

Some special cases of a general convergence rate theorem in the law of large numbers

Tibor Tómacs

Department of Applied Mathematics, Eszterházy Károly College
e-mail: tomacs@ektf.hu

Abstract

Tómacs in [6] proved a general convergence rate theorem in the law of large numbers for arrays of Banach space valued random elements. We shall study this theorem in case Banach space of type φ and for two special arrays.

Key Words: Convergence rates; Arrays of Banach space valued random variables; Banach space of type φ

1. Introduction and notation

Let \mathbb{N} be the set of the positive integers and \mathbb{R} the set of real numbers. Let Φ_0 denote the set of functions $f: [0, \infty) \rightarrow [0, \infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the Δ_2 -condition ($f \sim \Delta_2$) if there exists a constant $c > 0$ such that $f(2t) \leq cf(t)$ for all $t > 0$.

Let B be a real separable Banach space with norm $\|\cdot\|$ and zero element $\mathbf{0}$. If X is a B -valued random variable (r.v.) and $\mathbf{E}\|X\| < \infty$ then $\mathbf{E}X$ stands for the Bochner integral of X .

Throughout the paper let $\{k_n, n \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of B -valued r.v.'s. It is rowwise independent, if X_{n1}, \dots, X_{nk_n} are independent r.v.'s for any fixed $n \in \mathbb{N}$. Let $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$. If $k_n = n$ for all n , then we denote S_{k_n} by S_n . This corresponds to the case of ordinary sequences.

The array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is said to be *bounded in probability* if for all $\varepsilon > 0$ there exists $A > 0$ such that $\mathbf{P}(\|X_{nk}\| \geq A) < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1, \dots, k_n$.

The following remark give a sufficient condition for the boundedness in probability.

Remark 1.1. If there exists a constant $M > 0$ such that $\mathbf{E} \|X_{nk}\| \leq M$ for every $n \in \mathbb{N}, k = 1, \dots, k_n$, then the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is bounded in probability. (The reader can readily verify this statement.)

Definition 1.2 (Gut [2]). We say that the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is *weakly mean dominated* (w.m.d.) by the r.v. X , if for some $\gamma > 0$,

$$\frac{1}{k_n} \sum_{k=1}^{k_n} \mathbf{P}(\|X_{nk}\| > t) \leq \gamma \mathbf{P}(|X| > t) \quad \text{for all } t \geq 0 \quad \text{and } n \in \mathbb{N}.$$

The following theorem a general convergence rate theorem, which is proved in [6].

Theorem 1.3 (Tórnács [6], Theorem 3.1). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Assume that there exists a sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ is bounded in probability. Let $\alpha, \vartheta, \varphi \in \Phi_0$, and assume that α is not bounded, $\vartheta, \varphi \sim \Delta_2$, $\vartheta \not\equiv 0$. Let $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$, $n = 0, 1, 2, \dots$. Assume that $\mathbf{E} \varphi(|X|) < \infty$, $\mathbf{E} \vartheta(|X|) < \infty$ and $\lim_{n \rightarrow \infty} \alpha(n)/\gamma_n = \infty$.*

Let either $\mu(n) = \beta(n-1)$ for all $n \in \mathbb{N}$ or $\mu(n) = \beta(n)$ for all $n \in \mathbb{N}$. In second case assume that there exists a constant $c > 0$ such that for $n \in \mathbb{N}$ large enough $c\beta(n) \leq \beta(n-1)$.

Let $n_0 \in \mathbb{N}$ be such that $\vartheta(\alpha(n)) > 0$ for all $n \geq n_0$. If there exist $j \in \mathbb{N}$ and $r > 0$ such that

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))} \right)^{2j} < \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbf{P}(\|S_n\| > \varepsilon \alpha(n)) < \infty \quad \text{for all } \varepsilon > 0.$$

In the following two corollaries of Theorem 1.3 we use some special notations: Following Gut [1], introduce the functions ψ and M_r with

$$\psi(t) = \text{Card}\{n \in \mathbb{N} : k_n \leq t\} \quad \text{for } t \geq 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if } t \geq 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if } 0 \leq t < 1,$$

where $r \in \mathbb{R}$, $\text{Card}A$ is the cardinality of the set A and $[.]$ denotes the integer function. Let $M = M_2$. Let $f \circ g$ be the composite function of functions f and g .

Remark 1.4. $M_r \circ \psi \in \Phi_0$ and

$$(M_r \circ \psi)(t) = M_r(\psi(t)) = \begin{cases} \sum_{i=1}^n k_i^{r-1} = M_r(n), & \text{if } k_n \leq t < k_{n+1}, \\ k_1^{r-1} = M_r(1), & \text{if } 0 \leq t < k_1. \end{cases}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas [5].

Corollary 1.5 (Tórnács [6], Corollary 3.2). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Let $M \circ \psi \sim \Delta_2$, $r, s, t > 0$, $rs > t$. Assume that $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$ is bounded in probability. Furthermore, if $r > 2$ we assume that $\{M(n)/M(n-1), n \in \mathbb{N}\}$ is bounded. If $\mathbf{E}M^{r/2}(\psi(|X|^{t/r})) < \infty$ and $\mathbf{E}|X|^s < \infty$, then*

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1} \mathbf{P} \left(\|S_{k_n}\| > \varepsilon k_n^{r/t} \right) < \infty \quad \text{for all } \varepsilon > 0.$$

The following corollary is a version of Corollary 4.1 of Hu et al. [3].

Corollary 1.6 (Tórnács [6], Corollary 3.3). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Let $r \in \mathbb{R}$, $0 < t < s$ and $M_r \circ \psi \sim \Delta_2$. Assume that $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$ is bounded in probability. If $\mathbf{E}M_r(\psi(|X|^t)) < \infty$ and $\mathbf{E}|X|^s < \infty$, then*

$$\sum_{n=1}^{\infty} k_n^{r-2} \mathbf{P} \left(\|S_{k_n}\| > \varepsilon k_n^{1/t} \right) < \infty \quad \text{for all } \varepsilon > 0.$$

In Section 2 we give a sufficient condition for the boundedness in probability and in Section 3 we study two concrete sequences k_n in Corollary 1.5 and 1.6.

2. The boundedness in probability in case Banach space of type φ

If B has an appropriate geometric property, then a moment condition can imply the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$.

Definition 2.1. A function $\varphi: [0, \infty) \rightarrow [0, \infty)$ is said to be an *Orlicz function* if it is continuous, convex, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. For an Orlicz function φ the *Orlicz space* $l_\varphi(B)$ consists of those B -valued sequences $\{u_n, n \in \mathbb{N}\}$ for which

$$\sum_{n=1}^{\infty} \varphi(\|u_n\|/a) < \infty \quad \text{for some } a > 0.$$

Let $\varepsilon_1, \varepsilon_2, \dots$ be independent r.v.'s with $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = -1) = 1/2$ for all $n \in \mathbb{N}$. B is said to be of *type φ* , if $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges in probability for all $\{u_n, n \in \mathbb{N}\} \in l_\varphi(B)$.

Definition 2.2. An Orlicz function φ is said to satisfy the Δ_2^0 -condition ($\varphi \sim \Delta_2^0$) if there exist constants $c > 0$ and $t_0 > 0$ such that $\varphi(2t) \leq c\varphi(t)$ is satisfied for all $0 \leq t \leq t_0$.

Lemma 2.3. Let φ be an Orlicz function and $\varphi \sim \Delta_2^0$. B is of type φ iff there exists a constant $c > 0$ such that

$$\mathbf{E} \left\| \sum_{k=1}^n X_k \right\| \leq c \mathbf{E} \inf_{y>0} \left\{ \frac{1}{y} \left(1 + \sum_{k=1}^n \varphi(y \|X_k\|) \right) \right\}$$

for all $n \in \mathbb{N}$ and every independent B -valued r.v.' X_1, \dots, X_n with $\mathbf{E}X_k = \mathbf{0}$, $k = 1, \dots, n$.

For the proof see Fazekas [4].

The following lemma is a generalization of Lemma 2.1 of Gut [2] and Lemma 2.7 (b) of Fazekas [5].

Lemma 2.4 (Tórnács [6], Lemma 4.4). Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of B -valued r.v.'s which is w.m.d. by the r.v. X and constant γ . If $\varphi \in \Phi_0$ then

$$\frac{1}{k_n} \sum_{k=1}^{k_n} \mathbf{E} \varphi(\|X_{nk}\|) \leq \max\{1, \gamma\} \mathbf{E} \varphi(|X|).$$

The following theorem show that in Theorem 1.3 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ if B is of type φ .

Theorem 2.5. Let $\varphi \in \Phi_0$ be a submultiplicative Orlicz function, $\varphi \sim \Delta_2^0$ and let B be a space of type φ . Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Assume that the sequence $\{k_n \varphi(1/\gamma_{k_n}), n \in \mathbb{N}\}$ is bounded for some sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers. If $\mathbf{E}X_{nk} = \mathbf{0}$ for every $n \in \mathbb{N}, k = 1, \dots, k_n$ and $\mathbf{E} \varphi(|X|) < \infty$, then $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability.

Proof. By Lemma 2.3 and 2.4 there exists a constant $c > 0$ such that

$$\begin{aligned} \frac{\mathbf{E} \|S_{k_n}\|}{\gamma_{k_n}} &\leq \frac{c}{\gamma_{k_n}} \mathbf{E} \inf_{y>0} \left\{ \frac{1}{y} \left(1 + \sum_{k=1}^{k_n} \varphi(y \|X_{nk}\|) \right) \right\} \\ &\leq c \mathbf{E} \left(1 + \sum_{k=1}^{k_n} \varphi(\|X_{nk}\|/\gamma_{k_n}) \right) \\ &\leq c(1 + \varphi(1/\gamma_{k_n}) \max\{1, \gamma\} k_n \mathbf{E} \varphi(|X|)). \end{aligned}$$

Thus Remark 1.1 implies the statement. □

3. Convergence rate theorems for two concrete sequences k_n

Lemma 3.1. $f \sim \Delta_2$ iff there exist constants $k > 1$ and $c > 0$ such that

$$f(kt) \leq cf(t) \quad \text{for all } t > 0. \quad (3.1)$$

Proof. If $f \sim \Delta_2$ then in case $k = 2$ we get (3.1). Now suppose that there exist constants $k > 1$ and $c > 0$ such that the inequality (3.1) is true for all $t > 0$. Then we can obtain with induction that

$$f(k^n t) \leq c^n f(t) \quad \text{for all } t > 0 \quad \text{and for all } n \in \mathbb{N}.$$

It follows that there exists $n_0 \in \mathbb{N}$ such that

$$f(2t) \leq f(k^{n_0} t) \leq c^{n_0} f(t) \quad \text{for all } t > 0.$$

Thus we get $f \sim \Delta_2$. □

The reader can readily verify the following lemma.

Lemma 3.2. Let $g: [k_1, \infty) \rightarrow \mathbb{R}$ be a nondecreasing function which has the property that $g(k_n) \geq M_r(n)$ for all $n \in \mathbb{N}$. Then $M_r(\psi(x)) \leq g(x)$ for all $x \geq k_1$.

Lemma 3.3. Let $r \in \mathbb{R}$. Assume that there exists strictly increasing sequence $\{a_n, n \in \mathbb{N}\}$ of positive integers and there exist constants $k > 1$, $c > 0$ such that

$$\frac{k_n}{k_{a_n}} \leq \frac{1}{k} \quad \text{and} \quad \frac{M_r(a_n)}{M_r(n-1)} \leq c \quad \text{for all } n \in \mathbb{N}.$$

Then $M_r \circ \psi \sim \Delta_2$.

Proof. Assume that $k_n \leq t < k_{n+1}$. Then Remark 1.4 implies

$$M_r(\psi(kt)) \leq M_r(\psi(kk_{n+1})) \leq M_r(\psi(k_{a_{n+1}})) = M_r(a_{n+1}) \leq cM_r(n) = cM_r(\psi(t)).$$

Similarly if $0 < t < k_1$ then

$$M_r(\psi(kt)) \leq M_r(\psi(kk_1)) \leq M_r(\psi(k_{a_1})) = M_r(a_1) \leq cM_r(0) = cM_r(\psi(t)).$$

It follows that $M_r(\psi(kt)) \leq cM_r(\psi(t))$ for all $t > 0$. Thus, by Lemma 3.1 we get the statement. □

Lemma 3.4. Let $l \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + (ln)^k}{1^k + 2^k + \dots + (n-1)^k} = \begin{cases} l^{k+1}, & \text{if } k > -1, \\ 1, & \text{if } k \leq -1. \end{cases}$$

Proof. It is easy to see that

$$x^{k+1} - (x-1)^{k+1} \leq (k+1)x^k \leq (x+1)^{k+1} - x^{k+1} \quad \text{for all } x \geq 1, k \geq 0$$

and

$$(x+1)^{k+1} - x^{k+1} \leq (k+1)x^k \leq x^{k+1} - (x-1)^{k+1} \quad \text{for all } x \geq 1, -1 < k < 0.$$

Apply these inequalities for $x = 1, 2, \dots, n$. Then we have

$$\lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + n^k}{n^{k+1}} = \frac{1}{k+1} \quad \text{for all } k > -1,$$

which implies the statement for $k > -1$.

It is well known that $\frac{1}{1^c} + \frac{1}{2^c} + \dots + \frac{1}{n^c}$ is convergent if $c > 1$. It follows that the statement is true in case $k < -1$ as well.

Finally in case $k = -1$ the inequalities

$$1 + \frac{\frac{l-1}{l}}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}} < \frac{1 + \frac{1}{2} + \dots + \frac{1}{ln}}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}} < 1 + \frac{l}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}}$$

imply the statement. □

Lemma 3.5. Let $k_1, d \in \mathbb{N}$, $q \in \mathbb{N} \setminus \{1\}$. If $k_n = k_1 q^{n-1}$ or $k_n = k_1 n^d$ then $M_r \circ \psi \sim \Delta_2$ for all $r \in \mathbb{R}$.

Proof. In the first case, when $k_n = k_1 q^{n-1}$, let $a_n = n+1$ and $k = q$. Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 q^{n-1}}{k_1 q^n} = \frac{1}{q} \leq \frac{1}{k}.$$

Let $Q = q^{r-1}$ and assume that $r > 1$. In this case $|1/Q| < 1$, thus we get

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{M_r(n+1)}{M_r(n-1)} = \frac{1 + Q + \dots + Q^n}{1 + Q + \dots + Q^{n-2}} = \frac{Q^2 - \frac{1}{Q^{n-1}}}{1 - \frac{1}{Q^{n-1}}} \rightarrow Q^2.$$

If $r < 1$ then $1/Q > 1$, thus

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{Q^2 - \frac{1}{Q^{n-1}}}{1 - \frac{1}{Q^{n-1}}} \rightarrow 1.$$

If $r = 1$ then $Q = 1$, so

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{n+1}{n-1} \rightarrow 1.$$

Thus we get that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r \in \mathbb{R}$. Hence conditions of Lemma 3.3 are satisfied, which implies the statement.

In the second case, when $k_n = k_1 n^d$, let $a_n = 2n$ and $k = 2^d$. Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 n^d}{k_1 (2n)^d} = \frac{1}{2^d} \leq \frac{1}{k}.$$

On the other hand it follows from Lemma 3.4 that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r \in \mathbb{R}$. So Lemma 3.3 implies the statement. \square

Theorem 3.6. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 n^d\}$ ($k_1, d \in \mathbb{N}$ are fixed) be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Let $t > 0$, $r \geq 2d/(d + 1)$, $s > t/r$ and $v = \max\{s, t(d + 1)/(2d)\}$. If $\{\|S_{k_1 n^d}\|/n^{d/s}, n \in \mathbb{N}\}$ is bounded in probability and $\mathbf{E}|X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} \mathbf{P}\left(\|S_{k_1 n^d}\| > \varepsilon n^{dr/t}\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.5 are satisfied. Let $k_n = k_1 n^d$. Then by Lemma 3.4 $\{M(n)/M(n - 1), n \in \mathbb{N}\}$ is bounded. Let $Y = M^{r/2}(\psi(|X|^{t/r}))$. Now we turn to the proof of $\mathbf{E}Y < \infty$. It is well known that

$$1^d + \dots + n^d = a_1 n^{d+1} + a_2 n^d + \dots + a_{d+2}$$

for some $a_1, a_2, \dots, a_{d+2} \in \mathbb{R}$. Let

$$g: [k_1, \infty) \rightarrow \mathbb{R}, \quad g(x) = \sum_{i=1}^{d+2} |a_i| (k_1^{i-2} x^{d+2-i})^{1/d}.$$

Then g is nondecreasing, $g(k_n) \geq M(n)$ and $g(x) \leq \text{const.}x^{(d+1)/d}$ for all $x \geq k_1$. Therefore by Lemma 3.2 we have

$$M^{r/2}\left(\psi(x^{t/r})\right) \leq \text{const.}x^{t(d+1)/(2d)} \quad \text{for all } x^{t/r} \geq k_1.$$

It follows that

$$Y = Y\mathbf{I}(|X|^{t/r} < k_1) + Y\mathbf{I}(|X|^{t/r} \geq k_1) \leq k_1^{r/2} + \text{const.}|X|^{t(d+1)/(2d)},$$

where $\mathbf{I}(A)$ denotes the indicator function of the set A . So $\mathbf{E}Y < \infty$. By Lemma 3.5 $M \circ \psi \sim \Delta_2$. It is easy to see that the other conditions of Corollary 1.5 hold true as well, on the other hand $M(n) \geq \text{const.}n^{d+1}$. So this theorem is consequence of Corollary 1.5. \square

Theorem 3.7. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 q^{n-1}\}$ ($k_1 \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}$ are fixed) be an array of rowwise independent B -valued r.v.'s which is w.m.d. by the r.v. X . Let $w \geq 0$, $t > 0$, $s > t$ and $v = \max\{s, t(w + 1)\}$. If $\{\|S_{k_1 q^{n-1}}\|/q^{n/s}, n \in \mathbb{N}\}$ is bounded in probability and $\mathbf{E}|X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} q^{nw} \mathbf{P}\left(\|S_{k_1 q^{n-1}}\| > \varepsilon q^{n/t}\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.6 are satisfied. Let $k_n = k_1 q^{n-1}$, $r = w + 2$ and $Y = M_r(\psi(|X|^t))$. Then $M_r(n) = k_1^{r-1} \frac{Q^n - 1}{Q - 1}$, where $Q = q^{r-1}$. Let

$$g: [k_1, \infty) \rightarrow \mathbb{R}, \quad g(x) = k_1^{r-1} \frac{Q^{1+\log(x/k_1)/\log q} - 1}{Q - 1}.$$

Then g is nondecreasing, $g(k_n) = M_r(n)$ and $g(x) \leq \text{const.} \cdot x^{r-1}$ for all $x \geq k_1$. Therefore by Lemma 3.2 we have

$$M_r(\psi(x^t)) \leq \text{const.} \cdot x^{t(w+1)} \quad \text{for all } x^t \geq k_1.$$

It follows that

$$Y = Y\mathbf{I}(|X|^t < k_1) + Y\mathbf{I}(|X|^t \geq k_1) \leq k_1^{r-1} + \text{const.} \cdot |X|^{t(w+1)}.$$

So $EY < \infty$. By Lemma 3.5 $M_r \circ \psi \sim \Delta_2$. The other conditions of Corollary 1.6 hold true as well. Thus Corollary 1.6 implies the statement. \square

References

- [1] GUT, A., On complete convergence in the law of large numbers for subsequences, *Ann. Probab.*, 13 (1985) 1286–1291.
- [2] GUT, A., Complete convergence for arrays, *Periodica Math. Hungar.*, 25 (1) (1992) 51–75.
- [3] HU, T.-C., ROSALSKY, A., SZYNAL, D., VOLODIN, A. I., On complete convergence for arrays of rowwise independent random elements in Banach spaces, *Stoch. Anal. Appl.*, 17 (6) (1999) 963–992.
- [4] FAZEKAS, I., A strong law of large numbers in Banach spaces of type Φ , *In Mathematical Statistics and Probability Theory*, Vol. A, 89–97, Reidel, Dordrecht, 1987.
- [5] FAZEKAS, I., Convergence rates in the law of large numbers for arrays, *Publ. Math. Debrecen*, 41/1-2 (1992) 53–71.
- [6] TÓMACS, T., Convergence rates in the law of large numbers for arrays of Banach space valued random elements, *Statist. Probab. Lett.*, 72 (2005) 59–69.

Tibor Tómacs

Department of Applied Mathematics

Eszterházy Károly College

P.O. Box 43

H-3301 Eger

Hungary