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Burkholder's inequality for multiindex martingales*

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Dedicated to the memory of my teacher Péter Kiss

Abstract

Multiindex versions of Khintchine's and Burkholder's inequalities are presented.

Key Words: Khintchine's inequality, Burkholder's inequality, random field, martingale.

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1. Introduction and notation

Burkholder's inequality is a powerful tool of martingale theory. Let $(Z_n, \mathcal{F}_n)$, $n = 1, 2, \dots$, be a martingale with difference $X_n = Z_n - Z_{n-1}$. Let $p > 1$. There exist finite and positive constants C_p and D_p depending only on p such that

$$C_p \left[\mathbb{E}\left(\sum\nolimits_{k=1}^{n} X_k^2\right)^{p/2}\right]^{1/p} \leqslant \left(\mathbb{E}|Z_n|^p\right)^{1/p} \leqslant D_p \left[\mathbb{E}\left(\sum\nolimits_{k=1}^{n} X_k^2\right)^{p/2}\right]^{1/p}, \quad (1.1)$$

see Burkholder's classical paper [1] and the textbook [2]. When the random variables $X_1, X_2, \dots$ are independent (1.1) is called the Marcinkiewicz-Zygmund inequality (and in this particular case it is valid also for $p = 1$).

Let $\varepsilon_i(t)$, $i = 1, 2, \dots$, be the Rademacher system on $[0, 1]$. If $X_k = \varepsilon_k a_k$, then we obtain Khintchine's inequality. There exist finite and positive constants A_p and B_p depending only on p such that for any real sequence a_k, $k = 1, 2, \dots$,

$$A_p \left(\sum\nolimits_{k=1}^{n} a_k^2\right)^{1/2} \leqslant \left[\int_0^1 \left|\sum\nolimits_{k=1}^{n} \varepsilon_k(t) a_k\right|^p dt\right]^{1/p} \leqslant B_p \left(\sum\nolimits_{k=1}^{n} a_k^2\right)^{1/2}. \quad (1.2)$$

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This inequality is valid for $p > 0$. Actually, the standard proof of (1.1) is based on (1.2), see [1]).

The two-index version of (1.1) is obtained in [8], see also [7].

The aim of this paper is to prove a multiindex version of Burkholder's inequality. The proof is based on the transform of a single parameter martingale. We also use the multiindex version of Khintchine's inequality (for the sake of completeness, we prove it).

In [9] the second inequality of (3.2) was presented (without proof) for $p > 2$. It was applied to obtain a Brunk-Prokhorov type strong law of large numbers for martingale fields (see [9], Proposition 14). For a recent overview of multiindex random processes see [6]. In [6] a certain version of the Burkholder inequality was presented for continuous parameter random fields without the details of the proof (p. 257, Theorem 4.1.2). We do not use that theorem, we give a simple proof based on well-known one-parameter results.

Our Burkholder type inequality can be used to prove convergence results for multiindex autoregressive type martingales (see [5], for the two-index case see [4]).

We use the following notation. Let d be a fixed positive integer. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers. The multidimensional indices will be denoted by $\mathbf{k} = (k_1, \dots, k_d), \mathbf{n} = (n_1, \dots, n_d), \dots \in \mathbb{N}_0^d$. Relations $\leqslant$, min are defined coordinatewise. I.e. $\mathbf{k} \leqslant \mathbf{n}$ means $k_1 \leqslant n_1, \dots, k_d \leqslant n_d$. Relation $\mathbf{k} < \mathbf{n}$ means $\mathbf{k} \leqslant \mathbf{n}$ but $\mathbf{k} \neq \mathbf{n}$.

Let $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ for $p > 0$. Then $\|X\|_{p_1} \leqslant \|X\|_{p_2}$ for $0 < p_1 \leqslant p_2$.

2. Khintchine's inequality

Theorem 2.1. *Let $\varepsilon_i(t)$, $i = 1, 2, \dots,$ be the Rademacher system on $[0, 1]$. Let $p > 0$. There exist finite and positive constants $A_{p,d}$ and $B_{p,d}$ depending only on p and d such that for any d-index sequence $a_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^d$,*

$$\begin{aligned} A_{p,d}\left(\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} a_{\mathbf{k}}^2\right)^{1/2} &\leqslant \left[\int_0^1 \cdots \int_0^1 \left|\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1)\cdots\varepsilon_{k_d}(t_d)a_{\mathbf{k}}\right|^p dt_1\dots dt_d\right]^{1/p} \\ &\leqslant B_{p,d}\left(\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} a_{\mathbf{k}}^2\right)^{1/2}. \end{aligned} \tag{2.1}$$

Proof. First we remark that for $d = 1$ inequality (2.1) is the original Khintchine's inequality.

Denote by ε_{i,n_i}, $n_i = 1, 2, \dots,$ $i = 1, 2, \dots, d$, independent sequences of independent Bernoulli random variables with $\mathbb{P}(\varepsilon_{i,n_i} = 1) = \mathbb{P}(\varepsilon_{i,n_i} = -1) = 1/2$ for each i and n_i. Let $s_{\mathbf{n}} = \left(\sum_{\mathbf{k}\leqslant\mathbf{n}} a_{\mathbf{k}}^2\right)^{1/2}$ and $S_{\mathbf{n}} = \sum_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{1,k_1}\cdots\varepsilon_{d,k_d}a_{\mathbf{k}}$. Then, by the Fubini theorem, inequality (2.1) is equivalent to

$$A_{p,d}\, s_{\mathbf{n}} \leqslant \|S_{\mathbf{n}}\|_p \leqslant B_{p,d}\, s_{\mathbf{n}}. \tag{2.2}$$

Now we prove that the products $\varepsilon_{1,k_1}\cdots\varepsilon_{d,k_d}$, $(k_1, \dots, k_d) \in \mathbb{N}^d$, are pairwise independent Bernoulli variables. By induction, it is enough to prove that $\varepsilon_{1,k_1}\varepsilon_{2,k_2}$,

$(k_1,k_2)\in\mathbb{N}^2$, are pairwise independent Bernoulli variables if ε_{1,k_1}, $k_1\in\mathbb{N}$, and ε_{2,k_2}, $k_2\in\mathbb{N}$, are independent sequences of pairwise independent Bernoulli variables. Indeed, if ε_1 and ε_2 are independent Bernoulli variables then their product is Bernoulli: $\mathbb{P}(\varepsilon_1\varepsilon_2=\pm1)=1/2$. Now turn to the independence. It is obvious that the independence of ε_1, ε_2, ε_3, and ε_4 implies the independence of $\varepsilon_1\varepsilon_2$ and $\varepsilon_3\varepsilon_4$. Moreover, the independence of ε_1, ε_2 and ε_3 implies the independence of $\varepsilon_1\varepsilon_3$ and $\varepsilon_2\varepsilon_3$:

$$\mathbb{P}\left(\varepsilon_1\varepsilon_3=\pm1,\varepsilon_2\varepsilon_3=\pm1\right)=\frac{1}{4}=\mathbb{P}\left(\varepsilon_1\varepsilon_3=\pm1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_3=\pm1\right).$$

Therefore $\|S_{\mathbf{n}}\|_2^2$ is the variance of the sum of pairwise indepenent random variables, so we have $s_{\mathbf{n}}=\|S_{\mathbf{n}}\|_2$. In particular, (2.2) is true for $p=2$.

Now we show that the products $\varepsilon_{1,k_1}\cdots\varepsilon_{d,k_d}$, $(k_1,\ldots,k_d)\in\mathbb{N}^d$, are not (completely) independent. Indeed, if ε_1, ε_2, ε_3, and ε_4 are independent Bernoulli variables, then $\varepsilon_1\varepsilon_3$, $\varepsilon_2\varepsilon_3$, $\varepsilon_1\varepsilon_4$, and $\varepsilon_2\varepsilon_4$ are not independent:

$$\mathbb{P}\left(\varepsilon_1\varepsilon_3=1,\varepsilon_2\varepsilon_3=1,\varepsilon_1\varepsilon_4=1,\varepsilon_2\varepsilon_4=1\right)=1/8\neq$$

$$\neq 1/16=\mathbb{P}\left(\varepsilon_1\varepsilon_3=1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_3=1\right)\mathbb{P}\left(\varepsilon_1\varepsilon_4=1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_4=1\right).$$

So relation (2.2) is really different from its one-index version.

Now we prove the second part of (2.2). We start with the case of $p\geq 2$. We use induction. For $d=1$ it is the original Khintchine's inequality. Assume (2.2) for $d-1$. Let

$$I_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d)=\sum\nolimits_{k_2=1}^{n_2}\cdots\sum\nolimits_{k_d=1}^{n_d}\varepsilon_{k_2}(t_2)\cdots\varepsilon_{k_d}(t_d)a_{k_1,k_2,\ldots,k_d}.$$

Then, by the original Khintchine's inequality,

$$\int_0^1\left|\sum\nolimits_{k_1=1}^{n_1}\varepsilon_{k_1}(t_1)I_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d)\right|^p\,dt_1\leqslant$$

$$\leqslant B_{p,1}^p\left(\sum\nolimits_{k_1=1}^{n_1}I_{k_1,n_2,\ldots,n_d}^2(t_2,\ldots,t_d)\right)^{p/2}.$$

From here

$$\begin{aligned}
\|S_{\mathbf{n}}\|_p^p &\leqslant B_{p,1}^p\int_0^1\cdots\int_0^1\left(\sum_{k_1=1}^{n_1}I_{k_1,n_2,\ldots,n_d}^2(t_2,\ldots,t_d)\right)^{p/2}\,dt_2\ldots dt_d\\
&\leqslant B_{p,1}^p\left\{\sum_{k_1=1}^{n_1}\left[\int_0^1\cdots\int_0^1\left(I_{k_1,n_2,\ldots,n_d}^2(t_2,\ldots,t_d)\right)^{p/2}\,dt_2\ldots dt_d\right]^{2/p}\right\}^{p/2}\\
&\leqslant B_{p,1}^p\left\{\sum_{k_1=1}^{n_1}\left[B_{p,d-1}\left(\sum_{k_2=1}^{n_2}\cdots\sum_{k_d=1}^{n_d}a_{k_1,k_2,\ldots,k_d}^2\right)^{1/2}\right]^2\right\}^{p/2}
\end{aligned}$$

$$= \quad (B_{p,d}\, s_{\mathbf{n}})^p\,,$$

where we used the triangle inequality in $L_{p/2}$ and (2.2) for $d-1$. So we proved the second part of (2.2) for $p \geq 2$.

As $\|S_{\mathbf{n}}\|_p \leqslant \|S_{\mathbf{n}}\|_2$ for $0 < p \leqslant 2$, the second part of (2.2) is true for $0 < p$.

Now turn to the first part of (2.2). We see that $s_{\mathbf{n}} = \|S_{\mathbf{n}}\|_2 \leqslant \|S_{\mathbf{n}}\|_p$ for $p \geq 2$. Therefore it is enough to prove the inequality for $0 < p < 2$. We follow the lines of [2], p. 367.

Let $0 < p < 2$. Choose $r_1, r_2 > 0$, $r_1 + r_2 = 1$, $pr_1 + 4r_2 = 2$. By Holder's inequality and the second part of (2.2), we have

$$s_{\mathbf{n}}^2 = \|S_{\mathbf{n}}\|_2^2 \leqslant \|S_{\mathbf{n}}\|_p^{pr_1} \|S_{\mathbf{n}}\|_4^{4r_2} \leqslant \|S_{\mathbf{n}}\|_p^{pr_1} B s_{\mathbf{n}}^{4r_2}.$$

From here

$$\|S_{\mathbf{n}}\|_p^{pr_1} \geqslant (1/B) s_{\mathbf{n}}^{2-4r_2} = (1/B) s_{\mathbf{n}}^{pr_1}.$$

Therefore the first part of (2.2) is true for $0 < p < 2$. □

3. Burkholder's inequality

Let $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale difference. It means that $\mathcal{F}_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, is an increasing sequence of σ-algebras, i.e. $\mathcal{F}_{\mathbf{k}} \subseteq \mathcal{F}_{\mathbf{n}}$ if $\mathbf{k} \leqslant \mathbf{n}$; $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and integrable; $\mathbb{E}(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{k}}) = 0$ if $\mathbf{k} < \mathbf{n}$.

To obtain Burkholder's inequality, we shall assume the so called condition (F4). I. e.

$$\mathbb{E}\left\{\mathbb{E}(\eta|\mathcal{F}_{\mathbf{m}})|\mathcal{F}_{\mathbf{n}}\right\} = \mathbb{E}\left\{\eta|\mathcal{F}_{\min\{\mathbf{m},\mathbf{n}\}}\right\} \tag{3.1}$$

for each integrable random variable η and for each $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$ (see, e.g., [6] and [3]).

Denote by $(Z_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, the martingale corresponding to the difference $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$. More precisely, let $Z_{\mathbf{n}} = 0$ and $\mathcal{F}_{\mathbf{n}} = \{\emptyset, \Omega\}$ if $\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{N}^d$ and $Z_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{N}^d$.

Theorem 3.1. *Let $(Z_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale and $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let $p > 1$. There exist finite and positive constants $C_{p,d}$ and $D_{p,d}$ depending only on p and d such that*

$$C_{p,d} \left[\mathbb{E}\left(\sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}\right]^{1/p} \leqslant \left(\mathbb{E}|Z_{\mathbf{n}}|^p\right)^{1/p} \leqslant D_{p,d} \left[\mathbb{E}\left(\sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}\right]^{1/p}. \tag{3.2}$$

Proof. We follow the lines of [8]. Let $u_{i,n_i} \in \{0, 1\}$, $n_i = 1, 2, \ldots$, $i = 1, 2, \ldots, d$. Let

$$T_{\mathbf{n}} = \sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} u_{1,k_1} \cdots u_{d,k_d} X_{\mathbf{k}} = \sum\nolimits_{k_1=1}^{n_1} u_{1,k_1} Y_{k_1},$$

where

$$Y_{k_1} = Y_{k_1,n_2,\dots,n_d} = \sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} X_{k_1,k_2,\dots,k_d}\,.$$

First we show that

$$\mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d\, \mathbb{E}|T_{\mathbf{n}}|^p. \tag{3.3}$$

We use induction. For $d = 1$ (3.3) is included in [1], p. 1502 (because T_n is a transform of the martingale Z_n and vice versa). Now we assume that (3.3) is true for $d-1$. Let $n_2, \dots, n_d$ be fixed, $\mathcal{F}_{k_1} = \mathcal{F}_{k_1,n_2,\dots,n_d}$. Then, using (3.1), we can show that $(Y_{k_1}, \mathcal{F}_{k_1})$, $k_1 = 1, 2, \dots,$ is a martingale difference. As the martingale $\sum_{k_1=1}^{n_1} Y_{k_1} = \sum_{k_1=1}^{n_1} u_{1,k_1}(u_{1,k_1} Y_{k_1})$ is a transform of the martingale $\sum_{k_1=1}^{n_1}(u_{1,k_1} Y_{k_1})$, by [1], p. 1502,

$$\mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} Y_{k_1}\right|^p \leqslant M_1 \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} (u_{1,k_1} Y_{k_1})\right|^p. \tag{3.4}$$

Now, using (3.1), we can show that for any fixed n_1 the $(d-1)$-index sequence $\left\{\sum_{k_1=1}^{n_1} X_{k_1,k_2,\dots,k_d}, \mathcal{F}_{n_1,k_2,\dots,k_d}\right\}$, $(k_2, \dots, k_d) \in \mathbb{N}^{d-1}$, is a martingale difference. Therefore, using (3.3) for $d-1$, we obtain

$$\begin{aligned}
\mathbb{E}|Z_{\mathbf{n}}|^p &= \mathbb{E}\left|\sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} \left[\sum\nolimits_{k_1=1}^{n_1} X_{k_1,\dots,k_d}\right]\right|^p \leqslant \\
&\leqslant M_{d-1}\, \mathbb{E}\left|\sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} \left[\sum\nolimits_{k_1=1}^{n_1} X_{k_1,\dots,k_d}\right]\right|^p = \\
&= M_{d-1}\, \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} Y_{k_1}\right|^p \leqslant M_{d-1}\, M_1\, \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} (u_{1,k_1} Y_{k_1})\right|^p = \qquad (3.5)\\
&= M_d\, \mathbb{E}|T_{\mathbf{n}}|^p.
\end{aligned}$$

In (3.5) we applied (3.4). So we proved (3.3).

Because $Z_{\mathbf{n}}$ and $T_{\mathbf{n}}$ are each other's transforms, (3.3) implies

$$N_d\, \mathbb{E}|T_{\mathbf{n}}|^p \leqslant \mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d\, \mathbb{E}|T_{\mathbf{n}}|^p. \tag{3.6}$$

Now we prove the first part of (3.2). By (2.1),

$$\begin{aligned}
\mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2} &\leqslant \frac{1}{A_{p,d}^p} \mathbb{E}\left[\int_0^1 \cdots \int_0^1 \left|\sum_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_{\mathbf{k}}\right|^p dt_1 \dots dt_d\right] \\
&= \frac{1}{A_{p,d}^p} \int_0^1 \cdots \int_0^1 \mathbb{E}\left[\left|\sum_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_{\mathbf{k}}\right|^p\right] dt_1 \dots dt_d \\
&\leqslant \frac{1}{A_{p,d}^p} \int_0^1 \cdots \int_0^1 \frac{1}{N_d} \mathbb{E}\left[\left|\sum_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}\right|^p\right] dt_1 \dots dt_d \\
&= \frac{1}{A_{p,d}^p} \frac{1}{N_d} \mathbb{E}|Z_{\mathbf{n}}|^p.
\end{aligned}$$

In the third step we applied (3.6).

We turn to the second part of (3.2). By (3.6),

$$\mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d\,\mathbb{E}\left[\left|\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1)\cdots\varepsilon_{k_d}(t_d)X_{\mathbf{k}}\right|^p\right].$$

From here, using (2.1),

$$\begin{aligned}\mathbb{E}|Z_{\mathbf{n}}|^p &\leqslant M_d \int_0^1\cdots\int_0^1 \mathbb{E}\left[\left|\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1)\cdots\varepsilon_{k_d}(t_d)X_{\mathbf{k}}\right|^p\right]\,dt_1\dots dt_d\\ &\leqslant M_d\,B_{p,d}^p\,\mathbb{E}\left(\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}.\end{aligned}$$

The proof is complete. □

4. Final comments

Burkholder's inequality is valid for martingales with values in $\mathbb{R}^t$ (t is a fixed positive integer). For $p>0$ and $\mathbf{x}=(x_1,\dots,x_t)\in\mathbb{R}^t$ let $\|\mathbf{x}\|_p=\left(\sum_{i=1}^t |x_i|^p\right)^{1/p}$.

Let $(\mathbf{X_n},\mathcal{F}_\mathbf{n})$, $\mathbf{n}\in\mathbb{N}^d$, be a martingale difference with values in $\mathbb{R}^t$. Assume that condition (F4) is satisfied. Let $(\mathbf{Z_n},\mathcal{F}_\mathbf{n})$, $\mathbf{n}\in\mathbb{N}^d$, be the martingale corresponding to the difference $(\mathbf{X_n},\mathcal{F}_\mathbf{n})$, $\mathbf{n}\in\mathbb{N}^d$.

Theorem 4.1. *Let $(\mathbf{Z_n},\mathcal{F}_\mathbf{n})$, $\mathbf{n}\in\mathbb{N}^d$, be a martingale with values in $\mathbb{R}^t$ and $(\mathbf{X_n},\mathcal{F}_\mathbf{n})$, $\mathbf{n}\in\mathbb{N}^d$, the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let $p>1$. There exist finite and positive constants C and D depending only on t, p and d such that*

$$C\left[\mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}}\|\mathbf{X_k}\|_2^2\right)^{p/2}\right]^{1/p} \leqslant \left(\mathbb{E}\|\mathbf{Z_n}\|_2^p\right)^{1/p} \leqslant D\left[\mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}}\|\mathbf{X_k}\|_2^2\right)^{p/2}\right]^{1/p}. \quad (4.1)$$

Proof. It is known that for any $p,q>0$ there exist $0<c,d<\infty$ such that $c\|\mathbf{x}\|_p\leqslant\|\mathbf{x}\|_q\leqslant d\|\mathbf{x}\|_p$ for all $\mathbf{x}\in\mathbb{R}^t$. Applying this observation and (3.2) we obtain (4.1). □

Using this theorem we can prove limit theorems for autoregressive type martingale fields. For details see [5] and [4] including the d-index case and the two-index case, respectively.

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