REMARKS ON THE CONCEPT OF SIMILARITY IN TEACHING GEOMETRY IN TEACHERS' TRAINING COLLEGE

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Abstract. In [12] and [13] (textbooks for teachers' training colleges written by B. Pelle) isometry and similarity are defined not in the classical way, but as a product of reflections, and as a product of central dilatation and isometry. We make some remarks on this way of definition, and we study some important theorems on similarity (e.g. fixed point, classification) by using this way of definition.

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1. Introduction

In the classical treatment of geometrical transformations isometry is defined as a transformation which preserves distance, and by similarity one means a transformation in which the ratio of each corresponding line segments is constant (e.g. [5], [17], [18]). In [12] and [13] (textbooks for teachers' training colleges) these concepts are defined in a different way. The basis of the structure is the group of the axioms of Reflection refering to the primitive concept of "reflection in plane"; then follows the concept of reflection in line. Space (plane) isometry is defined as a product of reflections in plane (in line). After the axioms of Metric and Parallelism, the theorems of parallel secants and the concept and properties of central dilatation, similarity is defined as a product of central dilatation and isometry. If we want to describe the difference between the two ways of definition, we can say that the classical one is based on a property, and the other one is a "constructive" way; it provides technique to give the transformation.

In this paper we examine the connection between the classical and "constructive" ways. We shall apply the latter way consistently throughout the study of similarity; we aspire to the complete analogy with the concepts and theorems involved in studying of isometry. Related to these purposes we suggest some complements, changes to the structure involved in [12] and [13]. There are some topics which are not detailed in [12] and [13], namely the theorems on the fixpoint and the classification of similarities, the concept of dilatation; we shall examine these topics also in the "constructive" way. We make these suggestions with the aim of forming a unified system of concepts and theorems for the students in this very important domain of geometry.

We note here that the axioms of Reflection involved in [12] and [13] are used instead of the classical axioms of Congruence only. So the concept of reflection has not such a central role as in [1] or in [14] (chapters 5., 6.).

2. Isometry

The axioms of Reflection which we use are a little bit different from the axioms involved in [12] and [13] ([12] pp. 21–22; [13] pp. 17–18), therefore we list them (R1–R5). In [8] we wrote some remarks on these axioms and the concept of orthogonality and reflection in line. We note that in this paper by space (plane) transformation we mean a bijective mapping from the space (plane) onto itself; two transformations are said to be equal, if they transform any point into the same point; by line-preserving mapping we mean a mapping, which transforms collinear points into collinear points; by fixed point of a mapping we mean a point which coincides with its image under the mapping; by fixed plane (line) of a mapping we mean a plane (line) whose points are fixed by the mapping; by plane-flag we mean the union of a halfplane and a ray on its boundary, and by space-flag we mean the union of a halfspace and a plane flag on its boundary.

R1: Any reflection in plane is a line-preserving involutory space-transformation, which has a fixed plane; and this plane separates every P–P' pair, if P is not on it.

R2: For any plane there is a unique reflection in plane, whose fixed plane is the given one.

R3: For any two points there is a unique reflection in plane, in which they are corresponding points.

R4: For any two rays, starting from the same point, there is a unique reflection in plane, which transforms the given rays into each other.

R5: If two products of reflections in plane transform a space-flag into the same one, then the products are equal.

Definition 2.1. By space (plane) isometry we mean a product of reflections in plane (in line). ([12] pp. 58, 198; [13] pp. 57, 190)

We make some remarks on this definition. Students in secondary school learn the classical definition (e.g. [4]), so the different definitons may cause confusion. To avoid this, we think that it is important to show them the equivalence of the definitions. It is easy to see that isometry, defined in 2.1, preserves distance; since we defined the distance of two points as the length of their line segment ([12] p. 41; [13] p. 34) and in the axiom of Metric we postulate that the lengths of congruent segments are equal ([12] p. 40; [13] p. 34). For the equivalence we need the following theorem.

Theorem 2.2. *If a space (plane) transformation preserves distance, then it can be got as an isometry.*

Proof of Theorem 2.2. First we shall prove the case on the plane. Since the given transformation preserves distance, then due to the triangle-inequality the images of three points are collinear iff the points are collinear. So it is a line-preserving transformation. Let us consider three noncollinear points and their images. The corresponding sides of the triangles are equal due to the distance-preserving property, so due to the "three sides" congruency theorem of triangles ([12] p. 55; [13] p. 54) there is an isometry, under which the images of the three points are the same as under the given transformation. Finally it is easy to see, that due to the line- and distance-preserving properties our previous statement is true for every point, so the isometry and the given transformation are equal. In the proof of the case on the space the only difference is that we have to take four noncoplanar points instead of three noncollinear points, and we have to refer to the congruence of tetrahedra instead of that of triangles.

Classically the previous proof is related to the theorem, which states that on the plane any two triangles whose corresponding sides are equal, are related by a unique isometry (e.g. [3], [15]). In the structure based on axioms of Reflection the analogue of this "fundamental" theorem is the following one ([13] p. 43, only the case on the plane).

Theorem 2.3. *Any two space (plane) flags are related by a unique isometry.*

This theorem can be proved easily by axioms of Reflection and their equivalents refering to the case on the plane. At the same time we also proved the following Theorem 2.4. We use axiom R5 instead of axiom XII. of [12] and [13] because of its great importance in these fundamental theorems. (In [8] we examined the connection between the two axioms.)

Theorem 2.4. *Any space (plane) isometry can be obtained as the product of at most four reflections in plane (at most three reflections in line).*

([12] p. 58, [13] p. 43., only the case on the plane.)

We start the classification of isometries with this theorem. Naturally, we finish it only after the axiom of Parallelism. After the classification it is worth remarking that any isometry can be obtained as the product of at most two of the following transformations: reflection in plane, reflection in line, reflection in point. (This statement is a simple corollary of classification.)

3. Central dilatation

In [12] and [13] the concept of central dilatation is defined after the Euclidean axiom of Parallelism and the theorems of parallel secants ([12] p. 110, [13] p. 105). Our definition is a little bit different from that, because we use negative ratio, too (as e.g. in [3], [4], [15]). We make this change for the sake of unity and brevity in

Paragraphs 5. and 6. Due to this change, there is a difference between the properties of central dilatation on the plane and on the space: it preserves orientation on space iff its ratio is positive, while it preserves orientation on plane with any ratio. For the sake of brevity in the definition we use oriented segments; we defined the operations related to them in the usual way.

Definition 3.1. By central dilatation we mean the following mapping. Suppose that there is a point O and a $\lambda(\neq 0)$ constant. The image of the point P is those P', for which $OP' = \lambda OP$.

We shall use the notation $\mathbf{N}_{O,\lambda}$ for this mapping. We make some other definitions. By invariant plane (line) of a mapping we mean a plane (straight line) which coincides with its image under the mapping. By the center of a mapping we mean a point, through which every straight line passing is invariant. To emphasize the analogies with the axioms of Reflection we list some properties of central dilatation.

I. Any central dilatation is a line-preserving space (plane) transformation, which has a center; this point separates every other P–P' pair, iff $\lambda < 0$.

II. For any point O and any constant $\lambda(\neq 0)$, there is a unique $\mathbf{N}_{O,\lambda}$.

III. For any three collinear points O, P and P', so that P and P' differ from O, there is a unique central dilatation with center O, under which the image of P is P'.

IV. For any point O and any two parallel lines a, a' which are off O, but coplanar with it, there is a unique central dilatation with center O, under which the image of a is a'. (Two coplanar lines are called parallel, if they coincide or do not meet.)

(We need the Euclidean axiom of Parallelism only for the proof of line-preserving property and statement IV.)

These properties are just the analogues of the first four axioms of Reflection. The analogue of the fifth one will occur at the concept of similarity, in Theorem 4.2. As in the case of axioms of Reflection, statements II., III. and IV. provide techniques to give a central dilatation; and the first one contains the most important (non metric) properties of central dilatation. We declare these the most important ones because of the following theorem.

Theorem 3.2. *If a mapping on the Euclidean space (plane) is a line-preserving transformation with a center, then it can be got as a central dilatation.*

Proof of Theorem 3.2. Since mapping is a line-preserving transformation, any line is coplanar with its image, and the images of parallel lines are also parallel. Planes passing through the center are invariant, and the images of parallel planes are also parallel. Let O denote the center. O is fixed, since it is the point of intersection of invariant lines. Let us first assume that there is another fixed point, say C. Let α be a plane that contains C, and let β be the plane that contains O, which is parallel to α. Since β is invariant and the mapping preserves parallelism,

α is also invariant. If every plane passing through C is invariant, then C is a center. Since there are two centers, we can fit two invariant lines on every point, so every point is fixed. In this case the mapping is the identity, which is a central dilatation. Let us now assume that O is the only fixed point. First we shall show that any line, which is off O, is not invariant, but it is parallel to its image. If it were invariant, its points would be fixed. If it intersected its image, their point of intersection would be fixed. Finally, the theorem of the parallel secants concludes that $\dfrac{OP'}{OP}$ is constant for any $P(\neq O)$. So the mapping is $\mathbf{N}_{O,\lambda}$, the proof is completed.

We think that it is also very important to emphasize the connection between the line-preserving property of the central dilatation and the Euclidean axiom of Parallelism in the lectures. In general, after the axiom of Parallelism, textbooks list some statements equivalent to the axiom, but generally the line-preserving property of the central dilatation is missing. In this treatment which is based on the axioms of Reflection and products, it would be important to mention this, too. The first reason for that is that the concept of similarity is (partially) based on the central dilatation. The other reason is, that the concept of isometry is based on the primitive concept of reflection in plane, whose line-preserving property is declared in an axiom (R1). We can prove easily the line-preserving property of the central dilatation by axiom of Parallelism ([12] p. 110, [13] p. 106). For the equivalence we need the following theorem.

Theorem 3.3. *If the statement of Euclidean axiom of Parallelism is false, then central dilatation is not line-preserving mapping.*

The proof of this theorem can be found e.g. in [7], where the basis of proof is a modell, while the following one does not use modell.

Proof of Theorem 3.3. Let P be a point, e a line, and $P \notin e$. We shall work on the plane of P and e. Let m be the line, for which $P \in m$, and $m \perp e$, let $C = m \cap e$, and f the line, for which $P \in f$, and $m \perp f$ (Fig. 1).

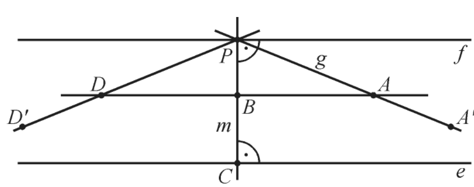

Figure 1.

It is known that f does not meet e. Let g be another line through P, which does not meet e. It is obvious that the reflected image of g under the reflection in line m, does not meet e either. Let A be a point on g between e and f, let D denote the image of A under the reflection in line m, and let $B = m \cap (AD)$, which is obviously an inner point of the segment PC. Let us consider the central dilatation with center P, which transforms B to C. The images of A and D under this central dilatation remain between e and f, and they are separated by m. So, according to axioms of Order, they are not collinear with C.

In [12] and [13] the product of central dilatations is worked out immediately after the investigation of the properties of central dilatation, before the definition of similarity ([12] p. 113, [13] p. 109). For the sake of unity, we choose the way which was used in [12], [13] and by us to observe isometries. Namely, we first deal with the general concept of similarity and the fundamental theorems related to it, and we shall observe special products only after these theorems.

4. Similarity

In [12] and [13] plane similarity is defined as a product of a central dilatation and an isometry ([12] p. 114, [13] p. 111). The definition for the case on the space is a little bit different: the factors of the product are in plural ([12] p. 200, [13] p. 192). We choose the latter way for both cases.

Definition 4.1. By space (plane) similarity we mean a product of central dilatations and space (plane) isometries.

We choose this way for two reasons. The first is that the analogy with the definition of isometry in 2.1 comes with the use of plural. The second is that this form gives immediately the closure of the set of similarities for composition. In [12] and [13] this statement ([12] p. 115, [13] p. 111) is derived from the following facts: a product of isometries is also an isometry ([12] p. 58, [13] p. 57), a product of central dilatations is either a central dilatation or a translation ([12] p. 113, [13] p. 110). In [12] and [13] the concept of the ratio of similarity is defined in the classical way, namely, it is the constant ratio of corresponding segments. In our opinion, another way of definition fits better the Definiton 4.1. Namely, the modulus of the product of the ratios of the central dilatations involved in the product in Definition 4.1 is taken as the ratio of similarity.

The equivalence of Definition 4.1 and the classical one comes from the following facts. From the properties of isometry and central dilatation we get the statement: for the similarity defined by 4.1 the ratio of each corresponding segments is constant. On the other hand, the following theorem is valid.

Theorem 4.2. *If the ratio of each corresponding segments related by a transformation is constant, then it can be got as a similarity.*

It is true, because it is easy to show that the given transformation is a product of an isometry and a central dilatation (e.g. [4], [10], [17]).

We note that in secondary school similarity is defined as in [12] and [13] for the case on the plane, namely, as a product of a central dilatation and an isometry (e.g. [4]). So this transformation is a similarity in the sense of Definition 4.1, too. On the other hand, from Theorem 4.2 we get that every similarity in the sense of Definition 4.1 is a product of a central dilatation and an isometry. This means that the two definitions are equivalent. The second statement which gives the equivalence will occur later in Theorem 4.4, which will be important for this treatment from another point of view, too.

After the examination of the different ways of definition, there follow the fundamental theorems on similarity. The simple properties of similarity (e.g. line-, ratio- and angle-preserving property) come directly from Definition 4.1, as the common properties of the factors of the product. The further observations are based on the following theorem, which is the analogue of Theorem 2.3 and axiom R5.

Theorem 4.3. *Suppose that* Z *and* V *are space (plane) flags, P and Q are points on their ray. Then there exists a unique similarity, which transforms* Z *to* V *and P to Q.*

In [12] and [13] there is not a theorem like this. In the classical treatment the equivalent statement of Theorem 4.3 is the one which says that any two triangles (tetrahedra) whose corresponding sides (edges) have a constant ratio, are related by a unique similarity (e.g. [3], [15]); or this one: on the plane any two segments are related by just two similarities, a direct one and an opposite one (e.g. [3], [9]). We use the above Theorem 4.3 instead of these theorems, because it fits better this structure than the classical theorems mentioned.

Proof of Theorem 4.3. First let us consider the isometry, $\mathbf{M}$, which transforms Z to V (Theorem 2.3). Then we consider the central dilatation, whose center is the starting point of the ray of V, and which transforms $\mathbf{M}(P)$ to Q. The product of these transformations has the desired properties. If there is another similarity, then it is equal to the first product, due to the ratio- and angle-preserving properties.

So for the sake of unity and consistency, in the sequel we shall use Theorem 4.3 for the investigation of similarities.

From the construction involved in the previous proof, we get the following two important consequences. The first is the analogue of Theorem 2.4.

Theorem 4.4. *Any similarity can be obtained as a product of an isometry and a central dilatation, whose ratio is the ratio of the given similarity (so it is positive).*

Theorem 4.5. *A similarity can be got as an isometry iff its ratio is 1.*

These theorems have already been mentioned above when we discussed equivalence, but if we observe this structure on its own this is the right place for them.

5. Classification of similarities

We start with the classification theorem for plane similarities.

Theorem 5.1. *Any plane similarity, which is not isometry, can be got either as a dilative rotation or as a dilative reflection.*

(We regard the central dilatation as a dilative rotation with rotation angle $0°$.)

This theorem is not in [12] and [13], but the two special transformations are mentioned in [13] ([13] p. 111). This theorem is usually proved after the theorem on the fixed point of similarity. We observe these two questions together.

There are many ways to prove the existence of the fixed point. The classical one—using parallelograms—is e.g. in [3], [6], [10], [15], [17]. There is another way to construct the fixed point—using circles—e.g. in [2], [9], [11], [16]. A proof based on continuity can be found e.g. in [2]. Also in [2] there is special construction for the case on the plane.

Here we give a proof of Theorem 5.1, which is in close connection with the structure that has been built above. It is based on the product-definition of similarity and isometry, and on Theorems 4.3 and 4.4. Some details in case II. are similar to the construction in [2]. Our proof is more lengthy than the previously mentioned ones, but our aim is to make a consistent structure. We note that in the proof we use orientated segments and angles, we defined the operations related to them in the usual way; we denote the reflection in line a by $\mathbf{T}_a$; we use the term "axis" for the fixed line of reflection in line; we make the products of transformations from right to left.

Proof of Theorem 5.1. Let $\mathbf{H}$ be a similarity which is not isometry. From Theorem 4.4 we get that $\mathbf{H} = \mathbf{N}_{O,\lambda}\mathbf{M}$, where $\mathbf{M}$ is an isometry, $\lambda > 0$, $\lambda \neq 1$. We shall consider six cases depending on the type of $\mathbf{M}$.

I. If $\mathbf{M}$ is either the identity, a rotation about O, or a reflection in line passing through O, then proof is complete.

II. If $\mathbf{M}$ is a reflection in line, $\mathbf{M} = \mathbf{T}_b$, $O \notin b$, then let m be the line, for which $O \in m$, $m \perp b$, and $B = b \cap m$ (Fig. 2.). Let C be the point, for which $BC = \dfrac{\lambda - 1}{\lambda + 1} OB$. C is fixed point of $\mathbf{H}$. Let a be the line, for which $C \in a$, $a \parallel b$. It is obvious that a is invariant line of $\mathbf{H}$, and $\mathbf{H}$ interchanges the halfplanes bounded by a. Let P be a point on a ($P \neq C$), and $P' = \mathbf{H}(P)$. Since C is fixed, $CP' = \lambda CP$. The similarity $\mathbf{N}_{C,\lambda}\mathbf{T}_a$ also has these properties. Then let us consider the plane flag which contains the ray $[CP)$ and one of the halfplanes bounded by a. From the results above we get that the images of this flag and P under $\mathbf{H}$ and $\mathbf{N}_{C,\lambda}\mathbf{T}_a$ are the same. So according to Theorem 4.3 $\mathbf{H} = \mathbf{N}_{C,\lambda}\mathbf{T}_a$.

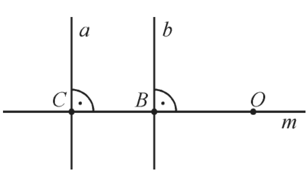

Figure 2.

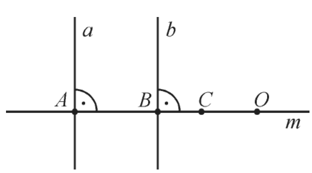

Figure 3.

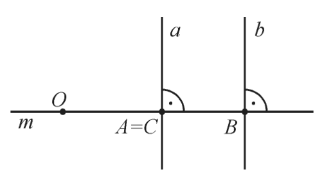

Figure 4.

We reduce the further cases to case II. in the following way. Translation—and rotation, too—is the product of two reflections in line, where one of the axes is partially arbitrary. We observe how to take it, so that the fixed point of the product of the reflection in this line and $\mathbf{N}_{O,\lambda}$ should be incident to the other axis. If it is satisfied, then $\mathbf{H}$ also fixes this point. In the case of glide reflection, we shall base our proof on the fact that it is the product of a translation and a reflection in line.

III. If $\mathbf{M}$ is a translation, $\mathbf{M} = \mathbf{T}_b\mathbf{T}_a$, $a \parallel b$, then let m and B be as in II., $A = m \cap a$, and let C be the fixed point of $\mathbf{N}_{O,\lambda}\mathbf{T}_b$ (Fig. 3.). C is on a iff $BC = BA$ (Fig. 4.), so iff $OB = \dfrac{1+\lambda}{1-\lambda}AB$. (Because, according to II., $BC = \dfrac{\lambda-1}{\lambda+1}OB$ and $\lambda \neq 1$.) Instead of the original axes we take new ones for which the previous equation stands for OB. (We can construct the new B, b by using O, λ and the original AB segment.) So by the new axes we get that $\mathbf{H}$ fixes the new C. According to II. $\mathbf{N}_{O,\lambda}\mathbf{T}_b$=$\mathbf{N}_{C,\lambda}\mathbf{T}_a$, so it also comes that $\mathbf{H} = \mathbf{N}_{C,\lambda}$.

Among rotations first we examine the half-turn, and then the other ones.

IV. If $\mathbf{M}$ is a half-turn, $\mathbf{M} = \mathbf{T}_b\mathbf{T}_a$, $a \perp b$, $a \cap b = K$, $K \neq O$, then let the new axes be (OK) and the line perpendicular to it through K (Fig. 5.) According to II., the fixed point of $\mathbf{N}_{O,\lambda}\mathbf{T}_b$, C, lies on a, so $\mathbf{H}$ also fixes it. Moreover $\mathbf{N}_{O,\lambda}\mathbf{T}_b = \mathbf{N}_{C,\lambda}\mathbf{T}_e$, where e is the line for which $C \in e$ and $e \parallel b$, so $\mathbf{H} = \mathbf{N}_{C,-\lambda}$.

V. If $\mathbf{M}$ is a rotation, $\mathbf{M} = \mathbf{T}_b\mathbf{T}_a$, $(a,b)\angle = \phi$, $\phi \neq 90°$, $a \cap b = K$, $K \neq O$, then let m, B and C be as in III., and let $\omega = ((KO), b)\angle$ (Fig. 6.).

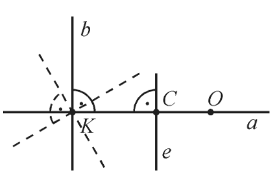

Figure 5.

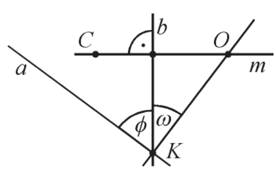

Figure 6.

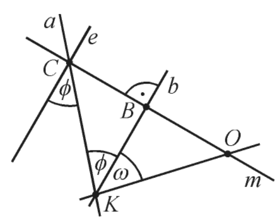

Figure 7.

C is on a iff $\dfrac{\tan\omega}{\tan\phi} = \dfrac{OB}{CB}$ (Fig. 7.), so iff $\tan\omega = \dfrac{1+\lambda}{1-\lambda}\tan\phi$. (Because, according to II., $BC = \dfrac{\lambda-1}{\lambda+1}OB$.) Instead of the original axes we take new ones for which the previous equation holds for $\tan\omega$. (We can construct the new ω, b by using O, λ and the original ϕ angle.) So by the new axes we get that $\mathbf{H}$ fixes the new C. According to II. $\mathbf{N}_{O,\lambda}\mathbf{T}_b = \mathbf{N}_{C,\lambda}\mathbf{T}_e$, where e is the same as in IV. (Fig. 7.), so $\mathbf{H} = \mathbf{N}_{C,\lambda}\mathbf{T}_e\mathbf{T}_a$, $C = e \cap a$ and the angle of rotation is 2ϕ.

VI. If $\mathbf{M}$ is a glide reflection, $\mathbf{M} = \mathbf{T}_b\mathbf{T}_a\mathbf{T}_c$, $a \perp c \perp b$, then according to III. there exists a point K for which $\mathbf{N}_{O,\lambda}\mathbf{T}_b\mathbf{T}_a = \mathbf{N}_{K,\lambda}$ (Fig. 8.). $((OK) \parallel c)$. According to II. there exists a point C and a line e for which $\mathbf{N}_{K,\lambda}\mathbf{T}_c = \mathbf{N}_{C,\lambda}\mathbf{T}_e$ and $C \in e$. So $\mathbf{H} = \mathbf{N}_{C,\lambda}\mathbf{T}_e$. (According to IV. $C \in (OB)$.)

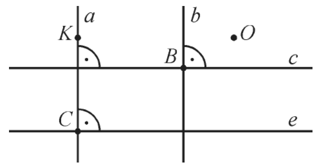

Figure 8.

Since there is not a further case for **M**, the proof is complete. It is obvious that the center of the central dilatation is the only fixed point of the product. In each case the proof also provides a way to construct this point.

There follows the classification theorem for space similarities. In [13] this theorem is included, but its proof is missing ([13] p. 192). Recall that dilative rotation on the space is the product of a rotation about a line and a central dilatation whose center lies on the axis of the rotation.

Theorem 5.2. *Any space similarity, which is not isometry, can be got as a dilative rotation.*

(We regard the central dilatation as a dilative rotation with rotation angle 0°.)

Proof of Theorem 5.2. The principle of the proof is the same as in the previous one so we do it breefly. First we put the given similarity into the form of $\mathbf{N}_{O,\lambda}\mathbf{M}$, and make classification according to the type of the **M** isometry. If **M** is either the identity, a reflection in plane, a translation, a rotation about a line, or a glide reflection, then we get—in the same way as in the corresponding case of the proof of Theorem 5.1—that the given similarity is a dilative rotation. (For reflection in plane and glide reflection the axis is the line passing through the fixed point and perpendicular to the fixed plane of the original reflection, the angle is 180°, and the ratio is $-\lambda$. For rotation about line the new axis is the line passing through the fixed point and parallel to the original one, the angle and the ratio do not change. For translation and identity we get central dilatation also with the original ratio.) For those isometries which do not have corresponding case in the proof of Theorem 5.1—namely, if **M** is either a rotatory reflection or a screw displacement—we get the desired result by using completed cases: either rotation about line and reflection in plane, or translation and rotation about line. We use the method which we used in case VI. in the proof of Theorem 5.1, where the question were reduced to cases II. and III. (For both cases the new axis is the line passing through the fixed point and parallel to the original one. For screw displacement the angle and the ratio do not change, for rotatory reflection the angle increases by 180° and the ratio is $-\lambda$.)

6. Dilatation

Finally, we deal with the concept of dilatation. We examine here the question mentioned at the end of Paragraph 3.: product of central dilatations.

In the classical treatment dilatation (or parallel similarity) is defined as a transformation, which transforms each line into a parallel line (e.g. [3], [5], [10], [18]). Here we give another definition which fits the structure using products (see Definitions 2.1 and 4.1).

Definition 6.1. By dilatation we mean a product of central dilatations and translations.

This definition is equivalent to the classical one, naturally. It is obvious that the dilatation 6.1 is a transformation and it transforms each line into a parallel line. On the other hand, it is involved e.g. in [3], [10], that if a transformation transforms each line into a parallel line, then it is either a central dilatation or a translation. (Those proofs refer to the case on the plane, but it is easy to extend them to the space.) Besides the equivalence of the definitions these facts prove the following theorem, too:

Theorem 6.2. *Any dilatation can be got either as a central dilatation or as a translation.*

It is worth emphasizing this theorem for another reason, too. This is the analogue of Theorems 2.4 and 4.4. We can get this theorem in our structure in a different way, too:

Proof of Theorem 6.2. According to Definition 4.1 the dilatations defined in 6.1 are similarities, so we can apply our results on classification of isometries and similarities. Since the product transforms each line into a parallel line, if it is an isometry, then it is either the identity, a translation or a reflection in point, and if it is not an isometry, then according to Theorems 5.1 and 5.2 it is a dilative rotation with rotation angle $0°$. Thus the theorem is proved, because every transformation mentioned except the translation is a central dilatation.

If we examine the question in details, first we find that it is enough to examine products with two factors. If we observe the products of isometries, we find that the set containing the identity, translations and reflections in point, contains the product of any two. So we have to examine only products with central dilatation whose ratio is not 1 or -1. The product of such central dilatation and translation is not isometry, so according to the previous proof it is a central dilatation. We get the center as the point of intersection of two lines passing through corresponding points. The other case, in which the product is not isometry, is the product of two central dilatations with product of ratios neither 1 nor -1. We get the center similarly to the previous case. If the product of ratios is 1, then the line passing through the centers and the halfplanes bounded by that line are invariant. So the product is either the identity or a translation depending on the centers whether they coincide or not. If the product of the ratios is -1, then the mentioned halfpanes interchange with their coplanar pair, so the product is a reflection in point. We get the center also in the way described above.

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