

Remarks on the concepts of affine transformation and collineation in teaching geometry in teachers' training college

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Abstract

In [11] and [12] (textbooks for teachers' training colleges written by B. Pelle) isometry and similarity are defined not in the classical way but as a product. We continue this way of definition refer to the affine transformation and collineation, and study the fundamental theorems of these mappings analogously to those of isometry and similarity.

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1. Introduction

In the last textbooks on geometry for teachers' training colleges, in [11] and in [12], (plane-) isometry is defined as a product of reflections in line, similarity is defined as a product of isometry and central dilatation, instead of the classical way which is based on the properties of distance- and ratio-preserving. We call this way of definition "constructive" because it provides technique to give the mapping. However, this way doesn't proceed in [11] for studying affine transformations and collineations. (These subjects are not involved in [12].)

As it is known, in the classical treatment in the Euclidean plane-geometry affine transformation is a line-preserving transformation of the plane, and axial affine transformation is defined as an affine transformation with an axis. In the classical projective plane-geometry collineation is defined as a line preserving transformation of the plane, and central (axial) collineation is a collineation with a center (or an axis).

In [11] the concept of the affine transformation is approached from projective geometry. At first, the concept of collineation is defined in the classical way ([11] p. 329.). Affine transformation is a collineation, which leaves the line at infinity

(the ideal line) invariant ([11] p. 332.). Axial affine transformation is a central collineation, whose center is a point at infinity (an ideal point), and the axis is an ordinary line ([11] p. 338.).

In our opinion building a uniform system is a very important didactical principle especially in teacher training. Therefore in our present paper we'll apply the "constructive" way – only in the plane geometry – for these topics, too. (In [9] we wrote our remarks on studying similarities in the constructive way.) We'll follow the reversed way for both cases mentioned above. At first, we'll define the axial affine transformation in the Euclidean plane in a metric way. Then we'll build the concept of the affine transformation on the concept of axial affine transformation and similarity. We'll deal with the concept of collineation in a similar way. So in all cases at first we define a special mapping, and then we get the general type by using this special one and the previous mapping. This way fulfills another important didactical principle, too, namely: progressing from the special case towards the general one. Beside providing a uniform definition, our further aim is to study the fundamental theorems related to the new concepts analogously to the studying of isometries and similarities, in order to form a uniform and consistent system in the "constructive" way. We'll point out several analogies which also strengthen unity and understanding. Meanwhile, we'll touch upon the connection between the classical and "constructive" ways, too.

2. Preliminaries

In this paper by transformation we mean a bijective mapping of the plane onto itself. Two transformations are said to be equal, if they transform any point into the same point. By line-preserving mapping we mean a mapping, which transforms collinear points into collinear points. If a point coincides with its image under a mapping, then we call it fixed point. If a straight line is fixed pointwise by a mapping, then we call it axis. If a line coincides with its image under a mapping, then we call it invariant line. By center of a mapping we mean a point, through which every line passing is invariant. By plane-flag we mean the union of a halfplane and a ray on its boundary. If three lines have common point or they are parallel to each other, then we call them concurrent lines. We use directed line segments; we define the operations related to them in the usual way. We use the concepts of the affine- and the cross ratio on the Euclidean plane almost totally in the usual metric sense. We study their elementary properties and the Pappos-Steiner theorem also in the classical way (e.g. in [4]). The only difference is that by the affine ratio (ABC) of the collinear points A , B and C we mean the ratio $\frac{AC}{BC}$ as in [7]. This definition is involved in [11] (p. 141), too, but there the ratio is negative, iff C does not separate A and B , on the contrary as in our case. After the usual introduction of the ideal elements of the Euclidean plane, we extend the concepts of the affine- and cross ratio to them preserving the previous properties. (For example: if P is an ideal point then $(ABP) := 1$, $(ABCP) := (ABC)$, $(PABC) := (CBA)$, etc.)

In [11] and [12] the concept of isometry is based on the axioms of Reflection

related to the primitive concept of “reflection in plane”. We’ll make comparison with the equivalents of these axioms related to the concept of reflection in line, therefore we list these theorems (R1-R5). (The theorems which we use are a little bit different from the statements involved in [11] and [12] ([11] pp. 21-22, 25-26; [12] pp. 17-18, 22-23). In [8] we examined the connection between the two ways.)

R1: Any reflection in line is a line-preserving, involutory transformation of the plane with an axis, and the axis separates every other $P - P'$ pair.

R2: For any line there is a unique reflection, whose axis is the given line.

R3: For any two points there is a unique reflection, in which they are corresponding points.

R4: For any two rays, starting from the same point, there is a unique reflection, which transforms the given rays into each other.

R5: If two products of reflections transform a plane-flag into the same one, then the products are equal.

3. Definitions

To construct the concept of the axial affine transformation and the central-axial collineation we connect to the concept of the central dilatation; and to construct the concept of the affine transformation and the collineation we connect to the concept of the similarity. The analogous definition of these concepts can help the understanding of the new concept, it makes them imaginable and contributes to the developing of a uniform system. It is a further advantage that the proofs of the corresponding properties can be done similarly; it also strengthens unity. It is an evident disadvantage of this way of definition that it is more complicated than the classical one.

Classically central dilatation is defined by giving its center and ratio. One can say that in the definition of the central dilatation we give the way how the position of a point changes in comparison with a fixed point. In the case of the axial affine transformation we compare it with an axis. In our opinion, if we want to make this new concept analogously then it fits best the previous definition of the central dilatation if we define the axial affine transformation by giving its axis, direction and ratio. This way occurs in the secondary school, too, only as supplementary subject, and only with orthogonal direction ([3]).

Definition 3.1.a. By general axial affine transformation we mean the following mapping on the Euclidean plane. Suppose that there are two intersecting lines t and e , and a $\lambda (\neq 0)$ constant. The image of the point P is those P' for which $P_t P' = \lambda P_t P$, where P_t denotes the point on t for which $e \parallel (P_t P)$.

The metrical condition related to P' can be written also in the form of $(P' P P_t) = \lambda (P \notin t)$. In the case of the central dilatation the analogous formulae of this equation is the next one: $(P' P C) = \lambda (P \neq C)$.

Definition 3.1.b. By special axial affine transformation we mean the following mapping on the Euclidean plane. Suppose that there are two parallel lines t and

e , e is directed, a $\lambda (> 0)$ constant, and a halfplane bounded by t is indicated. The image of the point P is those P' for which $d(P, P') = \lambda d(t, P)$, and the PP' segment is either similarly or oppositely directed as e , depending on P whether it is in the indicated halfplane or not.

We use the terms “general” and “special” according to [7]. (For the special axial affine transformation in [1], [5], [10], [13] there is the term “shear”. If $e \perp t$, we call it orthogonal axial affine transformation (e.g. [4]), or “strain” (e.g. [10]).)

If we extend the definitions of axial affine collineation, central dilatation and translation to the ideal elements by using the line-preserving property as usual, we can emphasize here the fact that each of them has both axis and center. Furthermore, the next statements are valid: any line-preserving transformation of the Euclidean plane which transforms any line into a parallel one, is either a central dilatation or a translation (e.g. in [1]); any line-preserving transformation of the Euclidean plane which transforms any point P so that the lines (PP') are parallel to each other, is either an axial affine transformation or a translation (e.g. in [13]). They mean that there isn't any other line-preserving transformation that has ideal center or axis. The motivation for making the concept of central-axial collineation is the following: construct a nonidentical line-preserving mapping that has both ordinary center and axis. If we want to preserve the unity, to make the definition in the manner of those of the central dilatation and axial affine transformation, the it is the most natural way if we define the new concept by giving the center, the axis and the ratio, and try to “join” the methods of the previous definitions.

The first attempt is the following: there is a line t , a point C , $C \notin t$, and a $\lambda (\neq 0)$ constant. The image of the point $P (\neq C, \notin t)$ is those P' for which either $\frac{CP'}{P_tP'} = \lambda \frac{CP}{P_tP}$ or $CP' = \lambda CP$, depending on (CP) whether it intersects t at point P_t or it is parallel to t .

It is not “perfect”, because the affine ratio never equals 1 on the Euclidean plane, so there are points without image-point, and there are points without origin-point. Namely, P has not image iff $(CP_tP) = \frac{1}{\lambda}$, and has not origin iff $(CP_tP) = \lambda$. It is obvious that in both cases the locus of these points is a line parallel to the axis. According to [11], we call them the “line of disappearing” and the “line of directions”, respectively. This problem is eliminated by using the Euclidean plane extended by ideal elements. According to the above mentioned theorems on the mappings that have ideal center or axis, we can introduce the concept of central-axial collineation by defining separately only the mapping that has both ordinary center and axis, and if one of them is an ideal one, we call the central-axial collineation related to them the appropriate previous mapping. If the axis doesn't contain the center we don't need this way because a unified definition can be given by using the remark after Definition 3.1.a. Unfortunately, if the center is on the axis we can't construct the mapping in such an elegant way.

Definition 3.2.a. By general central-axial collineation we mean the following mapping on the extended Euclidean plane. Suppose that there is a line t , a point C , $C \notin t$, and a $\lambda (\neq 0)$ constant. (P_t will denote the point on t for which $C \in (P_tP)$.) The image of the point $P (\neq C, \notin t)$ is those P' on t for which $(P'PCP_t) = \lambda$. If

$P = C$ or $P \in t$, then $P' = P$. (This cross ratio is called the “characteristical cross ratio” ([6]).) If t is the ideal line we get the central dilatation with center C and ratio λ ; if C is an ideal point we get the general axial affine transformation with axis t , ratio $\frac{1}{\lambda}$ and direction C .

Definition 3.2.b. Suppose that there is a line t and a point C on the extended Euclidean plane, $C \in t$. If C and t are both ideal elements by the special central-axial collineations with center C and axis t we mean the translations with direction C . If C is an ideal point and t is an ordinary line by the special central-axial collineations with center C and axis t we mean the special axial affine transformations with axis t . If C and t are both ordinary elements suppose that there is a λ segment, and an (euclidean) halfplane bounded by t is indicated. In this case by special central-axial collineation we mean the following mapping. (T_P will denote the foot of the perpendicular from the point P to t .) The image of the ordinary point $P (\notin t)$ is those P' for which (CPP') equals either $\frac{\lambda}{PT_P}$ or $-\frac{\lambda}{PT_P}$, depending on P whether it is in the indicated halfplane or not. The image of the ideal point $P (\notin t)$ is those P' in the not-indicated halfplane for which $P' \in (CP)$ and $P'T_{P'} = \lambda$. If $P \in t$, then $P' = P$. (The λ and PT_P segments are not directed. We use the terms “general” and “special” again according to [7]. (In [1] for the general case there is the term “homology” and for the special case “elation”).

In the general case the above-mentioned line of disappearing is the origin of the ideal line, and the line of directions is the image of it. In the special case these lines also exist: P has ideal image (origin) iff $PT_P = \lambda$ and P is in the indicated (not-indicated) halfplane.

The following definitions are the analogues of those of isometry and similarity involved in [11], [12].

Definition 3.3. By affine transformation we mean a finite product of axial affine transformations and similarities on the Euclidean plane.

Definition 3.4. By collineation we mean a finite product of central-axial collineations on the extended Euclidean plane.

4. Properties of the axial affine transformation and the central-axial collineation

First we mention properties related to the ratio. Both general axial affine transformation and general central-axial collineation of ratio 1 are the identity, just as the central dilatation with ratio 1. In the general cases we get the inverse mapping simply by changing the ratio to $\frac{1}{\lambda}$, just as in the case of the central dilatation. (In the special cases we have to interchange the roles of the indicated and not-indicated halfplanes.) The general axial affine transformation with ratio -1 and with direction orthogonal to the axis is a reflection in line, just as the central dilatation of ratio -1 is a reflection in point.

To emphasize other analogies with the reflection in line and the central dilatation, we list some other properties of the axial affine transformation (I-IV) and the central-axial collineation (I*-IV*).

I. Any axial affine transformation is a line-preserving transformation of the Euclidean plane with an axis; this line separates every other $P - P'$ pair iff $\lambda < 0$.

II. For any lines t, e and constant $\lambda (\neq 0)$, (if $e \parallel t$, then e is directed, $\lambda > 0$ and a halfplane bounded by t is indicated), there is a unique axial affine transformation with axis t , direction e and ratio λ .

III. For any line t and pair of points $P - P'$ which are off t , there is a unique axial affine transformation with axis t , under which the image of P is P' .

IV. For any lines t, e and pair of lines $a - a'$ which are not parallel to e and differ from t but concurrent with it, there is a unique axial affine transformation with axis t , direction e , under which the image of a is a' .

I*. Any central-axial collineation is a line-preserving transformation of the extended Euclidean plane with center and axis; this point and line separate every other $P - P'$ pair iff $\lambda < 0$.

II*. For any point C , any line t , $C \notin t$ and any constant $\lambda (\neq 0)$, there is a unique central-axial collineation with center C , axis t and ratio λ . (If $C \in t$ then it is quite difficult to formulate this property.)

III*. For any point C , any line t and any pair of points $P - P'$, which are off t and differ from C but collinear with it, there is a unique central-axial collineation with center C , axis t , under which the image of P is P' .

IV*. For any point C , any line t and any pair of lines $a - a'$ which are off t and differ from t but concurrent with it, there is a unique central-axial collineation with center C , axis t , under which the image of a is a' .

These properties are just the analogues of the theorems R1-R4 on reflection in line and the corresponding properties of the central dilatation. The analogue of R5 will occur later in Theorems 5.1, 5.3. Statements I and I* contain the most important (non metric) properties of the mappings. The second, third and fourth statements provide techniques to give the mapping. (The only nontrivial proof is that of the line-preserving property: it can be done by the theorems of parallel secants and by the Pappos-Steiner theorem.) Also by using the mentioned theorems we get that the axial affine transformation preserves the affine ratio and the central-axial collineation preserves the cross ratio.

Besides the analogies mentioned, there are further view-points to compare the central dilatation, the axial affine transformation and the central-axial collineation. The first comparison is connected to the properties listed in statements I and in I*. We declared them the most important ones because they determine the mappings. The next two theorems are the analogues of the statement that any line-preserving transformation of the Euclidean plane with a center, is a central dilatation.

Theorem 4.1. *Any line-preserving transformation of the Euclidean plane with an axis, is an axial affine transformation.*

Theorem 4.2. *Any line-preserving transformation of the extended Euclidean plane with a center and an axis, is a central-axial collineation.*

To prove these theorems we can use classical ways. In the case of Theorem 4.1 we get that for the (nonidentical) mapping the lines (PP') are invariant and parallel to each other, and either $(P'PP_t)$ or $\frac{d(P,P')}{d(t,P)}$ is constant, depending on the lines (PP') whether they intersect the axis or not. In the case of Theorem 4.2 due to the previous results we have to prove only in the case of ordinary center and axis. We get that for the (nonidentical) mapping either $(P'PCP_t)$ or the segment $(CPP')PT_P$ is constant depending on the center whether it is off or on the axis. Theorem 4.1 implies that the Definition 3.1 of the axial affine transformation is equivalent to the classical one.

Before the Definition 3.2 we have already mentioned the center and axis of the mappings. According to the emerged conjecture based on the examples above, this is the right time in this treatment to touch upon the fact that the existence of center and axis are inseparable: if a mapping is a line-preserving transformation of the extended Euclidean plane with a center, then it has an axis, too, and conversely. (In the classical projective geometry this theorem is usually deduced from Desargues's theorem (e.g. in [7], [11]); other, direct type of proof can be found e.g. in [2], [13]. In this treatment we apply the latter one.) By applying the results, it follows:

Theorem 4.3. *Any line-preserving transformation of the extended Euclidean plane with a center or an axis, is a central-axial collineation.*

From this theorem we also get the equivalence of Definition 3.2 and the classical one for the central-axial collineation.

5. Fundamental theorems of affine transformations and collineations

The basic properties of the affine transformation and the collineation follow directly from the definitions, as the common properties of the factors of the product (line-preserving transformation of the Euclidean plane, preserves the order and the affine ratio; line-preserving transformation of the extended Euclidean plane, preserves the separatedness and the cross ratio). We also immediately get that the affine transformations – and the collineations as well – form a group. The first group contains the group of the similarities as a subgroup, and the second one similarly contains that of the affine transformations.

Theorem 5.1. *Let us consider on the Euclidean plane two flags, a point on each ray and halfplane. There exists a unique affine transformation, which transforms the first flag to the other one, the points on the first flag to the correspondent points on the other one.*

In the classical treatment the equivalent theorem of Theorem 5.1 is the one, which states that any two triangles are related by a unique affine transformation. (In [11] there is not a theorem like this.) We use Theorem 5.1 instead of this

theorem, because this is the analogue of theorem R5 and the fundamental theorems of (plane-) isometries and similarities: there exists a unique isometry (similarity), which transforms a given flag to an other given one (and a given point on the first ray to a given point on the other one).

Proof of Theorem 5.1. The flags are denoted by Z and V , the points on their rays are P and Q , the points on their halfplanes are R and S , respectively. First let us consider the similarity \mathbf{H} , which transforms Z to V and P to Q . Then we consider the axial affine transformation whose axis is the line of the ray of V , which transforms $\mathbf{H}(R)$ to S . The product of these transformations has the desired properties. If there is another affine transformation, then it is equal to the first product, due to the line- and affine ratio preserving properties. \square

From the construction involved in the previous proof we get the next theorem.

Theorem 5.2. *Any affine transformation can be obtained as a product of a similarity and an axial affine transformation.*

This is the analogue of the “obtaining” theorems for plane-similarities (isometries): any similarity (isometry) can be obtained as a product of an isometry and a central dilatation (at most three reflections in line).

In the sequel we’ll draw up the analogues of Theorems 5.1 and 5.2 for collineations. At first, we have to generalize the concepts of ray, halfplane and flag. On the extended Euclidean plane any two points divide the line consisting them into two “segments”; we’ll use the term “ray” for the union of the interior of a segment and one of its boundary-points. (We call this point as “starting point”, and the other boundary point as “end-point”.) Any two lines divide the plane into two parts; we’ll use the term “halfplane” for the interior of these parts. On the extended Euclidean plane by “flag” we mean the union of a “halfplane” and a “ray” on its boundary, whose end-point is the point of intersection of the boundary-lines of the “halfplane”. (Fig. 1. shows some flag-types.) It is clear that the image of any flag under a collineation is again a flag.

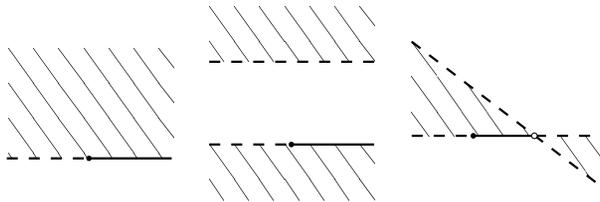


Figure 1.

Theorem 5.3. *Let us consider on the extended Euclidean plane two flags, a point on each ray and halfplane. There exists a unique collineation, which transforms the*

first flag to the other one, the points on the first flag to the correspondent points on the other one.

This is the analogue of theorem R5 and Theorem 5.1. However, the proof will not be the analogue of that of Theorem 5.1.

Proof of Theorem 5.3. First let us transform each flag by such a central-axial collineations, whose “line of disappearing” is that boundary-line of the flag which doesn’t contain its ray. Now, the mentioned boundary-line of the image-flags is the ideal line, hence there is an affine transformation which transforms the first new flag to the second one, and the points on the first one to the correspondent points on the second one. So the original flags and the points are related by a product of two central-axial collineations and an affine transformation. If there is another collineation, then it is equal to the first product, due to the line- and cross ratio preserving properties. \square

In the classical treatment its equivalent theorem is this one: any two “quadrangles” on the projective plane are related by a unique collineation. It is obvious from Fig. 2. that the quadrangles $APMK$ and $BQNL$ determine the same collineation as the given flags and points, and conversely.

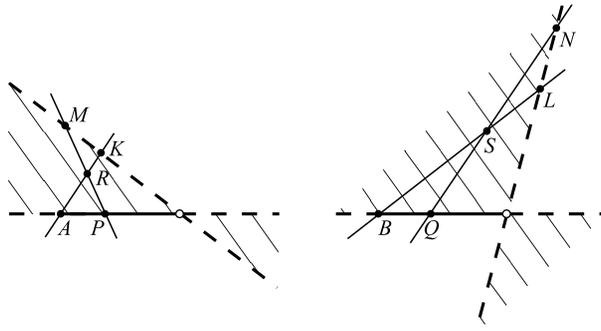


Figure 2.

The next theorem is the analogue of Theorem 5.2. It is not the direct corollary of Theorem 5.3 as Theorem 5.2 was that of Theorem 5.1, here we need a little bit different construction. This will be the proof, which is the analogue of that of Theorem 5.1.

Theorem 5.4. Any collineation can be obtained as a product of an affine transformation and a central-axial collineation.

Proof of Theorem 5.4. (Fig. 3.) Let us consider a collineation K , the line f which is the image of the ideal line under K , a flag V whose boundary-lines are

f and an ordinary line which contains the ray and intersects f in an ideal point. Also consider the flag Z which is the origin of V under \mathbf{K} , the points P and R on the ray and halfplane of Z , finally the points Q and S which are the image-points of P and R under \mathbf{K} . It is easy to see that all these points and the starting points of the rays are ordinary points. (f may be either the ideal line or an ordinary one.) Now let us consider the affine transformation which transforms the ray of Z to that of V , P to Q and R to S . Then we consider the central-axial collineation with center S , whose axis is the line of the ray of V , which transforms the ideal line to f . According to Theorem 5.3 the product of these two transformations is equal to \mathbf{K} .

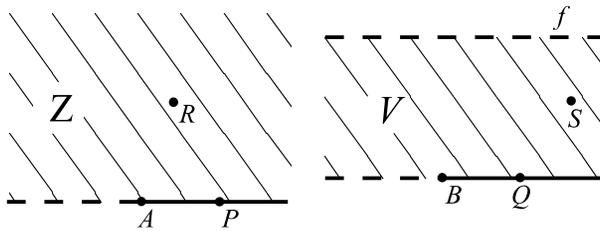


Figure 3.

Now we make a little change on the previous construction. First let us consider a similarity \mathbf{H} , which transforms the ray of Z to that of V and P to Q . Then we consider the central-axial collineation with axis (BQ) which transforms $\mathbf{H}(R)$ to S and the ideal line to f . This concludes that any collineation can be obtained as a product of a similarity and a central-axial collineation. \square

Finally there follow the analogues of Theorems 4.1 and 4.4.

Theorem 5.5. *Any line-preserving transformation of the Euclidean plane is an affine transformation.*

Theorem 5.6. *Any line-preserving transformation of the extended Euclidean plane is a collineation.*

Proving the Theorem 5.5 at first we get in the classical way that the mapping preserves the affine ratio. Then, by using the method of the proof of Theorem 5.1, we get that the transformation is a product of a similarity and an axial affine transformation. The proof of Theorem 5.6 has two cases. If the ideal line is invariant, then according to Theorem 5.5 the given transformation is an affine transformation. Otherwise let us consider the product of the given transformation and a central-axial collineation which transforms the image of the ideal line under the given transformation to the ideal line. According to the first case this product is an affine transformation. Thus the given transformation is a collineation. These

theorems imply the equivalence of Definitions 3.2, 3.4 and the classical ones, too. Moreover, in our treatment they mean that it is impossible to widen the set of affine transformations and collineations if we demand the properties mentioned in the theorems.

6. Closing remarks

We think it is important to remark here the following: we don't state that the constructive way is better for studying these mappings than the classical one. The constructive way emphasizes other didactical principles; this has advantageous and disadvantageous consequences, too:

1. The constructive way – as we have mentioned in the Introduction – always progresses from the special case towards the general one. Hence, naturally, the treatment becomes more lengthy, less “economical”, and the main invariants of the mappings appear later, not immediately in the definitions.

2. The constructive giving of a mapping is in accordance with the most frequent way of giving of the functions: it gives the domain and the rule of the correspondence. Therefore, the definitions are more concrete but also more complicated than in the classical way, where we define classes of functions having certain properties.

3. In order to form a uniform system, we define the new mappings on metrical basis so as to connect strongly with the way of definition of the previous mappings. This is the most natural way for the students, because the metrical thinking about the geometrical mappings is very strong. However, this way may lead to the false idea, that these concepts – and even the whole geometry – can be studied only on metrical basis. It could help if we mention the connection between the classical and constructive ways after the appropriate theorems (4.1, 4.3, 5.5, 5.6) because the classical way loosens the connection to metrical concepts. The study of the possibility of approaching a domain from an other point of view is also an important principle in teacher training. Another kind of problem arises in connection with the collineations. We defined this mapping on the Euclidean plane extended by ideal elements, which is often called as “projective plane”, because from the point of view of incidence, order and continuity it is a model of the “real (classical) projective plane”. However, during our considerations we always made distinction between the “ideal” and “ordinary” elements, moreover we used metrical concepts based on the Euclidean Axioms of Reflection. On the real projective plane there aren't either indicated elements or euclidean based metric. That's why we didn't use the term “projective plane”. If the students learn about the axiomatical real projective plane during their further studies, the usage of the term “projective plane” in different ways may cause confusion. The problem would be partly solved, if we defined the central-axial collineation only on the Euclidean plane except the line of disappearing. So we could stay within the frames of the Euclidean geometry. But this change would make the treatment very complicated. Another solution could be given on the basis of the real projective geometry: at first, we study the projective plane and then we make a model for the Euclidean plane (eg.: [2],

[7]). Clearly, this way can't help us either because it would be a higher course in geometry.

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