LINEAR DIOPHANTINE EQUATION WITH THREE CONSECUTIVE BINOMIAL COEFFICIENTS

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Abstract. In this note, we study the diophantine equation $A\binom{n}{k}+B\binom{n}{k+1}+C\binom{n}{k+2}=0$ in positive integers (n,k), where A, B and C are fixed integers.

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1. Introduction

D. Singmaster (see [3]) found infinitely many positive integer solutions (n,k) to the diophantine equation

$$\binom{n}{k}=\binom{n-1}{k+1}. \tag{1}$$

All such solutions arise in a natural way from the sequence of Fibonacci numbers $(F_m)_{m\geq 0}$ given by $F_0=0$, $F_1=1$ and $F_{m+2}=F_{m+1}+F_m$ for $m\geq 0$. Goetgheluck (see [1]) extended the above result and found infinitely many positive integer solutions (n,k) for the diophantine equation

$$2\binom{n}{k}=\binom{n-1}{k+1}.$$

These solutions arise in a natural way from the positive integer solutions of the Pell equation $x^2-3y^2=-2$. Several other diophantine equations involving binomial coefficients have been considered in [2], [4] and [5].

In this note, we fix three integers A, B, C, not all zero, and look at the positive integer solutions (n,k) of the equation $A\binom{n}{k}+B\binom{n}{k+1}+C\binom{n}{k+2}=0$. To avoid degenerate cases, we shall assume that $1\leq k<k+2\leq n-1$. We shall also assume that $AC\neq 0$. Indeed, say if $A=0$, then the above equation simplifies to

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$$B\binom{n}{k+1}+C\binom{n}{k+2}=0. \tag{2}$$

Obviously, equation (2) has no solution if $BC > 0$. Suppose that $BC < 0$ (say, up to changing signs, that $B < 0$ and $C > 0$) and that $\gcd(B, C) = 1$. Then equation (2) implies $B(k+2) + C(n-k-1) = 0$, which can be rewritten as $n = ((C-B)k + C - 2B)/C = k + 1 - B(k+2)/C$. Thus, n is an integer if and only if $k \equiv -2 \pmod{C}$. Moreover, the conditions $1 \le k < k+2 \le n-1$ are always fulfilled if $k > 1$ and $k \ge -2(1+C/B)$, and therefore (2) has infinitely many solutions.

The case when $C = 0$ can be reduced to the case when $A = 0$ by using the symmetry of the binomial coefficients and the substitution $(A, C, k) \longmapsto (C, A, n-k-2)$.

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2. Main Result

It is clear that we may assume that $\gcd(A, B, C) = 1$ and that $A > 0$. Our main result is the following.

Theorem. *Let A, B and C be integers with $A > 0$, $C \ne 0$ and $\gcd(A, B, C) = 1$. If the diophantine equation*

$$A\binom{n}{k}+B\binom{n}{k+1}+C\binom{n}{k+2}=0. \tag{3}$$

admits infinitely many integer solutions $1 \le k < k+2 \le n-1$, then one of the following holds:
(i) $B = A + C$ and $C < 0$, case in which all the solutions (n, k) are on the line

$$A(k+2) + C(n-k) = 0,$$

(ii) $A = A_0^2,\ B = -2A_0C_0,\ C = C_0^2$ hold with some positive coprime integers A_0 and C_0, case in which all solutions (n, k) with $1 \le k < k+2 \le n-1$ of (3) are of the form

$$k+2=\frac{t(t+C_0)}{A_0(A_0+C_0)} \qquad \text{and} \qquad n-k=\frac{t(t-A_0)}{C_0(A_0+C_0)} \tag{4}$$

for some positive integer t.
(iii) $B \neq A + C$, $D = B^2 - 4AC > 0$ *is not a perfect square, and*

$$X^2 - DY^2 = E \tag{5}$$

holds, where $X = (B^2 - 4AC)(n - k) - A(B - 2C)$, $Y = 2A(k + 2) + B(n - k) - A$, $E = 4A^2C(A - B + C)$, *case in which all positive integer solutions* (n, k) *of equation (3) can be found by solving the Pell like equation (5).*

Proof. After simplifications, equation (3) becomes

$$A(k + 1)(k + 2) + B(k + 2)(n - k) + C(n - k)(n - k - 1) = 0.$$

Writing $k + 2 = x,\ n - k = y$ we get

$$Ax(x - 1) + Bxy + Cy(y - 1) = 0,$$

or, equivalently,

$$Ax^2 + Bxy + Cy^2 - Ax - Cy = 0. \tag{6}$$

We shall assume that $D := B^2 - 4AC \neq 0$, and we shall return to the case when $D = 0$ later.

With the substitution $x = u + \alpha,\ y = v + \beta$, we get that the above relation becomes

$$(Au^2 + Buv + Cv^2) + (2A\alpha + B\beta - A)u + (B\alpha + 2C\beta - C)v \tag{7}$$

$$= -(A\alpha^2 + B\alpha\beta + C\beta^2) + A\alpha + C\beta.$$

We choose α and β such that the coefficients of the linear terms in u and v in equation (7) vanish. These lead to the system of equations

$$2A\alpha + B\beta = A,$$

$$B\alpha + 2C\beta = C,$$

whose rational solution is

$$\alpha = \frac{C(B - 2A)}{B^2 - 4AC},$$

$$\beta = \frac{A(B - 2C)}{B^2 - 4AC}.$$

Note that we may divide by $D = B^2 - 4AC$, because $D \neq 0$. With the above formulas for α and β, we get that

$$-(A\alpha^2 + B\alpha\beta + C\beta^2) + A\alpha + C\beta = \frac{-AC(A-B+C)}{B^2-4AC},$$

and so equation (7) becomes

$$Au^2 + Buv + Cv^2 = \frac{-AC(A-B+C)}{B^2-4AC}.$$

This last equation implies that

$$(2Au + Bv)^2 - (B^2 - 4AC)v^2 = \frac{-4A^2C(A-B+C)}{B^2-4AC},$$

and since

$$2Au + Bv = (2Ax + By) - (2A\alpha + B\beta)$$
$$= (2Ax + By) - \frac{2AC(B-2A) + AB(B-2C)}{B^2-4AC} = 2Ax + By - A,$$

while

$$v = y - \beta = \frac{(B^2-4AC)y - A(B-2C)}{B^2-4AC},$$

it follows that if we write

$X := (B^2 - 4AC)y - A(B - 2C)$,

$Y := 2Ax + By - A$,

$E := 4A^2C(A - B + C)$,

we get that $X, Y \in \mathbb{Z}$ and

$$X^2 - DY^2 = E. \tag{8}$$

We thus see that if $D < 0$, then the diophantine equation (3) has at most finitely integer solutions $1 \leq k < k+2 \leq n-1$. We now assume that $D > 0$. If $E = 0$, then since $AC \neq 0$, it follows that $B = A + C$. In this case, $D = B^2 - 4AC = (A-C)^2$, and so pairs of integers $X,\ Y$ satisfying equation (8) satisfy either

$$X = (C - A)Y \qquad \text{or} \qquad X = (A - C)Y.$$

In terms of the variables x and y, the above lines become

$$x + y = 1 \qquad \text{or} \qquad Ax + Cy = 0.$$

It is clear that the first one admits no integer solutions $x = k + 2$ and $y = n - k$ for $1 \le k < k + 2 \le n - 1$, while the second one admits infinitely many such solutions if and only if $C < 0$ (whereas if $C > 0$, then the second one does not admit any such solutions either). Finally, if $E \ne 0$, then equation (8) admits only finitely many solutions (or none) if D is a perfect square, while if D is not a perfect square, the above equation (8) is a Pell like equation, which either has no solutions, or it has infinitely many, and in this later case all integer solutions (X, Y) of such equation belong to finitely many binary recurrent sequences whose roots are the fundamental unit ζ of norm 1 in the quadratic order $\mathbb{K} = \mathbb{Q}[\sqrt{D}]$ and its conjugate ζ_1, respectively.

Finally, we deal with the case $D = 0$. In this case, $B^2 = 4AC$, so $B = 2B_0$, and $B_0^2 = AC$. Since $\gcd(A, B, C) = 1$, and $A > 0$, it follows that $\gcd(A, C) = 1$, and then that $A = A_0^2$ and $C = C_0^2$ hold with some positive integers A_0 and C_0. Hence, $B_0 = \pm A_0 C_0$. When $B_0 = A_0 C_0$, it is clear that the left hand side of equation (3) is positive whenever $1 \le k < k + 2 \le n - 1$. Thus, $B_0 = -A_0 C_0$, and therefore $B = -2A_0 C_0$. Equation (6) becomes

$$A_0^2 x^2 - 2A_0 C_0 xy + C_0^2 y^2 = A_0^2 x + C_0^2 y,$$

which can be rewritten as

$$(A_0 x - C_0 y)^2 = A_0^2 x + C_0^2 y = A_0(A_0 x - C_0 y) + C_0(A_0 + C_0) y.$$

Setting $t := A_0 x - C_0 y$, we get that

$$C_0(A_0 + C_0) y = t^2 - A_0 t,$$

leading to

$$y = \frac{t(t - A_0)}{C_0(A_0 + C_0)},$$

and since $A_0 x = C_0 y + t$, we get that

$$x = \frac{t(t + C_0)}{A_0(A_0 + C_0)},$$

which lead to formulae (4) via the fact that $x = k + 2$, and $y = n - k$. Note that since x, t, and y are integers, it follows that t is in certain arithmetical progressions modulo $A_0 C_0 (A_0 + C_0)$, and from the fact that $x \ge 3$ and $y \ge 3$, it follows that either $t > G_1 := G_1(A_0, C_0)$, or $t < G_2 := G_1(A_0, C_0)$, where G_1 and G_2 are two constants which depend on A_0 and C_0 and which can be easily computed by solving the coresponding quadratic inequalities.

This completes the proof of the Theorem.

3. Examples

Example 1. The equation

$$\binom{n}{k} - \binom{n}{k+1} - 2\binom{n}{k+2} = 0 \tag{9}$$

is a particular case of equation (3) for $A = 1$, $B = -1$ and $C = -2$. Since $B = A + C$, all solutions of equation (9) satisfy

$$(k+2) - 2(n-k) = 0,$$

which is equivalent to $2n - 3k = 2$. The integer solutions of the above equation are given by $n = 1 + 3t$ and $k = 2t$ with some integer t, and since n and k must be positive, we must have $t > 1$. Conversely, one verifies easily that

$$\binom{3t+1}{2t} - \binom{3t+1}{2t+1} - 2\binom{3t+1}{2t+2} = 0$$

holds for all positive integers t.

Example 2. The equation

$$\binom{n}{k+2} - 2\binom{n}{k+1} + \binom{n}{k} = 0 \tag{10}$$

has $A = C = 1$ and $B = 2$, therefore $D = 0$. Moreover, $A_0 = C_0 = 1$, so all solutions (n, k) of the above diophantine equation (10) have

$$k + 2 = \frac{t(t+1)}{2} \qquad \text{and} \qquad n - k = \frac{t(t-1)}{2},$$

which gives

$$k = \frac{t^2 + t - 4}{2} \qquad \text{and} \qquad n = t^2 - 2.$$

Since $n > k > 0$, it follows that either $t \geq 3$, or $t \leq -3$. Conversely, one may check that if t is any integer which is ≤ -3, or ≥ 3, then

$$\binom{t^2-2}{\frac{t^2+t-4}{2}} - 2\binom{t^2-2}{\frac{t^2+t-2}{2}} + \binom{t^2-2}{\frac{t^2+t}{2}} = 0.$$

Example 3. The equation

$$\binom{n}{k+2} = \binom{n}{k+1} + \binom{n}{k} \tag{11}$$

reduces to equation (3) for $A = 1$, $B = 1$, and $C = -1$. In this case, $D = B^2 - 4AC = 5$, $E = 4A^2C(A-B+C) = 4$, $X = (B^2-4AC)(n-k) - A(B-2C) = 5(n-k) - 3$, and $Y = 2A(k+2) + B(n-k) - A = 2(k+2) + (n-k) - 1$. Since $X^2 - 5Y^2 = 4$, it follows that $X = L_m$ and $Y = F_m$ hold with some even positive integer m, where $(L_\ell)_{\ell\geq 0}$ is the Lucas sequence given by $L_0 = 2$, $L_1 = 1$, and $L_{\ell+2} = L_{\ell+1} + L_\ell$ for all $\ell \geq 0$, and $(F_\ell)_{\ell\geq 0}$ is the Fibonacci sequence. We now get that $n - k = (X+3)/5 = (L_m+3)/5$, and that $k + 2 = (Y - (n-k) + 1)/2 = (5F_m - L_m + 2)/10$. Hence, $k = (5F_m - L_m - 18)/10$, and $n = (5F_m + L_m - 12)/10$. Since n and k are integers, we need that $5|L_m + 3$, and that $10|5F_m - L_m + 2$. Thus, $5|L_m + 3$ and $2|F_m + L_m$. The second relation is always fulfilled, while the first one is fulfilled precisely if $m \equiv 0 \pmod 4$. Thus, $n = (5F_{4t} + L_{4t} - 12)/10$, and $k = (5F_{4t} - L_{4t} - 18)/10$. Since $k > 0$, we also need that $5F_{4t} > L_{4t} + 18$, which forces $t \geq 2$. One can now easily verify that

$$\binom{\frac{5F_{4t}+L_{4t}-12}{10}}{\frac{5F_{4t}-L_{4t}+2}{10}} = \binom{\frac{5F_{4t}+L_{4t}-12}{10}}{\frac{5F_{4t}-L_{4t}-8}{10}} + \binom{\frac{5F_{4t}+L_{4t}-12}{10}}{\frac{5F_{4t}-L_{4t}-18}{10}}$$

holds for all integers $t \geq 2$. Note also that since

$$\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1},$$

it follows that the diophantine equation (11) reduces to the diophantine equation (11), which in turn is a consequence of our Theorem.

Remark. We remark that at instance (iii) of our Theorem, it could be possible that the Pell equation (5) has integer solutions (X, Y), and yet none such that the additional congruence $X \equiv -A(B-2C) \pmod{B^2 - 4AC}$ (necessary in order for $n - k$ to be an integer) is satisfied.

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