A COMMENT ON THE DARBOUX TRANSFORMATION

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Abstract. It is known that the Darboux transformation (DT) allows us to construct isospectral potentials in the frame of the Schrödinger equation. Here we give a simple mathematical deduction for the DT.

Introduction

In the one-dimensional stationary case the Schrödinger equation is given by [1, 2]

$$-\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi \tag{1}$$

which is written in natural units taking $\frac{\hbar}{2m} = 1$. The values of λ represent the energy spectrum allowed for determinated boundary conditions and corresponding to the standard potential $u(x)$. With the very useful Darboux transformation (DT) [3–6] we can generalize any specific standard potential and thus generate new interaction models with the same energy levels. The DT is related to the Sturm–Liouville theory [7–10], and it is easy to see the implicit presence of DT in supersymmetric quantum mechanics [1, 2, 5, 11–15]. We suppose that (1) accepts the particular solution ψ_1 for the eigenvalue λ_1

$$-\psi_1{}'' + u(x)\psi_1 = \lambda_1\psi_1 \tag{2}$$

then we employ ψ_1 as á'seed function" to construct the DT [3–5, 16]:

$$\phi(x) = \psi' - \sigma_1(x)\psi \qquad \sigma_1 = \frac{d}{dx}ln\psi_1 \tag{3}$$

therefore (1) adopts the structure:

$$-\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi \tag{4}$$

with the generalized isospectral potential:

$$U(x) = u(x) - 2\frac{d}{dx}\sigma_1 \tag{5}$$

That is, the Schrödinger equation is covariant with respect to DT. Selecting other"seed functions" we can generate many DT-s and thus a great family of generalized potentials with the same energy spectrum.

In the next section we show a simple procedure to motivate (3), (4) and (5), that is, we exhibit how the basic expressions of the DT are born.

Darboux transformation

If in (1) we introduce the new dependent variable $y(x) = \psi/\theta(x)$, where θ is an arbitrary function for the time being, then this equation takes the form:

$$y'' + 2\frac{\theta'}{\theta}y' + \left(\lambda - \lambda_1 + \frac{\theta''}{\theta} - \frac{\psi''}{\psi_1}\right) y = 0 \tag{6}$$

because from (2) we have that $u = \lambda_1 + \psi_1''/\psi_1$. Therefore it is natural the election $\theta = \psi_1$, that yields:

$$y = \frac{\psi}{\psi_1} \tag{7}$$

and reduces this equation to the form:

$$y'' + 2\frac{\psi_1'}{\psi_1}y' + (\lambda - \lambda_1)y = 0 \tag{8}$$

if the definition of y written above is applied in deducing each of the equations of (7) and (8). Now we apply $\frac{d}{dx}$ to (8) and introduce the notation:

$$\eta(x) = \frac{d}{dx}y(x), \qquad \sigma_1 = \frac{\psi_1'}{\psi_1} \tag{9}$$

for thus to obtain the equation:

$$\eta'' + 2\sigma_1\eta' + (\lambda - \lambda_1 + 2\sigma_1')\eta = 0 \tag{10}$$

Finally, in (10) we make a transformation similar to (7):

$$\eta = \frac{\phi}{\psi_1} \tag{11}$$

Then this equation adopts the structure of (4) with the generalized isospectral potential $U(x) = \sigma_1^2 - \sigma_1' + \lambda_1 = u - 2\sigma_1'$, in according with (5). Besides, from (7), (9) and (11) we have that $\phi = \psi_1\eta = \psi_1 y' = \psi_1 \frac{d}{dx}(\psi/\psi_1)$, which reproduces (3) q.e.d.

In the literature on DT there is not an explicit motivation for these important transformations of mathematical physics. Thus, the present Note was dedicated to a simple demonstration of the basic expressions of DT.

References

[1] de Lange O. L., Raab, R. E., *Operators methods in quantum mechanics*, Clarendon Press, Oxford (1991)

[2] Schwabl, F., *Quantum mechanics*, Springer-Verlag, Berlin (1992)

[3] Darboux, G., Compt. Rend. Acad. Sc. (Paris) 94 (1882) 1456.

[4] Khare, A., Sukhatme, U. J. Phys. A: Math. Gen 22 (1989) 2847.

[5] V. B. Matveev and M-A. Salle, *Darboux transformations and solitons*, Springer-Verlag, Berlin (1991)

[6] J. Morales, J. J. Peña and J. López Bonilla J. Math. Phys. 42 (2001) 966.

[7] C. Lanczos, *Linear differential operators*, D. van Nostrand, London (1961)

[8] H. Hochstadt, *The functions of mathematical physics*, Dover NY (1986)

[9] J. B. Seaborn, *Hypergeometric functions and their applications*, Springer-Verlag, Berlin (1991)

[10] Z. Ahsan, *Differential equations and their applications*, Prentice Hall, India (2000)

[11] A. A. Andrianov, N. V. Borisov and M. J. Ioffe, Theor. Math. Fiz. 61 (1) (1984) 17 and 61 (2) (1984) 183.

[12] A. A. Andrianov, N. V. Borisov and M. J. Ioffe, Phys. Lett. B181 (1986) 141.

[13] R. W. Haymaker and A. R. P. Rau Am. J. Phys. 54 (1986) 928.

[14] F. Cooper, A. Khare and U. Sukhatme Phys. Rep. 251 (1995) 267.

[15] H. R. Hauslin Helv. Phys. Acta 61 (1988) 901.

[16] M. Crum Quat. J. Math. 6 (1955) 121.

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