n}$ will denote a subspace of trigonometric polynomials coinciding either with $\mathcal{T}_n \backslash span\{\cos n\theta\}$ or $\mathcal{T}_n \backslash span\{\sin n\theta\}$, so that $dim\left(\tilde{\mathcal{T}_n}\right) = 2n$. Let $\theta_1, \ldots, \theta_{2n}$ be the $2n$ distinct zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $(|a| + |b| > 0)$. Then, by Theorem 5.2, there exist weights $\lambda_1, \ldots, \lambda_{2n}$, uniquely determined, such that

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j) = I_\omega(f), \quad \forall\, f \in \tilde{\mathcal{T}_n}.$$

Let us next see that $I_{2n}(f)$ is also exact in $\mathcal{T}_{2n-1}$ (observe that $\tilde{\mathcal{T}_n} \subset \mathcal{T}_{2n-1}$). To do that, we will follow the classical pattern. Indeed, take $T \in \mathcal{T}_{2n-1}$ and let $L_n \in \tilde{\mathcal{T}_n}$ such that

$$T(\theta_j) = L_n(\theta_j), \quad j = 1, \ldots, 2n.$$

Then $T - L_n \in \mathcal{T}_{2n-1}$ and $(T - L_n)(\theta_j) = 0$ for all $j = 1, \ldots, 2n$. Hence we can write $T(\theta) - L_n(\theta) = T_n(\theta)V(\theta)$, with $V \in \mathcal{T}_{n-1}$ i.e., $T(\theta) = L_n(\theta) + T_n(\theta)V(\theta)$. Consequently

$$\begin{aligned} I_\omega(T) &= \int_{-\pi}^{\pi} T(\theta)\omega(\theta)d\theta = \int_{-\pi}^{\pi} (L_n(\theta) + T_n(\theta)V(\theta))\,\omega(\theta)d\theta \\ &= \int_{-\pi}^{\pi} L_n(\theta)\omega(\theta)d\theta = I_\omega(L_n), \end{aligned}$$

since $I_\omega(T_nV) = 0$ (by definition, $T_n(\theta)$ is orthogonal to any function in $\mathcal{T}_{n-1}$). Therefore,

$$I_\omega(T) = I_\omega(L_n) = \sum_{j=1}^{2n} \lambda_j L_n(\theta_j) = \sum_{j=1}^{2n} \lambda_j T(\theta_j) = I_n(T).$$

Finally, the positive character of the weights $\{\lambda_j\}_{j=1}^{2n}$ follows from Theorem 5.3. However, we can also give an explicit integral representation. Thus, for $j = 1, \ldots, 2n$, set

$$l_j(\theta) = \frac{T_n(\theta)}{2T_n'(\theta_j)\sin\left(\frac{\theta-\theta_j}{2}\right)}$$

so that $l_j(\theta_k) = \delta_{j,k}$ and $l_j^2(\theta) \in \mathcal{T}_{2n-1}$ for $j = 1, \ldots, 2n$. Thus

$$I_\omega\left(l_j^2(\theta)\right) = I_{2n}\left(l_j^2(\theta)\right) = \sum_{k=1}^{2n} \lambda_k l_j^2(\theta_k) = \lambda_j$$

yielding

$$\lambda_j = \int_{-\pi}^{\pi} \left[\frac{T_n(\theta)}{2T_n'(\theta_j)\sin\left(\frac{\theta-\theta_j}{2}\right)}\right]^2 \omega(\theta)d\theta, \quad j = 1, \ldots, 2n. \tag{5.5}$$

□

Theorems 5.4 and 5.5 may be summarized in the following characterization result,

Corollary 5.6. *Let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ so that $\theta_j \neq \theta_k$ if $j \neq k$, and $\{\theta_j\} \subset (-\pi, \pi]$. Then, $I_{2n}(f) = I_\omega(f)$ for all $f \in \mathcal{T}_{2n-1}$, if and only if,*

1. *$I_{2n}(f)$ is exact in a certain subspace $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_{2n-1}$ whith dimension $2n$.*
2. *There exist real numbers a and b not both zero such that $\{\theta_j\}_{j=1}^{2n}$ are the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ being a bi-orthogonal system for the weight function $\omega(\theta)$.*

Furthermore, when these conditions are satisfied the weights $\{\lambda_j\}_{j=1}^{2n}$ are positive.

Remark 5.7. The quadrature rules characterized in Corollary 5.6 were earlier introduced by Szegő in [14] and they are sometimes refered as "quadratures with the highest degree of trigonometric precision".

Next, we will see how we can also give an explicit representation of the weights $\{\lambda_j\}_{j=1}^{2n}$ in Corollary 5.6, in terms of a bi-orthonormal system similar to the well known Christoffel numbers for the Gaussian formulas (see e.g. [8]). Indeed, we have

Theorem 5.8. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthonormal system for $\omega(\theta)$ and let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ be a $2n$-point quadrature rule with the highest degree of trigonometric precision. Then, for $j = 1, \ldots, 2n$,*

$$\lambda_j = \frac{1}{f_0^2 + \sum_{k=1}^{n-1}\left(f_k^2(\theta_j) + g_k^2(\theta_j)\right) + \left(\frac{1-|\delta_{2n}|}{2}\right) f_n^2(\theta_j) + \left(\frac{1+|\delta_{2n}|}{2}\right) g_n^2(\theta_j)} \tag{5.6}$$

where, as usual, $\delta_{2n} = \rho_{2n}(0)$, $\rho_{2n}(z)$ being the monic Szegő polynomial of degree $2n$ and $\{\theta_j\}_{j=1}^{2n}$ being the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $|a| + |b| > 0$.

Proof. Set $T_n(\theta) = \prod_{k=1}^{2n} \sin\left(\frac{\theta-\theta_k}{2}\right) = af_n(\theta)+bg_n(\theta) \in \mathcal{T}_n$, $|a|+|b|>0$. Suppose without loss of generality that $a \neq 0$ so that $f_n(\theta_j) = \frac{-b}{a} g_n(\theta_j)$. Then, from the Christoffel-Darboux identity (4.7) it follows

$$\begin{aligned} \mathcal{K}_{n-1}(\theta,\theta_j) &= \frac{1}{2}\frac{k_{2n-1}}{k_{2n}} ctg\left(\frac{\theta_j-\theta}{2}\right)\left[f_n(\theta)g_n(\theta_j) - f_n(\theta_j)g_n(\theta)\right] - \\ & - (r_n f_n(\theta) f_n(\theta_j) + s_n g_n(\theta) g_n(\theta_j)) \\ &= \frac{1}{2a}\frac{k_{2n-1}}{k_{2n}} g_n(\theta_j) ctg\left(\frac{\theta_j-\theta}{2}\right) T_n(\theta) - \\ & - \left[\frac{1-|\delta_{2n}|}{2} f_n(\theta)f_n(\theta_j) + \frac{1+|\delta_{2n}|}{2} g_n(\theta)g_n(\theta_j)\right] \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{K}_{n-1}(\theta,\theta_j) + \left[\frac{1-|\delta_{2n}|}{2} f_n(\theta)f_n(\theta_j) + \frac{1+|\delta_{2n}|}{2} g_n(\theta)g_n(\theta_j)\right] &= \\ \frac{-1}{2a}\frac{k_{2n-1}}{k_{2n}} g_n(\theta_j)\cos\left(\frac{\theta-\theta_j}{2}\right)\frac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]}. & \end{aligned} \tag{5.7}$$

As θ tends to θ_j, we get

$$\begin{aligned} f_0^2 + \sum_{k=1}^{n-1}\left(f_k^2(\theta_j)+g_k^2(\theta_j)\right) + \left(\frac{1-|\delta_{2n}|}{2}\right) f_n^2(\theta_j) + \left(\frac{1+|\delta_{2n}|}{2}\right) g_n^2(\theta_j) &= \\ \frac{-1}{a}\frac{k_{2n-1}}{k_{2n}} g_n(\theta_j) T_n^{'}(\theta_j). & \end{aligned} \tag{5.8}$$

Now, due to the orthogonality conditions it follows from (5.7) that

$$1 = \frac{-1}{2a}\frac{k_{2n-1}}{k_{2n}} g_n(\theta_j) \int_{-\pi}^{\pi} \cos\left(\frac{\theta-\theta_j}{2}\right) \frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)} \omega(\theta) d\theta. \tag{5.9}$$

The combination of expressions (5.8) and (5.9) implies

$$\begin{aligned} \frac{1}{f_0^2+\sum_{k=1}^{n-1}\left(f_k^2(\theta_j)+g_k^2(\theta_j)\right)+\frac{1-|\delta_{2n}|}{2} f_n^2(\theta_j)+\frac{1+|\delta_{2n}|}{2} g_n^2(\theta_j)} &= \\ \frac{1}{2T_n^{'}(\theta_j)} \int_{-\pi}^{\pi} \cos\left(\frac{\theta-\theta_j}{2}\right) \frac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]} \omega(\theta) d\theta & \end{aligned} \tag{5.10}$$

On the other hand, from Corollary 5.6 one knows that the weights λ_j can be expressed as

$$\lambda_j = \int_{-\pi}^{\pi} \tilde{s}_j(\theta)\omega(\theta)d\theta, \quad j = 1, \ldots, 2n$$

where $\tilde{s}_j(\theta)$ are trigonometric polynomials of degree n at most given by (3.16) or (3.17). Thus, from (3.16) it follows

$$\tilde{s}_j(\theta) = \frac{1}{2T_n^{'}(\theta_j)\sin\eta_n} \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n,$$

with $\eta_n = \frac{1}{2}\sum_{j=1}^{2n}\theta_j$ and $\alpha_j = \eta_n - \frac{\theta_j}{2}$ for $j = 1,\ldots,2n$. Hence, $\sin\left(\frac{\theta+\alpha_j}{2}\right) = \sin\left(\frac{\theta-\theta_j}{2}+\eta_n\right) = \sin\left(\frac{\theta-\theta_j}{2}\right)\cos\eta_n + \cos\left(\frac{\theta-\theta_j}{2}\right)\sin\eta_n$ and one can write

$$\begin{aligned}\lambda_j &= \frac{1}{2T_n^{'}(\theta_j)\sin\eta_n}\left[\cos\eta_n\int_{-\pi}^{\pi}T_n(\theta)\omega(\theta)d\theta + \right.\\ &\left. + \sin\eta_n\int_{-\pi}^{\pi}\cos\left(\frac{\theta-\theta_j}{2}\right)\frac{T_n(\theta)}{\sin\frac{\theta-\theta_j}{2}}\omega(\theta)d\theta\right]\\ &= \frac{1}{2T_n^{'}(\theta_j)}\int_{-\pi}^{\pi}\cos\left(\frac{\theta-\theta_j}{2}\right)\frac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]}\omega(\theta)d\theta.\end{aligned} \tag{5.11}$$

Clearly, if we now start from (3.17) the same representation (5.11) is achieved. Thus, from (5.10) and (5.11) the proof follows. □

Example 5.9. As a simple illustration of formula (5.6), let us consider $\omega(\theta) \equiv 1$. As we have already seen, a bi-orthogonal system is given by $\{1\}\cup\{\cos n\theta, \sin n\theta\}_{n=1}^{\infty}$. Thus, we have the following bi-orthonormal system:

$$f_0 = \frac{1}{\sqrt{2\pi}},\ \ f_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}},\ \ g_n(\theta) = \frac{\sin n\theta}{\sqrt{\pi}},\ \ n = 1,2,\ldots.$$

Taking $a, b \in \mathbb{R}$, $|a|+|b| > 0$, the nodes of the corresponding $(2n)$-th quadrature rule are the zeros of $T_n(\theta) = af_n(\theta)+bg_n(\theta)$. Thus, when $a = 0$ and $b = 1$, i.e., $\sin n\theta = 0$, the zeros are $\theta_k = \frac{k\pi}{n}$ for all $k \in \mathbb{Z}$, i.e., the $2n$ zeros $\theta_j = \frac{(j-n)\pi}{n} = -\pi + \frac{2\pi j}{2n}$, $j = 0,1,\ldots,2n-1$, are equally spaced on the interval $[-\pi,\pi]$ with step size, $h = \frac{\pi}{n}$. Moreover, since now $\rho_n(z) = z^n$ for all $n = 0,1,\ldots$, then $\delta_{2n} = \rho_{2n}(0) = 0$ and formula (5.6) becomes, for all $j = 1,\ldots,2n$:

$$\lambda_j = \frac{1}{\frac{1}{2\pi}+\sum_{k=1}^{n-1}\left(\frac{\cos^2(k\theta_j)}{\pi}+\frac{\sin^2(k\theta_j)}{\pi}\right)+\frac{1}{2}\left(\frac{\cos^2(n\theta_j)}{\pi}+\frac{\sin^2(n\theta_j)}{\pi}\right)} = \frac{\pi}{n}. \tag{5.12}$$

Furthermore, from (5.12) we see that independently of the expression of the nodes $\{\theta_j\}_{j=1}^{2n}$ all the weights $\{\lambda_j\}_{j=1}^{2n}$ are equal to $\frac{\pi}{n}$. This result was deduced in a different manner in [11].

Paralleling rather closely Gaussian quadrature formulas, we will give a final result involving the Hermite-type interpolation problem stated in Theorem 3.2 which could be used to give an estimation of the error for $I_{2n}(f)$. Indeed, one has

Theorem 5.10. *Let a and b real numbers not both zero and let $\{\theta_j\}_{j=1}^{2n}$ the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ being a bi-orthogonal system. Let $H_{2n-1}(f,\cdot) \in \mathcal{T}_{2n-1}$ such that:*

$$H_{2n-1}(f;\theta_j) = f(\theta_j) \quad j = 1,\ldots,2n$$

$$H_{2n-1}^{'}(f;\theta_j) = f^{'}(\theta_j) \quad j = 1,\ldots,2n,\ j \neq k \in \{1,\ldots,2n\}.$$

Then, $I_\omega\left(H_{2n-1}(f,\cdot)\right)$ coincides with the $(2n)$-th quadrature rule with the highest degree of trigonometric precision with nodes $\{\theta_j\}_{j=1}^{2n}$. Furthermore, this formula does not depend on the parameter $k \in \{1,\ldots,2n\}$ previously fixed.

Proof. The existence and unicity of the Hermite trigonometric interpolant $H_{2n-1}(f,\theta)$ is guaranteed by Theorem 3.2. Furthermore, by (3.7) we can write,

$$H_{2n-1}(f,\theta) = \sum_{j=1}^{2n} t_j(\theta) f(\theta_j) + \sum_{j=1, j\neq k}^{2n} s_j(\theta) f^{'}(\theta_j) \tag{5.13}$$

where $t_j(\theta)$ and $s_j(\theta)$ are trigonometric polynomials in $\mathcal{T}_{2n-1}$ satisfying the interpolation condition (3.8). Hence,

$$I_\omega\left(H_{2n-1}(f,\cdot)\right) = \sum_{j=1}^{2n} A_j f(\theta_j) + \sum_{j=1, j\neq k}^{2n} B_j f^{'}(\theta_j) \tag{5.14}$$

where $A_j = I_\omega(t_j)$ for $j = 1, \ldots, 2n$ and $B_j = I_\omega(s_j)$, $j = 1, \ldots, 2n$, $j \neq k$. Now, taking into account that $T_n(\theta) = af_n(\theta) + bg_n(\theta)$ is orthogonal to $\mathcal{T}_{n-1}$, it can be deduced from (3.9) that

$$B_j = \frac{\sin\left(\frac{\theta_j - \theta_k}{2}\right)}{2\left[T_n'(\theta_j)\right]^2} I_\omega\left(T_n(\theta)\frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)\sin\left(\frac{\theta-\theta_k}{2}\right)}\right) = 0, \quad j = 1, \ldots, 2n, \ j \neq k.$$

Thus, $I_\omega\left(H_{2n-1}(f,\cdot)\right) = \sum_{j=1}^{2n} A_j f(\theta_j) = \tilde{I}_{2n}(f)$ and since for any $T \in \mathcal{T}_{2n-1}$, $H_{2n-1}(T,\theta) = T(\theta)$, we have

$$\tilde{I}_{2n}(T) = I_\omega\left(H_{2n-1}(T,\cdot)\right) = I_\omega(T), \quad \forall\, T \in \mathcal{T}_{2n-1}.$$

Now the proof follows by Corollary 5.6. □

Remark 5.11. Quadrature rules of the form $I_n(f) = \sum_{j=1}^{n} \lambda_n f(\theta_j)$ to estimate weighted 2π-periodic integrals $I_\omega(f)$ have been constructed making use of the zeros of certain trigonometric polynomials associated to a bi-orthogonal system. For this reason we have been forced to deal with an even number of nodes and weights. Now, we might wonder if a quadrature $I_n(f)$ with n an arbitrary natural number and with the highest degree of trigonometric precision $(n-1)$ could be also constructed. It seems clear that we can not use zeros of real trigonometric polynomials anymore, since the number of these is always even. Actually, this question does not appear in the paper by Szegő [14]. In a forthcoming paper a positive answer will be given by introducing convenient technical modifications of Szegő's paper [14]. However we can also find an answer in the paper by Jones et. al. [10] which, for the sake of completeness, will be surveyed in the next Section. As a consequence, a connection between the concepts of bi-orthogonality and para-orthogonality introduced in [14] and [10] respectively will be also made.

6. A connection with the unit circle. Para-orthogonal polynomials

In this Section we shall be concerned with the approximation of integrals on the unit circle, i.e., integrals of the form $\int_{\mathbb{T}} f(z)d\mu(z)$, μ being a positive measure on $\mathbb{T}$, by means of an n-point quadrature rule:

$$I_n(f) = \sum_{j=1}^{n} A_j f(z_j), \quad z_j \neq z_k,\ j \neq k, \quad \{z_j\}_{j=1}^n \subset \mathbb{T}. \tag{6.1}$$

By a slight abuse of notation we shall set $\mu(z) = \mu(\theta)$ for $z = e^{i\theta}$. As before, and for the sake of simplicity, we will also assume that μ is an absolutely continuous measure i.e., $d\mu(\theta) = \omega(\theta)d\theta$ so that we consider integrals of the form

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \tag{6.2}$$

where $f(e^{i\theta})$ is in general a complex function. Thus $f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$ with $f_j(\theta)$ for $j = 1,2$, both real 2π-periodic functions. Here, taking into account the basic fact that any continuous function on $\mathbb{T}$ can be uniformly approximated on $\mathbb{T}$ by Laurent polynomials, the nodes $\{z_j\}_{j=1}^n$ and weights $\{A_j\}_{j=1}^n$ are to be determined by requiring that $I_n(f)$ is exact in $\Lambda_{-p,q}$ (domain of validity) with p and q as large as possible (clearly this means that $I_\omega(L) = I_n(L)$, for all $L \in \Lambda_{-p,q}$). Now, assume that for the weight function $\omega(\theta)$ and an even integer n we have found an n-point quadrature rule $I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j)$ with the highest degree of trigonometric precision (recall that $\lambda_j > 0$ and $\theta_j \neq \theta_k$ if $j \neq k$, $\{\theta_j\}_{j=1}^n \subset (-\pi,\pi]$). Take $L \in \Lambda_{-(n-1),n-1}$ so that $L(e^{i\theta}) = L_1(\theta) + iL_2(\theta)$ with $L_1, L_2 \in \mathcal{T}_n$. Then

$$\begin{aligned} I_\omega(L) &= \textstyle\int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = \int_{-\pi}^{\pi} L_1(\theta)\omega(\theta)d\theta + i\int_{-\pi}^{\pi} L_2(\theta)\omega(\theta)d\theta \\ &= \textstyle\sum_{j=1}^n \lambda_j L_1(\theta_j) + i\sum_{j=1}^n \lambda_j L_2(\theta_j) \\ &= \textstyle\sum_{j=1}^n \lambda_j \left(L_1(\theta_j) + iL_2(\theta_j)\right) = \sum_{j=1}^n \lambda_j L(e^{i\theta_j}) \\ &= \textstyle\sum_{j=1}^n \lambda_j L(z_j), \quad z_j = e^{i\theta_j},\ j = 1,\ldots,n. \end{aligned}$$

(observe that $z_j \neq z_k$ if $j \neq k$). Thus, provided that n is even a quadrature rule with domain of validity $\Lambda_{-(n-1),n-1}$ for $I_\omega(f)$ has been constructed.

Conversely, let $I_n(f) = \sum_{j=1}^n A_j f(z_j)$, $z_j \neq z_k$ if $j \neq k$, be exact in $\Lambda_{-(n-1),n-1}$ and set $z_j = e^{i\theta_j}$, $\theta_j \in (-\pi,\pi]$, $\theta_j \neq \theta_k$ if $j \neq k$. Set $T \in \mathcal{T}_{n-1}$, then $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{n-1}^H$ so that

$$\int_{-\pi}^{\pi} T(\theta)d\theta = \int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = \sum_{j=1}^n A_j L(e^{i\theta_j}) = \sum_{j=1}^n A_j T(\theta_j) = I_n(T)$$

with $I_n(f) = \sum_{j=1}^n A_j f(\theta_j)$. Thus, we see that the problem of constructing an n-point quadrature formula for $\omega(\theta)$ with the highest degree of trigonometric precision with n arbitrary would be solved. As immediate consequences we would also have:

1. Any quadrature rule $I_n(f) = \sum_{j=1}^n A_j f(z_j)$ with distinct nodes on $\mathbb{T}$ which is exact in $\Lambda_{-(n-1),n-1}$ has positive weights A_j, $j = 1,\ldots,n$.

2. There can not exist an n-point quadrature rule as before which is exact in $\Lambda_{-n,n}$.

Thus, in the sequel, given the integral $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$ we shall be concentrated on the construction of $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \neq z_k$ if $j \neq k$, $z_j \in \mathbb{T}$ for $j = 1, \ldots, n$ by imposing

$$I_n(L) = I_\omega(L),\ \forall L \in \Lambda_{-(n-1),n-1}. \tag{6.3}$$

According to [10], $\Lambda_{(n-1),n-1}$ will be called "the maximun domain of validity" for $I_n(f)$, provided that (6.3) holds. Now, set $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta$ for any $k \in \mathbb{Z}$ (trigonometric moments) so that (6.3) gives rise to the equality

$$\sum_{k=1}^{n} A_k z_k^j = \mu_{-j},\ \ -(n-1) \leqslant j \leqslant n-1. \tag{6.4}$$

This leads to a study of the solutions of (6.4) which represents a nonlinear system with $2n$ unknowns and $2n-1$ equations. We will proceed as in the preceding section by analyzing the properties of the nodal polynomial for $I_n(f)$, $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. First, take into account that in case the zeros $\{z_j\}_{j=1}^{n}$ of $B_n(z)$ satisfy $z_j \neq 0$ and $z_j \neq z_k$ if $j \neq k$, then by taking n consecutive equations in (6.4), the weights $\{A_j\}_{j=1}^{n}$ are to be uniquely determined in terms of the nodes $\{z_j\}_{j=1}^{n}$. Indeed, let p and q be nonnegative integers suche that $p + q = n - 1$ and take in (6.4) the n equations

$$\sum_{k=1}^{n} A_k z_k^j = \mu_{-j},\ \ -p \leqslant j \leqslant q. \tag{6.5}$$

Clearly, (6.5) is a linear system for the unknowns $A_1, \ldots, A_n$ admitting a unique solution because the determinant of the matrix of the system satisfies

$$\begin{vmatrix} z_1^{-p} & z_2^{-p} & \cdots & z_n^{-p} \\ z_1^{-p+1} & z_2^{-p+1} & \cdots & z_n^{-p+1} \\ \vdots & \vdots & & \vdots \\ z_1^{q} & z_2^{q} & \cdots & z_n^{q} \end{vmatrix} = (z_1 \cdots z_n)^p \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix} \neq 0.$$

(Recall that we are assuming $z_j \neq 0$ and $z_j \neq z_k$ if $j \neq k$). Secondly, we can also deduce the following necessary conditions for the polynomials $B_n(z)$:

Theorem 6.1. *Let $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \in \mathbb{T}$ and $z_j \neq z_k$ if $j \neq k$ satisfying $I_n(L) = I_\omega(L)$, for all $L \in \Lambda_{-(n-1),n-1}$. Set $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. Then,*

1. *$B_n(z)$ is invariant.*

2. $$\langle B_n(z), z^k \rangle_\omega = 0,\ 1 \leqslant k \leqslant n-1,\ \langle B_n(z), 1 \rangle_\omega \neq 0,\ \langle B_n(z), z^n \rangle_\omega \neq 0. \tag{6.6}$$

Proof. 1. It trivially follows since by hypothesis the zeros of $B_n(z)$ lie on $\mathbb{T}$.

2. Set $1 \leqslant k \leqslant n-1$. Then

$$\langle B_n(z), z^k\rangle_\omega = \int_{-\pi}^{\pi} B_n(e^{i\theta})\overline{e^{ik\theta}}\omega(\theta)d\theta = \int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta$$

where $L(z) = z^{-k}B_n(z) \in \Lambda_{-k,n-k} \subset \Lambda_{-(n-1),n-1}$ and $L(z_j) = 0$. Then, because of the exactness of $I_n(f)$ in $\Lambda_{-(n-1),n-1}$ we have

$$\begin{aligned}\langle B_(z), z^k\rangle_\omega &= \textstyle\int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = I_n(L)\\ &= \textstyle\sum_{j=1}^{n} A_j L(z_j) = 0,\ 1 \leqslant k \leqslant n-1.\end{aligned}$$

If $\langle B_n(z), 1\rangle_\omega = 0$, then $\langle B_n(z), z^k\rangle_\omega = 0$ for $0 \leqslant k \leqslant n$, yielding $B_n(z) = \rho_n(z)$, and hence the zeros lie in $\mathbb{D}$, contrary to assumption. Similarly, if $\langle B_n(z), z^n\rangle_\omega = 0$ then $B_n(z) = \rho_n^*(z)$ and hence the zeros lie in $\mathbb{E}$, contrary to assumption. Thus $\langle B_n(z), 1\rangle_\omega \neq 0$ and $\langle B_n(z), z^n\rangle_\omega \neq 0$.

□

Remark 6.2. From the above considerations including the fact that

$$\langle B_n(z), 1\rangle_\omega \neq 0,\ \langle B_n(z), z^n\rangle_\omega \neq 0$$

when I_n is exact in $\Lambda_{-(n-1),n-1}$ and that the zeros of the n-th Szegő polynomial lie in $\mathbb{D}$, it follows that there can not exist an n-point quadrature formula with nodes on $\mathbb{T}$ to be exact either in $\Delta_{-(n-1),n}$ or in $\Delta_{-n,n-1}$.

Polynomials $B_n(z)$ satisfying (6.5) will play a crucial role in the construction of our quadratures $I_n(f)$ with the maximun domain of validity. This caused (see [10]) the following

Definition 6.3. A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is said to be para-orthogonal with respect to $\omega(\theta)$ if and only if the orthogonality conditions (6.6) are satisfied.

Now, several questions immediately arise. Indeed, for a given weight function $\omega(\theta)$ and a natural number n, does a para-orthogonal polynomial of exact degree n exist? If so, how can it be characterized? What about its zeros? The two first questions are answered in [10] where the concepts of “para-orthogonality” and “invariancy” were earlier introduced. Thus, in [10] one can find the following

Theorem 6.4. *A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is para-orthogonal and invariant if and only if*

$$B_n(z) = C_n\left[\rho_n(z) + \tau\rho_n^*(z)\right],\ \ C_n \neq 0,\ |\tau| = 1. \tag{6.7}$$

Now, by recalling that the sequences $\{\rho_n(z)\}_{n=0}^{\infty}$ and $\{\rho_n^*(z)\}_{n=0}^{\infty}$ satisfy the recurrence relations

$$\begin{aligned}&\rho_0(z) = \rho_0^*(z) = 1\\ &\rho_n(z) = z\rho_{n-1}(z) + \delta_n\rho_{n-1}^*(z) \quad n = 1,2,3,\dots \\ &\rho_n^*(z) = \overline{\delta_n} z\rho_{n-1}(z) + \rho_{n-1}^*(z) \quad n = 1,2,3,\dots\end{aligned} \tag{6.8}$$

where, as usual, $\delta_n = \rho_n(0)$ for all $n = 1, 2, \dots$ ($|\delta_n| < 1$), then we have

$$B_n(z) = C_n\left[\rho_n(z) + \tau\rho_n^*(z)\right] = \left(1 + \tau\overline{\delta_n}\right) C_n \left[z\rho_{n-1}(z) + \left(\frac{\tau + \delta_n}{1 + \tau\overline{\delta_n}}\right)\rho_{n-1}^*(z)\right],$$

yielding (observe that $|1 + \tau\overline{\delta_n}| \neq 0$)

$$B_n(z) = \tilde{C}_n\left[z\rho_{n-1}(z) + \lambda_n\rho_{n-1}^*(z)\right], \ \tilde{C}_n \neq 0, \ |\lambda_n| = 1 \tag{6.9}$$

(here, $\lambda_n = \frac{\tau+\delta_n}{1+\tau\overline{\delta_n}} \in \mathbb{T}$). Conversely, any polynomial $B_n(z)$ satisfying (6.9) can be expressed as in (6.7), were now $\tau = \frac{\delta_n - \lambda_n}{\overline{\delta_n}\lambda_n - 1} \in \mathbb{T}$. In short, we have obtained an alternative characterization of the para-orthogonal and invariant polynomials as shown in the following

Theorem 6.5. *A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is para-orthogonal and invariant if and only if*

$$B_n(z) = C_n\left[z\rho_{n-1}(z) + \tau\rho_{n-1}^*(z)\right], \ C_n \neq 0, \ |\tau| = 1.$$

Remark 6.6. >From this theorem we see that to compute a para-orthogonal polynomial of degree n, only the Szegő polynomial of degree $n-1$ is required.

Next, we will make a connection between certain sequences of para-orthogonal polynomials and bi-orthogonal systems of trigonometric polynomials for the same weight function $\omega(\theta)$. For this purpose, let $B_{2n}(z)$ be a polynomial of degree $2n$, para-orthogonal and invariant. Then, from the begining of Section 2, one can write (by virtue of invariance)

$$B_{2n}(e^{i\theta}) = a_n e^{in\theta} f_n(\theta), \ \ a_n \neq 0 \tag{6.10}$$

$f_n(\theta)$ being a real trigonometric polynomial of precise degree n.

Theorem 6.7. *Let $f_n(\theta) \in \mathcal{T}_n$ as given by (6.10). Then $\langle f_n(\theta), T(\theta)\rangle_\omega = 0$ for all $T \in \mathcal{T}_{n-1}$.*

Proof. Clearly, it will be enought to show that, $\langle \rho_n(z), z^j\rangle_\omega = 0$ for $-(n-1) < j \leqslant n-1$ ($z = e^{i\theta}$). By (6.10) and since $a_n \neq 0$, the above becomes

$$\langle e^{-in\theta} B_{2n}(e^{i\theta}), e^{ij\theta}\rangle_\omega = 0 \quad , \quad -(n-1) \leqslant j \leqslant n-1. \tag{6.11}$$

Now, by Theorem 6.4, $B_{2n}(z) = \rho_{2n}(z) + \tau\rho_{2n}^*(z)$ (observe that the constant $C_{2n} \neq 0$ is now irrelevant) so that (6.11) can be written as

$$\begin{gathered}\langle e^{-in\theta}\left(\rho_{2n}(e^{i\theta}) + \tau\rho_{2n}^*(e^{i\theta})\right), e^{ij\theta}\rangle_\omega = \\ \langle \rho_{2n}(z), z^{n+j}\rangle_\omega + \tau\langle \rho_{2n}^*(z), z^{n+j}\rangle_\omega = 0,\end{gathered}$$

because both inner products are zero by the orthogonality properties of $\rho_{2n}(z)$ and $\rho_{2n}^*(z)$. □

Now, as a direct consequence of Theorem 4.9, we can establish the fundamental property concerning the localization of the zeros of $B_n(z)$. Indeed, one has

Theorem 6.8. *Let $B_n(z)$ be a para-orthogonal and invariant polynomial of degree n. Then $B_n(z)$ has exactly n distinct zeros on the unit circle $\mathbb{T}$.*

Proof. Assume first that n is even, say $n = 2m$ so that by (6.10)

$$e^{-im\theta} B_{2m}(e^{i\theta}) = a_m h_m(\theta), \quad a_m \neq 0, \quad h_m \in \mathcal{T}_m.$$

Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system of trigonometric polynomials. Then, by Theorem 6.7, $h_m(\theta) = \alpha_m f_m(\theta) + \beta_m g_m(\theta)$, $|\alpha_m| + |\beta_m| > 0$ and the proof follows by Theorem 4.9. Suppose now that n is odd, i.e. $n = 2m + 1$. Since $B_{2m+1}(z)$ is invariant, one knows that $B_{2m+1}(z)$ has at least one zero λ on $\mathbb{T}$ of odd multiplicity. Thus, $B_{2m+1}(z) = (z - \lambda)\tilde{B}_{2m}(z)$ with $\tilde{B}_{2m}(z)$ a polynomial of degree $2m$. Furthermore, it can be easily checked that $\tilde{B}_{2m}(z)$ is also invariant and para-orthogonal for the weight function $\tilde{\omega}(\theta) = |e^{i\theta} - \lambda|^2 \omega(\theta)$. Hence, $\tilde{B}_{2m}(z)$ has $2m$ distinct zeros on $\mathbb{T}$. Furthermore, any zero of $\tilde{B}_{2m}(z)$ is different from λ, otherwise its multiplicity would be two. This concludes the proof. □

Remark 6.9. In [10] another different and longer proof of Theorem 6.8 is presented. Here we have taken advantage of the properties of bi-orthogonal systems introduced in Section 4 to give a simpler proof.

Let $\{B_{2n}(z)\}_{n=0}^{\infty}$ be a sequence of para-orthogonal and invariant polynomials such that for each $n \geqslant 1$, $B_{2n}(z)$ has exactly degree $2n$. Because of invariance again, it can be written

$$B_{2n}(z) = a_n e^{in\theta} f_n(\theta), \quad a_n \neq 0, \ f_n \in \mathcal{T}_n.$$

Then, by Theorem 6.7, $\{f_n(\theta)\}_{n=0}^{\infty}$ ($f_0(\theta) = f_0 \neq 0$) represents a nontrivial orthogonal system of trigonometric polynomials, in the sense that for each n, $f_n(\theta)$ has the precise degree n and $\langle f_n(\theta), f_m(\theta)\rangle_\omega = K_n \delta_{n,m}$, $K_n > 0$. Now, we could ask if it is possible to find another orthogonal system $\{g_n(\theta)\}_{n=1}^{\infty}$ so that $f_0 \cup \{f_n(\theta), g_n(\theta)\}_{n=1}^{\infty}$ constitutes a bi-orthogonal system of trigonometric polynomials. To fix ideas, set

$$B_{2n}(z) = B_{2n}(z, \tau_n) = \rho_{2n}(z) + \tau_n \rho_{2n}^*(z)$$

where $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of complex numbers on $\mathbb{T}$. Certainly, we can write $\tau_n = \frac{\overline{\gamma_n}}{\gamma_n}$, $\gamma_n \in \mathbb{C}$, $\gamma_n \neq 0$ so that if $\tau_n = e^{i\eta_n}$, then $\gamma_n = r_n e^{-i\eta_n/2}$, $\eta_n \in \mathbb{R}$, $r_n > 0$. On the other hand, setting $z = e^{i\theta}$:

$$\begin{aligned} z^{-n} B_{2n}(z) &= \frac{\rho_{2n}(z) + \tau_n \rho_{2n}^*(z)}{z^n} = \frac{1}{\gamma_n}\left[\frac{\gamma_n \rho_{2n}(z) + \overline{\gamma_n} \rho_{2n}^*(z)}{z^n}\right] \\ &= \frac{1}{\gamma_n}\left[\frac{\gamma_n \rho_{2n}(z) + \overline{\gamma_n} z^{2n} \rho_{(2n)*}(z)}{z^n}\right] \\ &= \frac{1}{\gamma_n}\left[\gamma_n z^{-n} \rho_{2n}(z) + \overline{\gamma_n z^{-n} \rho_{2n}(z)}\right] \\ &= \frac{2}{\gamma_n} \Re\left(\gamma_n z^{-n} \rho_{2n}(z)\right). \end{aligned}$$

Consider now $B_{2n}(z,-\tau_n) = \rho_{2n}(z) - \tau_n \rho_{2n}^*(z)$. Then, again by Theorem 6.7, one has

$$e^{-in\theta} B_{2n}(e^{i\theta}, -\tau_n) = \tilde{\lambda}_n g_n(\theta), \ \ \tilde{\lambda}_n \neq 0, \ g_n \in \mathcal{T}_n$$

and $\{g_n(\theta)\}_{n=1}^{\infty}$ is an orthogonal system of trigonometric polynomials. Therefore it holds that

$$\langle g_n(\theta), g_m(\theta)\rangle_\omega = \tilde{K}_n \delta_{n,m}, \ \tilde{K}_n > 0 \quad ; \quad \langle g_n(\theta), f_m(\theta)\rangle_\omega = 0, \ n \neq m.$$

Let us also see that $\langle f_n(\theta), g_n(\theta)\rangle_\omega = 0$ for $n = 0, 1, \ldots$. As above, it can be easily shown that

$$g_n(\theta) = \tilde{C}_n \Im\left(\gamma_n z^{-n} \rho_{2n}(z)\right) = \tilde{C}_n \tilde{g}_n(\theta)$$

with $\tilde{C}_n \neq 0$ and $\tilde{g}_n \in \mathcal{T}_n$. Hence,

$$\langle f_n(\theta), g_n(\theta)\rangle_\omega = 0 \ \Leftrightarrow \ \langle \tilde{f}_n(\theta), \tilde{g}_n(\theta)\rangle_\omega = 0.$$

Now, for $z = e^{i\theta}$,

$$\begin{array}{rcl} \int_{-\pi}^{\pi} \left(\gamma_n z^{-n} \rho_{2n}(z)\right)^2 \omega(\theta) d\theta & = & \int_{-\pi}^{\pi} \left[\tilde{f}_n(\theta) + i\tilde{g}_n(\theta)\right]^2 \omega(\theta) d\theta \\ & = & \int_{-\pi}^{\pi} \tilde{f}_n^2(\theta)\omega(\theta)d\theta - \int_{-\pi}^{\pi} \tilde{g}_n^2(\theta)\omega(\theta)d\theta + \\ & + & 2i \int_{-\pi}^{\pi} \tilde{f}_n(\theta)\tilde{g}_n(\theta)\omega(\theta)d\theta. \end{array}$$

Thus, by assuming that $\gamma_n^2 \int_{-\pi}^{\pi} z^{-2n} \rho_{2n}^2(z)\omega(\theta)d\theta$ $(z = e^{i\theta})$ is a real number it follows that

$$\int_{-\pi}^{\pi} \tilde{f}_n(\theta)\tilde{g}_n(\theta)\omega(\theta)d\theta = \langle \tilde{f}_n(\theta), \tilde{g}_n(\theta)\rangle_\omega = 0.$$

But

$$\begin{array}{rl} \gamma_n^2 \int_{-\pi}^{\pi} z^{-2n} \rho_{2n}^2(z)\omega(\theta)d\theta & = \gamma_n^2 \int_{-\pi}^{\pi} \rho_{2n}(z) \frac{z^{2n} + \cdots + \delta_{2n}}{z^{2n}} \omega(\theta)d\theta \\ & = \gamma_n^2 \int_{-\pi}^{\pi} \rho_{2n}(z) \frac{\delta_{2n}}{z^{2n}} \omega(\theta)d\theta \\ & = \gamma_n^2 \delta_{2n} \langle \rho_{2n}(z), z^{2n}\rangle_\omega. \end{array}$$

Since $\langle \rho_{2n}(z), z^{2n}\rangle_\omega = \langle \rho_{2n}(z), \rho_{2n}(z)\rangle_\omega = \| \rho_{2n}(z) \|_\omega^2 > 0$, then the positivity of

$$\gamma_n^2 \int_{-\pi}^{\pi} z^{-2n} \rho_{2n}^2(z)\omega(\theta)d\theta$$

reduces to $\gamma_n^2 \delta_{2n} \in \mathbb{R}$, or equivalently $\overline{\gamma_n^2 \delta_{2n}} \in \mathbb{R}$. In terms of the parameter $\tau_n = \frac{\overline{\gamma_n}}{\gamma_n} \in \mathbb{T}$, this condition implies $\tau_n \overline{\delta_{2n}} \in \mathbb{R}$. In other words, we have proved the following

Theorem 6.10. *Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of complex numbers in $\mathbb{T}$ such that $\tau_n \overline{\delta_{2n}} \in \mathbb{R}$ and consider the sequences of polynomials $\{B_{2n}(z,\tau_n)\}_{n=1}^{\infty}$ and $\{B_{2n}(z,-\tau_n)\}_{n=1}^{\infty}$ so that for each $n = 1, 2, \ldots$, $B_{2n}(z, \pm\tau_n)$ is a para-orthogonal and invariant polynomial of degree $2n$. Then*

1. *$e^{-in\theta}B_{2n}(e^{i\theta},\tau_n)=\lambda_n f_n(\theta)$ and $e^{-in\theta}B_{2n}(e^{i\theta},-\tau_n)=\tilde{\lambda}_n g_n(\theta)$ with λ_n and $\tilde{\lambda}_n$ nonzero complex numbers and $f_n(\theta)$ and $g_n(\theta)$ being trigonometric polynomials of the precise degree n.*

2. *Choose $f_0\neq 0$, then $f_0\cup\{f_n(\theta),g_n(\theta)\}_{n=1}^{\infty}$ represents a bi-orthogonal system for $\omega(\theta)$.*

Now, from Theorem 4.10 or Corollary 4.12 one immediately gets

Corollary 6.11. *Under the same assumptions as in Theorem 6.10, the zeros of the para-orthogonal polynomials $B_{2n}(z,\tau_n)$ and $B_{2n}(z,-\tau_n)$ interlace.*

On the other hand, a converse to Theorem 6.10 can be also given. Indeed, we have:

Theorem 6.12. *Let $f_0\cup\{f_k,g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system for $\omega(\theta)$ and take a and b real numbers not both zero. Then, for $n\geqslant 1$*

$$H_n(\theta)=af_n(\theta)+bg_n(\theta)=e^{-in\theta}B_{2n}(e^{i\theta})$$

and $B_{2n}(z)$ is a para-orthogonal and 1-invariant polynomial of degree $2n$.

Proof. We can write

$$f_n(\theta)=a_0+\sum_{j=1}^{n}(a_j\cos j\theta+b_j\sin j\theta),\quad g_n(\theta)=\alpha_0+\sum_{j=1}^{n}(\alpha_j\cos j\theta+\beta_j\sin j\theta)$$

with $|a_n|+|b_n|>0$, $|\alpha_n|+|\beta_n|>0$ and

$$f_n(\theta)=\sum_{k=-n}^{n}c_kz^k\in\Lambda_{-n,n},\quad g_n(\theta)=\sum_{k=-n}^{n}d_kz^k\in\Lambda_{-n,n},\quad z=e^{i\theta}$$

where for $k=1,\dots,n$,

$$\begin{array}{lll} c_0=a_0, & c_k=\frac{a_k-ib_k}{2} & c_{-k}=\frac{a_k+ib_k}{2}\\ d_0=\alpha_0, & d_k=\frac{\alpha_k-i\beta_k}{2}, & d_{-k}=\frac{\alpha_k+i\beta_k}{2}. \end{array} \tag{6.12}$$

Hence, by the transformation $z=e^{i\theta}$ it follows that

$$\begin{aligned} B_{2n}(\theta) &= z^n\left[a\textstyle\sum_{k=-n}^{n}c_kz^k+b\sum_{k=-n}^{n}d_kz^k\right]=\sum_{j=0}^{2n}\left(ac_{j-n}+bd_{j-n}\right)z^j\\ &=\textstyle\sum_{j=0}^{2n}e_jz^j\in\Pi_{2n}, \end{aligned}$$

and it is clear from (6.12) that $\overline{e_{2n-j}}=e_j$ for $j=0,\dots,2n$. This proves the 1-invariance property. Now, from the orthogonality conditions satisfied by $f_n(\theta)$ and $g_n(\theta)$ it follows for $j=1,\dots,2n-1$ that

$$\begin{aligned} \langle B_{2n}(\theta),e^{ij\theta}\rangle_\omega &= \langle e^{in\theta}\left[af_n(\theta)+bg_n(\theta)\right],e^{ij\theta}\rangle_\omega=\\ &= a\langle f_n(\theta),e^{i(j-n)\theta}\rangle_\omega+b\langle g_n(\theta),e^{i(j-n)\theta}\rangle_\omega=0, \end{aligned}$$

i.e., $\langle B_{2n}(z), z^j \rangle_\omega = 0$ for all $j = 1, \ldots, 2n-1$. We will prove next that $\langle B_{2n}(z), 1\rangle_\omega \neq 0$ and $\langle B_{2n}(z), z^{2n}\rangle_\omega \neq 0$. Firstly observe that

$$\begin{aligned} \langle B_{2n}(z), 1\rangle_\omega &= a\langle f_n(\theta), e^{-in\theta}\rangle_\omega + b\langle g_n(\theta), e^{-in\theta}\rangle_\omega, \\ \langle B_{2n}(z), z^{2n}\rangle_\omega &= a\langle f_n(\theta), e^{in\theta}\rangle_\omega + b\langle g_n(\theta), e^{in\theta}\rangle_\omega \ . \end{aligned} \tag{6.13}$$

Writing $\cos n\theta = \frac{e^{in\theta}+e^{-in\theta}}{2}$, $\sin n\theta = \frac{e^{in\theta}-e^{-in\theta}}{2i}$, $f_n(\theta) = a_n \cos n\theta + b_n \sin n\theta + H_{n-1}(\theta)$ and $g_n(\theta) = \alpha_n \cos n\theta + \beta_n \sin n\theta + \tilde{H}_{n-1}(\theta)$, where $H_{n-1}(\theta), \tilde{H}_{n-1}(\theta) \in \mathcal{T}_{n-1}$, we deduce that

$$\begin{array}{rcl} \langle f_n(\theta), f_n(\theta)\rangle_\omega &=& \langle f_n(\theta), a_n\cos n\theta + b_n \sin n\theta + H_{n-1}(\theta)\rangle_\omega = \\ &=& \frac{b_n+ia_n}{2i}\langle f_n(\theta), e^{in\theta}\rangle_\omega + \frac{-b_n+ia_n}{2i}\langle f_n(\theta), e^{-in\theta}\rangle_\omega = \\ &=& h_n > 0, \\ \\ \langle g_n(\theta), g_n(\theta)\rangle_\omega &=& \frac{\beta_n+i\alpha_n}{2i}\langle g_n(\theta), e^{in\theta}\rangle_\omega + \frac{-\beta_n+i\alpha_n}{2i}\langle g_n(\theta), e^{-in\theta}\rangle_\omega = \\ &=& h_n^{'} > 0, \\ \\ \langle f_n(\theta), g_n(\theta)\rangle_\omega &=& \langle f_n(\theta), \alpha_n\cos n\theta + \beta_n \sin n\theta + \tilde{H}_{n-1}(\theta)\rangle_\omega = \\ \\ &=& \frac{\beta_n+i\alpha_n}{2i}\langle f_n(\theta), e^{in\theta}\rangle_\omega + \frac{-\beta_n+i\alpha_n}{2i}\langle f_n(\theta), e^{-in\theta}\rangle_\omega = 0, \\ \\ \langle g_n(\theta), f_n(\theta)\rangle_\omega &=& \frac{b_n+ia_n}{2i}\langle g_n(\theta), e^{in\theta}\langle + \frac{-b_n+ia_n}{2i}\langle g_n(\theta), e^{-in\theta}\rangle_\omega = 0 \ . \end{array}$$

These relations can be summarized as

$$A\begin{pmatrix} \langle f_n(\theta), e^{in\theta}\rangle_\omega \\ \\ \langle f_n(\theta), e^{-in\theta}\rangle_\omega \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 2ih_n \end{pmatrix}, \ A\begin{pmatrix} \langle g_n(\theta), e^{in\theta}\rangle_\omega \\ \\ \langle g_n(\theta), e^{-in\theta}\rangle_\omega \end{pmatrix} = \begin{pmatrix} 2ih_n^{'} \\ \\ 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} \beta_n + i\alpha_n & -\beta_n + i\alpha_n \\ \\ b_n + ia_n & -b_n + ia_n \end{pmatrix}, \quad det(A) = 2i[a_n\beta_n - \alpha_n b_n] \neq 0$$

since $f_n(\theta), g_n(\theta)$ are linearly independent trigonometric polynomials. The solutions of these systems are given by

$$\langle f_n(\theta), e^{in\theta}\rangle_\omega = \frac{\beta_n - i\alpha_n}{a_n\beta_n - \alpha_n b_n} h_n \neq 0, \ \ \langle f_n(\theta), e^{-in\theta}\rangle_\omega = \overline{\langle f_n(\theta), e^{in\theta}\rangle_\omega},$$

$$\langle g_n(\theta), e^{in\theta}\rangle_\omega = \frac{-b_n + ia_n}{a_n\beta_n - \alpha_n b_n} h_n^{'} \neq 0, \ \ \langle g_n(\theta), e^{-in\theta}\rangle_\omega = \overline{\langle g_n(\theta), e^{in\theta}\rangle_\omega} \ .$$

Now, from (6.13) it follows that

$$\langle B_{2n}(z), 1\rangle_\omega = \frac{1}{a_n\beta_n - \alpha_n b_n}\left[(a\beta_n h_n - b\beta_n h_n^{'}) + i(a\alpha_n h_n - ba_n h_n^{'})\right]$$

and $\langle B_{2n}(z), z^{2n}\rangle_\omega = \overline{\langle B_{2n}(z), 1\rangle_\omega}$. Again, since $f_n(\theta), g_n(\theta)$ are linearly independent it is easy to observe that $\langle B_{2n}(z), 1\rangle_\omega \neq 0$ and hence $\langle B_{2n}(z), z^{2n}\rangle_\omega \neq 0$. This completes the proof. □

After having established certain connections between para-orthogonal polynomials and bi-orthogonal trigonometric polynomials we are now in a position to construct an n-point quadrature rule for $I_\omega(f)$ with nodes on $\mathbb{T}$ and having the "maximum domain of validity", $\Lambda_{-(n-1),n-1}$. Indeed, we have (see [10])

Theorem 6.13. *Let $z_1, \ldots, z_n$ be the n distinct zeros of $B_n(z)$ a given polynomial of degree n, para-orthogonal and invariant. Then, there exist positive numbers $A_1, \ldots, A_n$ such that*

$$I_n(f) = \sum_{j=1}^{n} A_j f(z_j) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \ \ \forall\ f \in \Lambda_{-(n-1),n-1}.$$

Now, by considering Theorems 6.1 and 6.13 together we obtain the following characterization (see [2]):

Corollary 6.14. *Let $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$ and let $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \in \mathbb{T}$, $j = 1, \ldots, n$ with $z_j \neq z_k$ if $j \neq k$ and set $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. Then $I_n(L) = I_\omega(L)$ for all $L \in \Lambda_{-(n-1),n-1}$ if and only if*

1. *$I_n(L) = I_\omega(L)$, for all $L \in \Lambda_{-p,q}$, p and q being nonnegative arbitrary integers such that $p + q = n - 1$.*
2. *$B_n(z)$ is para-orthogonal and invariant.*

Furthermore, when the conditions are satisfied the weights $\{A_j\}_{j=1}^{n}$ are positive and independent of p and q.

Remark 6.15. The quadrature rules $I_n(f)$, $n = 1, 2, \ldots$ as given above are called "Szegő quadrature formulas" and were earlier introduced in [10]. They represent the analogue on the unit circle of the Gauss-Christoffel formulas. For an alternative approach of Szegő quadratures making use of the so-called orthogonal Laurent polynomials on the unit circle, see the recent paper by the authors [3]. For further details concerning these quadratures see also [4], [5] and [9].

To conclude, it should be remarked that given the integral $\int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$, f being a 2π-periodic function, it clearly follows from Corollary 6.14 how to construct an n-point quadrature rule with distinct nodes on $[-\pi, \pi]$ which is exact in $\mathcal{T}_{n-1}$, n being an arbitrary natural number. As a simple illustration, let us consider again the weight function $\omega(\theta) \equiv 1$. Then, $B_n(z) = z^n - \tau$, $|\tau| = 1$ and the nodes of the n-th Szegő formula are the n-th roots of τ, that is $z_j = \sqrt[n]{\tau}$, $j = 1, \ldots, n$. Thus,

$$A_j = \frac{1}{B_n^{'}(z_j)} \int_{-\pi}^{\pi} \frac{B_n(z)}{z - z_j} d\theta = \frac{1}{n z_j^{n-1}} \frac{1}{i} \int_{\mathbb{T}} \frac{z^n - \tau}{z(z - z_j)} dz = \frac{2\pi\tau}{n z_j^n},$$

by the Residue Theorem. Since
$z_j^n = \tau$, we obtain $A_j = \frac{2\pi}{n}$, $j = 1, \ldots, n$ as previously deduced in Example 5.9.

7. Numerical examples

In order to illustrate the numerical effectiveness of the quadrature rules considered through the paper, in this section we are going to be concerned with the computation of the two-parameter integral,

$$I(m,\alpha) = \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha + \sin^2\theta} d\theta, \; m \geq 0, \; m \in \mathbb{N}, \; \alpha > 0. \tag{7.1}$$

Observe that for $\alpha = 0$, the integral diverges. Thus, for values of α close to zero, the denominator of the integrand is also close to zero as θ tends to $\pm\pi$. Certainly, this could generate some kind of unstability when undertaking the approximation of $I(m,\alpha)$ by means of a certain quadrature rule with nodes close to $\pm\pi$.

On the other hand, for m large enough, the integral is highly oscillating on $[-\pi,\pi]$. Indeed, setting $f(\theta) = \frac{\cos m\theta}{\alpha+\sin^2\theta}$, then $f(\theta)$ clearly changes sign at the points for which $f(\theta) = 0$, i. e., at $\theta_k = \frac{(2k+1)\pi}{2m}$, $-m \leq k \leq m-1$.

Under these considerations, we propose the following in order to compute approximately the integral $I(m,\alpha)$. Note that because of simmetry, one can write

$$I(m,\alpha) = 2\int_{0}^{\pi} \frac{\cos m\theta}{\alpha + \sin^2 m\theta} d\theta. \tag{7.2}$$

First, we have approximated (7.2) by means of the $n-$point Gauss-Legendre formula for the interval $[0,\pi]$ and the Trapezoidal rule for $n = 10, 12, 14, 16$. Here n denotes both the number of nodes in the Gauss-Legendre formulas and the number of subintervals in $[0,\pi]$. The results are displayed in the following tables.

Quadrature rules	n=10	n=12	n=14	n=16
Gauss-Legendre	2.26414	0.300761	0.00937743	0.000154023
Trapezoidal	0.0224394	0.000660554	0.000194449	5.72404E-7

Table 1: $(m = 14, \; \alpha = 1)$

Quadrature rules	n=10	n=12	n=14	n=16
Gauss-Legendre	8.93136E-5	7.12412E-7	1.17708E-8	4.65022E-10
Trapezoidal	4.20833E-8	1.30695E-10	4.05799E-12	1.26807E-15

Table 2: $(m = 8, \; \alpha = 4)$

Take into account that the trapezoidal rule coincides with the quadrature formula with the highest degree of trigonometric precision (Szegő formula). This fact might explain why the results provided by the Trapezoidal rule are better than those given by Gauss-Legendre formula. However, when α is closer to zero, the results of both quadrature rules, as it could be expected, are rather poor. This is shown in Table 3 corresponding to $m = 12$ and $\alpha = 0.25$.

Quadrature rules	n=6	n=8	n=10	n=12
Gauss-Legendre	5.05696	5.60122	0.516198	0.0190433
Trapezoidal	11.2748	1.64061	0.239269	0.0349069

Table 3: $(m = 12,\ \alpha = 0.25)$

In order to overcome this drawback, we are going to take the factor $\frac{1}{\alpha+\sin^2\theta}$ as a weight function. For this purpose, set $T(\theta) = \alpha + \sin^2\theta$, so that $T(\theta)$ is a positive trigonometric polynomial of degree two. Then, by Theorem 2.6, one can write,

$$T(\theta) = \left|g\left(e^{i\theta}\right)\right|^2,\ g \in \Pi_2.$$

Since $T(\theta) = \alpha + \sin^2\theta = \alpha + \frac{1}{2}(1 - \cos 2\theta)$, then by setting $\beta = 2\alpha + 1 > 1$ and $z = e^{i\theta}$,

$$2T(\theta) = \beta - \frac{1}{2}(z^2 + z^{-2}),$$

yielding,

$$4T(\theta) = \frac{-z^4 + 2\beta z^2 - 1}{z^2}.$$

Furthermore, since $T(\theta) > 0$ and $z \in \mathbb{T}$, then

$$4T(\theta) = |4T(\theta)| = |z^4 - 2\beta z^2 + 1|. \tag{7.3}$$

If we set $z^4 - 2\beta z^2 + 1 = 0$, then $z^2 = \beta \pm \sqrt{\beta^2 - 1}$. Let $\gamma = \beta + \sqrt{\beta^2 - 1}$, then, it is easy to check that $\frac{1}{\gamma} = \beta - \sqrt{\beta^2 - 1}$. Therefore, one has

$$z^4 - 2\beta z^2 + 1 = (z^2 - \gamma)(z^2 - \gamma^{-1}). \tag{7.4}$$

On the other hand, since $z = e^{i\theta}$ and $\gamma \in \mathbb{R}$, we have:

$$\begin{aligned} |z^2 - \gamma^{-1}|^2 &= (z^2 - \gamma^{-1})\overline{(z^2 - \gamma^{-1})} = (z^2 - \gamma^{-1})(z^{-2} - \gamma^{-1}) \\ &= (z^2 - \gamma^{-1})\left(\tfrac{\gamma - z^2}{\gamma z^2}\right) = -\tfrac{1}{\gamma z^2}(z^2 - \gamma)(z^2 - \gamma^{-1}) \end{aligned}$$

>From (7.3) and (7.4), one has:

$$0 < |z^2 - \gamma^{-1}|^2 = -\frac{1}{\gamma z^2}4T(\theta) = \left|-\frac{1}{\gamma z^2}4T(\theta)\right| = \frac{4}{\gamma}T(\theta).$$

Thus,

$$T(\theta) = \frac{\gamma}{4}|z^2 - \gamma^{-1}|^2 = \frac{\gamma}{4}|g(z)|^2,\ g(z) = z^2 - \gamma^{-1},\ z = e^{i\theta}.$$

Now, taking into account that for integrals of the form: $\int_{-\pi}^{\pi} f(e^{i\theta}) \frac{d\theta}{2\pi|h(e^{\theta})|^2}$, with h a monic polynomial with all its zeros in $\mathbb{D}$, the coefficients of the $n-$point Szegő quadrature formulas are explicitly known ([9]), we will transform our integral $I(m,\alpha)$ as follows: $(z = e^{i\theta})$

$$\begin{aligned} I(m,\alpha) &= \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha + \sin^2\theta} d\theta = \int_{-\pi}^{\pi} \cos m\theta \frac{d\theta}{\frac{\gamma}{4}|g(z)|^2} \\ &= \int_{-\pi}^{\pi} \left(\frac{4\pi}{\gamma}(z^m + z^{-m}) \right) \left(\frac{d\theta}{2\pi|g(z)|^2} \right). \end{aligned}$$

Therefore, we can write:

$$I(m,\alpha) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta) d\theta, \tag{7.5}$$

where $f(z) = \frac{4\pi}{\gamma}(z^m + z^{-m})$ and the weight function is given by $\omega(\theta) = \frac{1}{2\pi|g(z)|^2}$, with $g(z) = z^2 - \gamma^{-1}$ and $z = e^{i\theta}$.

In this case, from Corollary 6.14, one knows that the nodes $\{z_j\}_{j=1}^n$ of the $n-$point Szegő quadrature formula are the zeros of the para-orthogonal polynomial $B_n(z) = \rho_n(z) + \tau\rho_n^*(z)$, $\rho_n(z)$ being the $n-th$ monic Szegő polynomial for $\omega(\theta) = \frac{1}{2\pi|g(z)|^2}$, with $|\tau| = 1$. Thus, from Example 4.8, $B_n(z,\tau) = z^{n-2}g(z) + \tau g^*(z) = z^{n-2}(z^2 - \gamma^{-1}) + \tau(1 - \gamma^{-1}z^2)$. On the other hand, the coefficients $\{\lambda_j\}_{j=1}^n$ of an $n-$point Szegő's formula are given by [9]:

$$\begin{aligned} \lambda_j^{-1} &= |g(z_j)|^2 \left(n - 2 + \frac{1 - \frac{1}{\sqrt{\gamma}}^{\,2}}{z_j - \frac{1}{\sqrt{\gamma}}^{\,2}} + \frac{1 - -\frac{1}{\sqrt{\gamma}}^{\,2}}{z_j + \frac{1}{\sqrt{\gamma}}^{\,2}} \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1 - \gamma^{-1}) \left(\frac{1}{z_j - \frac{1}{\sqrt{\gamma}}^{\,2}} + \frac{1}{z_j + \frac{1}{\sqrt{\gamma}}^{\,2}} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1 - \gamma^{-1}) \left(\frac{z_j - \frac{1}{\sqrt{\gamma}}^{\,2} + z_j + \frac{1}{\sqrt{\gamma}}^{\,2}}{|g(z_j)|^2} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1 - \gamma^{-1}) \left(\frac{1 - \frac{2}{\sqrt{\gamma}}\Re(z_j) + \frac{1}{\gamma} + 1 + \frac{2}{\sqrt{\gamma}}\Re(z_j) + \frac{1}{\gamma}}{|g(z_j)|^2} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + 2(1 - \gamma^{-1})(1 + \gamma^{-1}) \frac{1}{|g(z_j)|^2} \right) \\ &= (n-2)|g(z_j)|^2 + 2(1 - \gamma^{-2}),\ j = 1, \ldots, n. \end{aligned}$$

Note that, if $m \leq n - 1$, then the $n-$point Szegő quadrature formula is exact since the integrand $f \in \Delta_{-m,m}$.

Now, by (7.5), $I(m,\alpha)$ is going to be approximated by an $n-$point Szegő formula $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$ so that the absolute errors can be exactly computed since $I(m,\alpha)$ can be calculated by the Residue's Theorem.

Indeed, since $I = \int_{-\pi}^{\pi} \frac{\sin m\theta}{\alpha+\sin^2\theta} d\theta = 0$, then

$$
\begin{aligned}
I(m,\alpha) &= \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha+\sin^2\theta} d\theta + i\int_{-\pi}^{\pi} \frac{\sin m\theta}{\alpha+\sin^2\theta} d\theta = \int_{-\pi}^{\pi} \frac{\cos m\theta + i\sin m\theta}{T(\theta)} d\theta \\
&= \int_{-\pi}^{\pi} \frac{z^m}{\frac{\gamma}{4}|g(z)|^2} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{8\pi}{\gamma} z^m\right) \frac{d\theta}{(z^2-\frac{1}{\gamma})(\frac{1}{z^2}-\frac{1}{\gamma})} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (-8\pi) \frac{z^{m+2}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)} d\theta = \frac{1}{2\pi i}\int_{\mathbb{T}} (-8\pi) \frac{z^{m+1}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)} dz \\
&= Res\left(h, \frac{1}{\sqrt{\gamma}}\right) + Res\left(h, \frac{-1}{\sqrt{\gamma}}\right),
\end{aligned}
$$

where $h(z) = (-8\pi)\frac{z^{m+1}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)}$.

Now,

$$
Res\left(h, \frac{1}{\sqrt{\gamma}}\right) = -8\pi \frac{\frac{1}{(\sqrt{\gamma})^{m+1}}}{\frac{2}{\sqrt{\gamma}}\left(\frac{1}{\gamma}-\gamma\right)} = \frac{4\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)},
$$

and

$$
Res\left(h, \frac{-1}{\sqrt{\gamma}}\right) = -8\pi \frac{\frac{(-1)^{m+1}}{(\sqrt{\gamma})^{m+1}}}{\frac{-2}{\sqrt{\gamma}}\left(\frac{1}{\gamma}-\gamma\right)} = (-1)^{m+1}\frac{-4\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)}.
$$

Hence,

$$
I(m,\alpha) = \frac{4\pi\gamma(1-(-1)^{m+1})}{(\sqrt{\gamma})^m(\gamma^2-1)} = \begin{cases} \frac{8\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)}, & \text{if m is even,} \\ 0, & \text{if m is odd.} \end{cases}
$$

Taking now $m = 12$ and $\alpha = 0.25$, the absolute errors for the corresponding $n-$point Szegő formula are displayed in Table 4 (Compare with Table 3).

n	Error- Szegő formula
n=4	3.18008
n=8	1.8473911237281646E-15
n=12	6.949821829035384E-15

Table 4: $(m = 12,\ \alpha = 0.25)$

The excellent behaviour of Segő formulas can be explained from [9, Theorem 3.3] taking into account that the integrand $f(z)$ in (7.5) has one only pole at the origin.

References

[1] N. I. Akhiezer The classical moment problem and some related questions in analysis, Hafner, New York, (1965).

[2] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad Orthogonality and quadrature on the unit circle, In C. Brezinski, L. Gori, and A. Ronveaux, editors, *Orthogonal Polynomials and their Applications* Vol. 9 of IMACS Annals on Computing and Applied Mathematics (1991), 205–210. Basel, J.C. Baltzer AG.

[3] R. Cruz-Barroso, L. Daruis, P. González-Vera and O. Njåstad Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle. *Submitted.*

[4] L. Daruis, P. González-Vera, A. Bultheel A conecction between quadrature formulas on the unit circle and the interval $[-1, 1]$, *J. Comp. Appl. Math* Vol. 132 (1) (2000), 1–14.

[5] L.q Daruis, P. González-Vera and O. Njåstad Szegő quadrature formulas for certain Jacobi-type weight functions, *Mathematics of Computation* Vol. 71, Num. 238 (2001), 683–701.

[6] P. J. Davis, P. Rabinowitz Methods of Numerical Integration, *Academic Press* New York, (1984).

[7] P. Delsarte, Y. Genin On the role of Orthogonal Polynomials on the Unit Circle in digital signal processing applications, Orthogonal polynomials, *(Columbus, OH, 1989), NATO Adv. Study Inst. Ser. C Math. Phys. Sci* Vol. 294 (1990), 115–133. Kluwer Acad. Punl., Dordrecht.

[8] W. Gautschi A survey of Gauss-Christoffel quadrature formulae *in E.B. Christoffel; The Influence of his Work in Mathematics and the Physical Sciences, eds. P. L. Butzer and F. Fehér, Birkhäuser, Basel* (1981), 72–147.

[9] P. González-Vera, J. C. Santos-León, O. Njåstad Some results about numerical quadrature on the unit circle, *Advances in Computational Mathematics* Vol. 5 (1996), 297–328.

[10] W. B. Jones, O. Njåstad and W. J. Thron Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, *Bull. London Math. Soc.* Vol. 21 (1989), 113-152.

[11] V. I. Krylov Approximate Calculation of Integrals, *The MacMillan Company* New York (1962).

[12] N. Levinson The Wiener RMS (root mean square) error criterion in filter design and prediction, *J.Math. Phys.* Vol. 25 (1947), 261–278.

[13] B. Simon Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, *AMS Colloquium Publications. American Mathematical Society, Providence, RI* Vol. 54.1 (2005).

[14] G. Szegő On bi-orthogonal systems of trigonometric polynomials, *Magyar Tud. Alcad. Kutató Int. Közl* Vol. 8 (1963), 255–273.

[15] G. Szegő Orthogonal polynomials, *Amer. Math. Soc. Coll. Publ.* Amer. Math. Soc. Providence, R.I. Vol. 23 (1975).

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