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ON SOME RESEARCH PROBLEMS IN MATHEMATICS

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Dedicated to the memory of Professor Péter Kiss

I. Introduction

The problem presented here is originated during our joint research activity with Z. Daróczy and some others for the Rényi–Parry expansions [1–11].

Let $\mathbb{C}^\infty$ denote the space of sequences $\underline{c} = (c_0, c_1, \ldots)$ the coordinates c_ν of which are complex numbers. The shift operator $\sigma\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ is defined by

$$\sigma(\underline{c}) = (c_1, c_2, \ldots).$$

Let $t_0 = 1,\ t_\nu \in \mathbb{C} \quad (\nu = 1, 2, \ldots)$ be bounded, and $\underline{t} = (t_0, t_1, \ldots)$. We define

$$R(z) = t_0 + t_1 z + \cdots . \tag{1.1}$$

Let l_1 be the linear space of the sequences $\underline{b} \in \mathbb{C}^\infty$, for which $\sum |b_\nu| < \infty$ holds.

The scalar product of a bounded sequence $\underline{c}$ and a $\underline{b} \in \mathbb{C}^\infty$ is defined as

$$\underline{c}\,\underline{b} = \underline{bc} = \sum_{\nu=0}^{\infty} b_\nu c_\nu .$$

Let

$$\mathcal{H}_{\underline{t}} = \left\{ \underline{b} \in l_1 \mid \sigma^l(\underline{b})\underline{t} = 0, \quad l = 0, 1, 2, \ldots \right\}. \tag{1.2}$$

It is clear that $\mathcal{H}_t$ is a closed linear subspace of l_1.

Let $\mathcal{H}_{\underline{t}}^{(0)} \subseteq \mathcal{H}_{\underline{t}}$ be the set of those $\underline{b} \in \mathcal{H}_t$ for which

$$|b_\nu| \le C(\varepsilon, \underline{b}) e^{-\varepsilon\nu} \quad (\nu \ge 0) \tag{1.3}$$

holds with suitable $\varepsilon > 0$ and $C(\varepsilon, \underline{b}) \quad (< \infty)$.

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If ρ is a root of $R(z)$, $|\rho|<1$, then $b_\nu=\rho^\nu$ satisfies $\sigma^l(\underline{b})\underline{t}=0 \quad (l=0,1,2,\dots)$ and even $|b_\nu|\le Ce^{-\varepsilon\nu}$, where ε can be defined from $e^{-\varepsilon}=|\rho|$, and $C=1$.

If ρ is a root of $R(z)$ of multiplicity m, then $b_\nu=\nu^j\rho^\nu \quad (\nu\ge 0)$ are solutions of $\sigma^l(\underline{b})\underline{t}=0 \quad (l\ge 0)$ for every $j=0,\dots,m-1$, furthermore (1.3) holds with suitable ε, and constant $C(\varepsilon,\underline{b})$. The sequences $b_\nu=\nu^j\rho^\nu \quad (\nu\ge 0)$ are called elementary solutions.

Let $\mathcal{H}_{\underline{t}}^{(e)}$ be the space of finite linear combinations of elementary solutions.

Let furthermore $\overline{\mathcal{H}}_t^{(e)}$ be the closure of $\mathcal{H}_t^{(e)}$.

It is obvious that $\overline{\mathcal{H}}_t^{(e)}\subseteq\mathcal{H}_t$.

Conjecture 1. $\overline{\mathcal{H}}_t^{(e)}=\mathcal{H}_t$.

Conjecture 2. *Assume that* $R(z)\ne 0$ *in* $|z|<1$. *Then* $\mathcal{H}_{\underline{t}}=\{\underline{0}\}$.

Theorem 1. *We have*

$$\mathcal{H}_{\underline{t}}^{(0)}=\mathcal{H}_{\underline{t}}^{(e)}.$$

Proof. $\mathcal{H}_{\underline{t}}^{(e)}\subseteq\mathcal{H}_{\underline{t}}^{(0)}$ obviously holds. We shall prove that $\mathcal{H}_t^{(0)}\subseteq\mathcal{H}_t^{(e)}$, i.e. that if $\sigma^l(\underline{b})\underline{t}=0 \quad (l=0,1,2,\dots)$, and

$$|b_\nu|<C(\underline{b},\varepsilon)\cdot e^{-\varepsilon\nu},$$

then there exist $\rho_1,\dots,\rho_k$ suitable roots of $R(z)$, $|\rho_s|\le 1/e^\varepsilon \quad (s=1,\dots,k)$ such that

$$b_\nu=\sum_{s=1}^{k}p_s(\nu)\rho_s^\nu \quad (\nu=0,1,2,\dots),$$

p_s are polynomials, deg $p_s=m_s-1$, where m_s is the multiplicity of the root ρ_s for $R(z)$.

Let $\underline{b}$ be a solution of

$$\sigma^l(\underline{b})\underline{t}=0 \quad (l=0,1,2,\dots), \qquad |b_\nu|\le C(\varepsilon,\underline{b})\cdot e^{-\varepsilon\nu}. \tag{1.4}$$

Let furthermore $\rho_1,\dots,\rho_p$ be all the roots of $R(z)$ in the disc $|z|<\dfrac{1}{e^\varepsilon}+\varepsilon_1$, where ε_1 is an arbitrary small positive number. Let m_s be the multiplicity of ρ_s, i.e.

$$R^{(j)}(\rho_s)=0 \quad (j=0,\dots,m_s-1), \quad R^{(m_s)}(\rho_s)\ne 0.$$

Let $\varphi(z)=\prod_{j=1}^{p}(z-\rho_j)^{m_j}$, $\quad\psi(z)=\prod_{j=1}^{p}(1-\rho_j z)^{m_j}$, and E be defined for a sequence $a_0a_1\dots$ such that $Ea_m=a_{m+1}$.

If $\underline{b}$ is a solution of the equation (1.4), and p is an arbitrary polynomial in $C[z]$, then $e_n = p(E)b_n$ is a solution of (1.4) as well.

Let

$$c_n := \psi(E)b_n \quad (n \in \mathbb{N}_0).$$

Let furthermore

$$B(z) = \sum_{\nu=0}^{\infty} \frac{b_\nu}{z^\nu}, \quad C(z) = \sum_{\nu=0}^{\infty} \frac{c_\nu}{z^\nu}. \tag{1.5}$$

Observe that

$$C(z) = \prod \left(1 - \frac{\rho_\nu}{z}\right)^{m_\nu} B(z) = \psi\left(\frac{1}{z}\right) B(z), \tag{1.6}$$

and that

$$\psi\left(\frac{1}{z}\right) z^M = \varphi(z), \quad M = m_1 + \cdots + m_p. \tag{1.7}$$

The function $B(z)$ is regular outside $|z| \leq e^{-\varepsilon}$, and bounded in $|z| \geq \dfrac{1}{e^\varepsilon} + \varepsilon_2$, where $\varepsilon_2 > 0$ is an arbitrary constant. We assume that $\dfrac{1}{e^\varepsilon} + \varepsilon_2 < 1$. In the ring $\dfrac{1}{e^\varepsilon} + \varepsilon_2 < |z| < 1$ we have

$$R(z)B(z) = \left(\sum_{u=0}^{\infty} t_u z^u\right)\left(\sum_{v=0}^{\infty} b_\nu \cdot z^{-v}\right) = \sum_{r=-\infty}^{\infty} \kappa_r z^r,$$

where

$$\kappa_r = \sum_{\substack{u-v=r \\ u,v\geq 0}} t_u b_v.$$

Due to (1.4), $\kappa_r = 0$ if $r < 0$, and $\kappa_r = O(1)$, for $r > 0$. Thus

$$R(z)B(z) = K(z), \quad K(z) = \kappa_0 + \kappa_1 z + \cdots,$$

$K(z)$ is regular in $|z| < 1$. Consequently, $B(z) = \dfrac{K(z)}{R(z)}$,

$$C(z) = \frac{K(z)\psi\left(\frac{1}{z}\right)}{R(z)}. \tag{1.8}$$

The right hand side of (1.8) is regular in $|z| < \dfrac{1}{e^{\varepsilon}} + \varepsilon_1$, and bounded there. Otherhand $B(z)$ and so $C(z)$ is bounded in $|z| \geq \dfrac{1}{e^{\varepsilon}} + \varepsilon_2$. If we choose $\varepsilon_2 < \varepsilon_1$, we conclude that $C(z)$ is bounded on the whole plane and so, it is constant, $C(z) = D$, $\sum \dfrac{b_\nu}{z^\nu} = B(z) = \dfrac{D}{\psi(1/z)}$, and so

$$\sum_{\nu=0}^{\infty} b_\nu z^\nu = \frac{D}{\prod(1-\rho_\nu z)^{m_\nu}}.$$

The right hand side can be splitted into partial fractions,

$$\frac{D}{\prod(1-\rho_\nu z)^{m_\nu}} = \sum_{\nu=1}^{p} \sum_{j=0}^{m_\nu} \frac{e_{\nu,j}}{(1-\rho_\nu z)^j}, \quad (e_{\nu,j} \in \mathbb{C}),$$

whence we obtain immediately that

$$c_n = \sum_{\nu=1}^{p} p_\nu(n) \rho_\nu^n \quad \deg p_\nu \leq m_\nu - 1,$$

and so the theorem holds.

II.

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a strictly monotonic sequence of positive numbers, $\lambda_1 > \lambda_2 > \cdots (> 0)$, and assume that $L_n = \lambda_{n+1} + \ldots$ is finite, furthermore that

$$\lambda_n \leq L_n \quad (n = 0, 1, 2, \ldots). \tag{2.1}$$

The condition (2.1) implies that

$$H = \left\{ x \;\middle|\; x = \sum \varepsilon_n \lambda_n, \quad \varepsilon_n \in \{0, 1\} \right\}$$

is the whole interval $[0, L_0]$. This assertion is due to Kakeya.

In some of our papers with Daróczy, we have investigated expansions generated by λ_n satisfying (2.1).

A sequence $\{\lambda_n\}$ is called interval filling, if (2.1) holds.

In a paper written jointly with Z. Daróczy and G. Szabó [12] we proved the following assertion.

Theorem 2. *Let λ_n be an interval filling sequence. Let $\{a_n\}_{n=1}^{\infty} \in l_1$ be a sequence with the following property: if*

$$\sum_{n=1}^{\infty} \varepsilon_n \lambda_n = 0, \quad \varepsilon_n \in \{-1, 0, 1\},$$

then

$$\sum \varepsilon_n a_n = 0.$$

We have $a_n = c\lambda_n$ with some constant c.

Conjecture 3. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be such a sequence of positive numbers for which $\lambda_1 > \lambda_2 > \cdots$, $\sum \lambda_n < \infty$, and $H = \{x \mid x = \sum \varepsilon_n \lambda_n, \quad \varepsilon_n \in \{0,1\}\}$ contains an interval. Assume furthermore that $\{a_n\}_{n=1}^{\infty} \in l_1$ such a sequence for which $\sum \delta_n \lambda_n = 0, \quad \delta_n \in \{-1, 0, 1\}$ implies that $\sum \delta_n a_n = 0$.*

Then $a_n/\lambda_n =$ constant.

Remarks. 1. If $H = \{\sum \varepsilon_n \lambda_n \mid \varepsilon_n \in \{0,1\}\}$ is totally disconnected, then each $x \in H$ has a unique expansion, therefore

$$\delta_1\lambda_1 + \delta_2\lambda_2 + \cdots = 0, \qquad \delta_j \in \{-1, 0, 1\}$$

implies that $\delta_1 = \delta_2 = \cdots = 0$, consequently every $\{a_n\} \in l_1$ is a solution.

2. Assume that $\Lambda := \{\lambda_n\}$ is interval filling and even that there is a non-trivial subsequence $\lambda_{n_j}(= w_j)$ for which $\Omega = \{w_j\}$ is interval-filling.

Let $\mathcal{M}$ denote the set of the following sequences $(e_1, e_2, \ldots) = \underline{e}$.

1. If $e_\nu \in \{-1, 0, 1\}$ for every ν and $e_\nu = 0$ for $\nu \notin \{n_1, n_2, \ldots\}$, then $\underline{e} \in \mathcal{M}$.

2. For every n, let λ_n be expanded in the system Ω with some digits $\{0, 1\}$:

$$\lambda_n = \sum_{j=1}^{\infty} \delta_{n+j}^{(n)} \lambda_{n+j},$$

where $\delta_m^{(n)} = 0$ if $m \notin \{n_1, n_2, \ldots\}$.

Then

$$\left(0, 0, \ldots, 0, \ -\overset{n}{1}, \ \delta_{n+1}^{(n)}, \ \delta_{n+2}^{(n)}, \ldots\right) \in \mathcal{M},$$

if $n \geq n_1$, where n_1 is a constant.

3. For every $n = 1, \ldots, n_1 - 1$ choose an arbitrary sequence $\left(e_1^{(n)}, e_2^{(n)}, \ldots\right)$ such that

(a) $e_l^{(n)} = 0$ if $l < n$,

(b) $e_u^{(n)} \neq 0$,

(c) $e_m^{(n)} \in \{-1, 0, 1\}$.

Let $\left(e_1^{(n)}, e_2^{(n)}, \dots\right) \in \mathcal{M}$.

Assertion: *Let* $\{a_n\} \in l_1$ *be a sequence for which*

$$\sum \varepsilon_n a_n = 0$$

whenever

$$\sum \varepsilon_n \lambda_n = 0$$

and $(\varepsilon_1, \varepsilon_2, \dots) \in \mathcal{M}$.

Then $a_n = c\lambda_n \quad (n \in \mathbb{N})$.

The assertion is an easy consequence of our Theorem 2.

Indeed, by using Theorem 2 for Ω, we obtain that $a_{n_j} = c\lambda_{n_j} \quad (j = 1, 2, \dots)$.

Let now $j \geq 1$ be fixed and consider the set of the integers $n \in [n_j + 1, n_{j+1} - 1]$. Since Ω is interval filling, therefore $\lambda_{n_j} \leq \lambda_{n_{j+1}} + \lambda_{n_{j+2}} + \cdots$ consequently for every n there is a suitable sequence defined in (ii).

We have $a_n = \sum \delta_{n+j}^{(n)} a_{n+j} = c \sum \delta_{n+j}^{(n)} \lambda_{n+j} = c\lambda_n$. Thus $a_n = c\lambda_n$ if $n \geq n_1$. From *(iii)*, we obtain that $a_n = c\lambda_n$ for $n = n_1 - 1,\ n_1 - 2, \dots, 1$.

The assertion is proved.

Let $\lambda_n := \Theta^n,\ \Theta \in \left(\frac{1}{2}, 1\right),\ L_0 = \dfrac{\Theta}{1 - \Theta}$. A sequence $\varepsilon_1, \dots, \varepsilon_N \in \{-1, 0, 1\}$ is said to be continuable if

$$\left|\varepsilon_1 \Theta + \cdots + \varepsilon_N \Theta^N\right| \leq \Theta^N L_0.$$

Let $t(0) = 2,\ t(\pm 1) = 1$ and

$$\tau(\varepsilon_1, \dots, \varepsilon_N) = \prod_{j=1}^{N} t(\varepsilon_j).$$

Let $m_N(\Theta) = \sum \tau(\varepsilon_1, \dots, \varepsilon_N)$, where the summation is extended over the continuable sequences. One can see easily that

$$m_N(\Theta) \geq c(4\Theta)^N, \quad c > 0.$$

Let $\mathcal{F}$ be a set of sequences $\underline{\varepsilon} = \varepsilon_1 \varepsilon_2 \dots,\ \varepsilon_\nu \in \{-1, 0, 1\}$, furthermore let $\mathcal{F}_N$ be the set of those sequences $\delta_1 \dots \delta_N \in \{-1, 0, 1\}^N$, which can be continued with suitable $\varepsilon_\nu \in \{-1, 0, 1\} \quad (\nu \geq N + 1)$ such that $\delta_1 \dots \delta_N\ \varepsilon_{N+1} \varepsilon_{N+2} \dots \in \mathcal{F}$.

Let

$$\pi_N(\Theta|\mathcal{F}) = \sum_{\delta_1 \ldots \delta_N \in \mathcal{F}_N} \tau(\delta_1, \ldots, \delta_N).$$

Conjecture 4. *If* $\underline{a} = a_1 a_2 \ldots \in l_1$ *and*

$$\sum \varepsilon_n a_n = 0 \quad \textit{whenever}$$

$\underline{\varepsilon} \in \mathcal{F}$, *and*

$$\pi_N(\Theta|\mathcal{F}) \to \infty \quad (N \to \infty)$$

then $a_n = c\Theta^n \quad (n = 1, 2, \ldots)$.

III.

Let $\Theta \in \left(\frac{1}{2}, 1\right)$, $q = 1/\Theta$, $L = \dfrac{\Theta}{1 - \Theta}$. Let $\eta \in [\Theta, \Theta L]$ and $T = T_\eta$ be the mapping $[0, L] \to [0, L]$ defined as follows.

If $x \in [0, L]$, then let

$$\varepsilon_1 = \varepsilon_1(x) = \begin{cases} 0, & \text{if } x < \eta, \\ 1, & \text{if } x \geq \eta, \end{cases}$$

and let $x_1 = Tx$ be defined from

$$x = \varepsilon_1 \Theta + \Theta x_1.$$

Continuing this process, $x_n = \varepsilon_{n+1}\Theta + \Theta x_{n+1} \quad (n = 1, 2, \ldots)$, an expansion of x

(3.1) $$x = \varepsilon_1 \Theta + \varepsilon_2 \Theta^2 + \cdots$$

is given. We say that it is a representation of level η of x.

We can see that $T\colon [0, q\eta) \to [0, q\eta)$. Let us consider the expansion of level η of $q\eta$, and η:

(3.2) $$q\eta = t_1 \Theta + t_2 \Theta^2 + \ldots, \quad \eta = \pi_1 \Theta + \pi_2 \Theta^2 + \cdots.$$

Let $\underline{t} = t_1 t_2 \ldots$, $\underline{\pi} = \pi_1 \pi_2 \ldots$.

Let

$$\mathcal{E} := \{\underline{\varepsilon}(\S) \mid \S \in [\prime, \Pi\eta)\}.$$

Let furthermore $\mathcal{F}$ be the set of those sequences $\underline{f} = f_1 f_2 \ldots \in \{0, 1\}^\infty$ for which:

(1) $\sigma^j(\underline{f}) < \underline{t} \quad (j = 0, 1, 2, \ldots)$,

(2) if $f_\nu = 1$, then
$$\sigma^{\nu-1}(\underline{f}) = f_\nu f_{\nu+1} \ldots \geq \underline{\pi}.$$

Theorem 3. *We have $\mathcal{E} = \mathcal{F}$.*

Remark. The expansion T for $\eta = \Theta$ was defined by A. Rényi [1]. W. Parry proved the relation $\mathcal{E} = \mathcal{F}$ for $\eta = \Theta$ in [2].

Proof of Theorem 3. The relation $\mathcal{E} \subseteq \mathcal{F}$ is obvious. Let $x = \varepsilon_1(x)\Theta + \cdots$, $x \in [0, q\eta]$. Since for every couples y_1, y_2, if $0 \leq y_1 < y_2 \leq q\eta$, then $\underline{\varepsilon}(y_1) < \underline{\varepsilon}(y_2)$, thus $\underline{\varepsilon}(x) < \underline{t}$. Since $x_n = \varepsilon_{n+1}(x)\Theta + \cdots < q\eta$, therefore $\sigma^n(\underline{\varepsilon}(x)) < \underline{t}$. If $\varepsilon_n(x) = 1$, then $x_{n-1} = \varepsilon_n(x)\Theta + \cdots \geq \eta$, thus $(\varepsilon_n(x), \ldots) \geq \pi$. Thus $\mathcal{E} \subseteq \mathcal{F}$ is true.

Let $\underline{f} \in \mathcal{F}$, $y : T = f_1\Theta + f_2\Theta^2 + \cdots$. We shall prove that $y \leq q\eta$ and that if $f_k = 1$, then $f_k\Theta + \cdots \geq \eta$. Hence it would follow that $\underline{\varepsilon}(y) = \underline{f}$.

Let $f_j = t_j \quad (j = 1, \ldots, k_1 - 1)$, $f_{k_1} = 0$, $t_{k_1} = 1$. Furthermore let $f_{k_1+j} = t_j$ for $(j = 1, \ldots, k_2 - 1)$, $f_{k_2} = 0$, $t_{k_2} = 1$, and so on. We allow the choice $k_\nu = 1$, when $(j = 1, \ldots, k_\nu - 1)$ is an empty condition.

Thus we have

$$y = t_1\Theta + \cdots + t_{k_1-1}\Theta^{k_1-1} + \Theta^{k_1}\left(t_1\Theta + \cdots + t_{k_2-1}\Theta^{k_2-1}\right) + + \Theta^{k_1+k_2}\left(t_1\Theta + \cdots + t_{k_3-1}\Theta^{k_3-1}\right) + \cdots . \tag{3.3}$$

If $t_k = 1$, then $t_k\Theta + t_{k+1}\Theta^2 + \cdots \geq \eta$, and so $t_1\Theta + \cdots + t_{k-1}\Theta^{k-1} \leq (q\Theta)(1 - \Theta^k)$.

From (3.3) we obtain that

$$y \leq (q\Theta)(1 - \Theta^{k_1}) + (q\Theta) \cdot \Theta^{k_1}(1 - \Theta^{k_2}) + \cdots = q\Theta.$$

The estimation from below is the same. Assume that $f_1 = \pi_1$, $f_j = \pi_j \quad j = 1, \ldots, (k_1 - 1)$, $f_{k_1} = 1$, $\pi_{k_1} = 0$, $f_{k_1+j} = \pi_j$, $j = 1, \ldots, k_2 - 1$, $f_{k_1+k_2} = 1$, $\pi_{k_2} = 0$, and so on. Then

$$y = \left(\pi_1\Theta + \cdots + \pi_{k_1-1}\Theta^{k_1-1}\right) + \Theta^{k_1-1}\left(\pi_1\Theta + \cdots + \pi_{k_2-1}\Theta^{k_2-1}\right) + + \Theta^{(k_1-1)+(k_2-1)}\left(\pi_1\Theta + \ldots + \pi_{k_3-1}\Theta^{k_3-1}\right) + \cdots$$

If k is such an integer for which $\pi_k = 0$, then $\eta = \pi_1\Theta + \cdots + \pi_{k-1}\Theta^{k-1} + \Theta^{k-1}\xi, \quad \xi < \eta$, and so

$$\pi_1\Theta + \cdots + \pi_{k-1}\Theta^{k-1} \geq \eta\left(1 - \Theta^{k-1}\right).$$

Therefore
$$y \geq \eta\left(1 - \Theta^{k_1-1}\right) + \eta\Theta^{k_1-1}\left(1 - \Theta^{k_2-1}\right) + \ldots = \eta.$$

Hence the assertion easily follows.

Theorem 4. *Let $\eta_1 < \eta_2$, $\eta_1, \eta_2 \in [\Theta, \Theta L]$. Furthermore let $\mathcal{H}(\eta_1, \eta_2)$ be the set of those $x \in [0, L]$ for which their expansions of level η_1 and of level η_2 are the same. Then the Lebesgue measure of $\mathcal{H}(\eta_1, \eta_2)$ is zero.*

We shall not prove this theorem presently.

IV.

Let $q > 1$ be a Pisot number, $\Theta = 1/q$, $k = [q]$, $\mathcal{A} = \{0, 1, \ldots, k\}$,

$$H := \left\{ \sum \varepsilon_n \Theta_n \;\middle|\; \varepsilon_n \in \mathcal{A} \right\} = [0, kL], \quad L = \frac{\Theta}{1 - \Theta}.$$

Let $\underline{\varepsilon}(x) = \varepsilon_1(x)\varepsilon_2(x)\ldots$ be the sequence of digits in the regular (that is the Rényi–Parry) expansion of $x\,(= \sum \varepsilon_n(\Theta)\Theta^n)$. Let $\underline{t} = t_1 t_2 \ldots$ be the sequence of digits in the quasi-regular expansion of 1.

The digit $\varepsilon_1(x)$ for the regular expansion of x is defined as

$$\varepsilon_1(x) = [qx],$$

while in the quasi-regular expansions by $[qx]$, if qx is not an integer, and by $qx - 1$ if it is an integer. Since q is a Pisot number, therefore $\sigma^k(\underline{t}) \quad (k = 0, 1, \ldots)$ is ultimately periodic, that is

$$\sigma^{k+p}(\underline{t}) = \sigma^k(\underline{t}) \tag{4.1}$$

holds with suitable $p > 0$, $k > 0$.

Let $\mathcal{B} = \{\lfloor_{/}, \lfloor_{\infty}, \ldots, \lfloor_{\nabla}\}$ be a set of distinct integers such that $b_0 = 0$, $-K_1 = \min b_\nu < 0$, $K_2 = \max b_\nu > 0$.

We would like to find those sequences $f_1, f_2, \ldots \in \mathcal{B}$ for which

$$O = f_1\Theta + f_2\Theta^2 + \cdots \tag{4.2}$$

holds.

Let $\gamma_0 = 0$, $\gamma_1 = -f_1$, $\gamma_j = q\gamma_{j-1} - f_j \quad (j = 1, 2, \ldots)$.

Then

$$\gamma_j = f_{j+1}\Theta + f_{j+2}\Theta^2 + \cdots \in [-K_1 L, K_2 L]. \tag{4.3}$$

The numbers γ_j are integers in $Q(q)$. Let the conjugates of q be $q = q_1$, $q_2, \ldots, q_n$. We have $|q_\nu| < 1 \quad (\nu = 2, \ldots, n)$.

Consequently,

$$\gamma_j(q_l) = -\left(f_1 q_l^{j-1} + \cdots + f_j\right), \quad |\gamma_j(q_l)| \le \frac{\max(K_1, K_2)}{1 - |q_l|}$$

$(j = 2, \ldots, n)$, $\gamma_j \in [-K_1 L, K_2 L]$.

Since the vectorials $\{\gamma_j(q_l) \mid l = 1, \ldots, n\}$ belong to a bounded domain, therefore they are taken from a finite set which is denoted by $\mathcal{F}$:

$$\mathcal{F} = \left\{\rho,\ \rho \text{ integer in } Q(q),\ |\rho(q_l)| \le \frac{\max(K_1, K_2)}{1 - |q_l|} \quad (l \ge 2), \quad \rho \in [-K_1 L, L_2 L]\right\}.$$

The construction of the graph $G(\mathcal{F})$

The edges of the graph are the elements of $\mathcal{F}$. We shall draw an edge from $\rho \in \mathcal{F}$ to $\rho q - f$ if $\rho q - f \in \mathcal{F}$. This (directed) edge is labeled with f.

It is clear that all solutions $f_1, f_2, \ldots$ of (4.2) can be getting by walking on the graph starting from 0, and noting the sequence of the labels of the graph.

By using this construction we can solve some interesting problems.

Problem. Let $\mathcal{A} = \{0, 1, \ldots, k\}$, $\underline{\varepsilon}(x)$ be the sequence of digits in the regular expansion of x. Let us determine those sequences $(\delta_1, \ldots, \delta_N) \in \mathcal{A}^N$ which can be continued appropriately, by $\delta_{N+j} \in \mathcal{A}$ $(j = 1, 2, \ldots)$ such that $x = \sum_1 \delta_\nu \Theta^\nu$.

This can be done as follows. We consider the set $\mathcal{B} = \mathcal{A} - \mathcal{A} = \{\sqcap - \sqsubseteq \mid \sqcap, \sqsubseteq \in \mathcal{A}\}$ and define $\mathcal{F}$ as earlier, then $G(\mathcal{F})$ by drawing the edge $\rho_1 \to \rho_2$, if $\rho_2 = q\rho_1 - f$. After then we delete the edge labeled with f, and substitute it with as many edges as many solutions $f = u - v$, $u, v \in \mathcal{A}$ has, and we label them with (u, v). Let $G^*(\mathcal{F})$ be this directed multigraph.

Let us walk on $G^*(\mathcal{F})$ starting from 0 and note the sequence of labels:

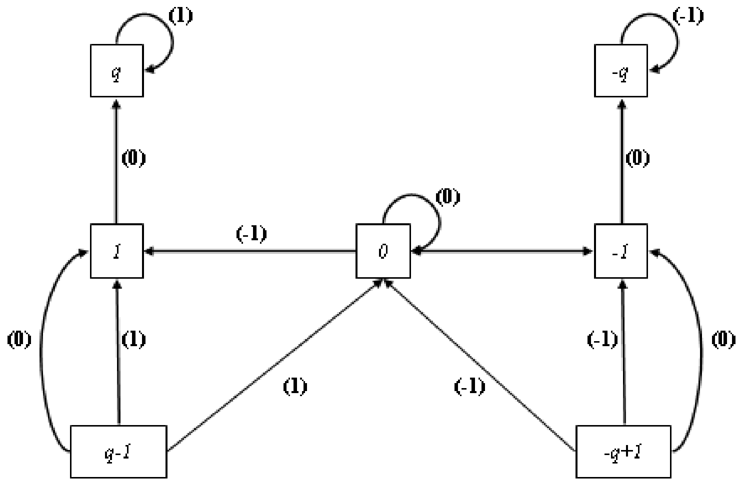
$$(u_1, v_1), (u_2, v_2), \ldots.$$

Let us consider only those routes for which $u_j = \varepsilon_j(x) \quad (j = 1, \ldots, N)$. Then the sequence of the second components will give a suitable continuable sequence $\delta_1, \ldots, \delta_N$, and all appropriate sequences can be getting on this way.

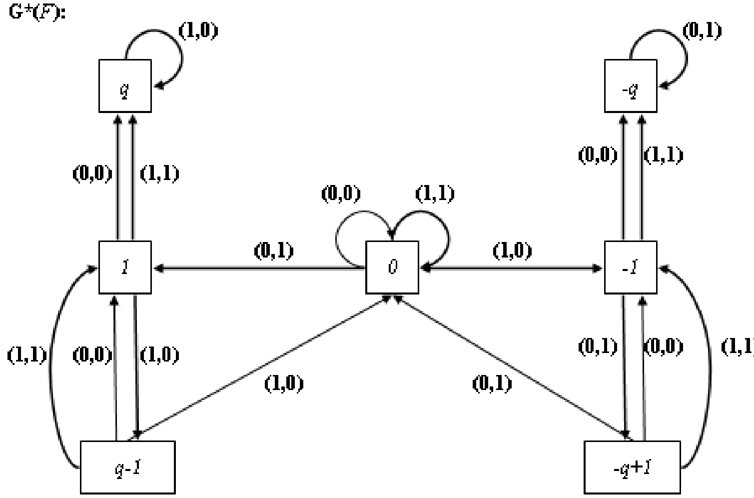
Let us see $G(\mathcal{F})$ and $G^*(\mathcal{F})$ in the simplest case

$$\Theta = \frac{\sqrt{5} - 1}{2}, \quad q = \frac{\sqrt{5} + 1}{2}, \quad \mathcal{A} = \{0, 1\}, \quad \mathcal{B} = \{-\infty, \prime, \infty\}.$$

G(*F*):


(1)
q
(-1)
-q
(0)
(0)
(0)
1
(-1)
0
-1
(0)
(1)
(-1)
(0)
(1)
(-1)
q-1
-q+1

G*(*F*):


(1,0)
q
(0,1)
-q
(0,0)
(1,1)
(0,0)
(1,1)
(0,0)
(1,1)
1
(0,1)
0
(1,0)
-1
(1,1)
(0,0)
(1,0)
(0,1)
(0,0)
(1,1)
(1,0)
(0,1)
q-1
-q+1

V.

Let $f\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative arithmetical function, $|f(n)| = 1 \quad (n \in \mathbb{N})$, and let $\delta_f(n) = f(n+1)\overline{f}(n)$.

E. Wirsing proved in 1984 that if $\delta_f(n) \to 1 \quad (n \to \infty)$, then $f(n) = n^{i\tau}$ [13], [14].

Daróczy and I proved the following assertion [15].

If G is a compact Abelian group, $f\colon \mathbb{N} \to G$ is completely additive, i.e. $f(mn) = f(m) + f(n)$ for every $m, n \in \mathbb{N}$, and $f(n+1) - f(n) \to 0 \quad (n \to \infty)$, then there is a continuous homomorphism $\Phi\colon \mathbb{R}_x \to G$ such that

$$f(n) = \phi(n) \quad (n \in \mathbb{N}).$$

Conjecture 5. *Let G be a compact Abelian group, $f\colon \mathbb{N} \to G$ be completely additive, and closure $f(\mathbb{N}) = G$ (closure $f(\mathbb{N})$ always is a closed subgroup in G). Let U be the set of those u for which there exists an infinite sequence of integers $n_\nu \nearrow$, such that $f(n_\nu + 1) - f(n_\nu) \to u$.*

Then U is a subspace in G, furthermore

$$f(n) := \Phi(n) + V(n),$$

where Φ is a continuous homomorphism, $\phi\colon \mathbb{R}_x \to G$, $V(\mathbb{N}) \subseteq U$, clos $V(\mathbb{N}) = U$.

We formulate our conjecture for complex valued completely multiplicative functions.

Conjecture 6. *Let f be completely multiplicative, $|f(n)| = 1 \quad (n \in \mathbb{N})$, $\delta_f(n) = f(n+1)\overline{f}(n)$. Let $\mathcal{A}_k = \{\alpha_1, \ldots, \alpha_k\}$ be the set of limit points of $\{\delta_f(n) \mid n = 1, 2, \ldots\}$. Then $\mathcal{A}_k = \{w | w^k = 1\}$, furthermore $f(n) = n^{i\tau} F(n)$, and*

(i) $$F(\mathbb{N}) = \mathcal{A}_k,$$

(ii) for every $w \in \mathcal{A}_k$ there is some infinite sequence n_ν such that $F(n_\nu + 1)\overline{F}(n_\nu) = w \quad (\nu = 1, 2, \ldots)$.

A weaker conjecture, namely that under the conditions of Conjecture 6 there is an s such that $F(\mathbb{N}) = \{\omega \mid \omega^s = 1\}$, was proved by E. Wirsing [18] in his brilliant paper.

VI.

Let $\mathcal{P}_k$ be the set of integers $n = p_1 \cdots p_k$ where $p_1, \ldots, p_k$ are distinct primes. Let α be a fixed irrational number. Let $e(\beta) := e^{2\pi i\beta}$. Let $q_1 < q_2 < \cdots < q_r$

be the whole sequence of the primes up to x. Let X_{q_j} $(j=1,\dots,r)$ be complex numbers,

$$Q_k(X_{q_1},\dots,X_{q_r}) := \left|\sum_{\substack{n\in\mathcal{P}_k\\ n=p_1\dots p_k<x}} X_{p_1}\cdots X_{p_k} e(n\alpha)\right|.$$

Let us define

$$\delta_k(x) = \max_{|X_{q_1}|\le 1,\dots,|X_{q_r}|\le 1} \frac{Q_k(X_{q_1},\dots,X_{q_r})}{\pi_k(x)},$$

$$\delta_k = \limsup_{x\to\infty} \delta_k(x).$$

Conjecture 7. *We have $\delta_k<1$ if $k\ge 2$. Furthermore $\delta_k\to 0$ $(k\to\infty)$.*

H. Daboussi proved several years ago that for every irrational α, for every multiplicative function f, such that $|f(n)|\le 1$ $(n\in\mathbb{N})$, the relation

$$\frac{1}{x}\left|\sum_{n\le x} f(n)e(n\alpha)\right| \to 0 \quad (x\to\infty).$$

The order of the convergence may depend on α, but does not depend on f. In our recent paper written jointly with Indlekofer [19] we proved:

If α is irrational, $w(n)$ is the number of the prime divisors of n, $\tilde{\mathcal{P}}_k = \{n \mid w(n)=k\}$, $\tilde\pi_k(x) = \#\left\{\tilde{\mathcal{P}}_k(x)\cap[1,x]\right\}$, $\eta>0$ is a small constant, then uniformly for multiplicative functions f restricted by the conditions $|f(n)|\le 1$ $(n\in\mathbb{N})$ we have

$$\max_k \frac{1}{\tilde\pi_k(x)}\left|\sum_{\substack{n\le x\\ n\in\tilde{\mathcal{P}}_K}} f(n)e(n\alpha)\right| \to 0 \quad \text{as} \quad \eta<\frac{k}{x_2}<2-\eta \qquad x\to\infty.$$

I hope that Conjecture 7 is true.

References

[1] Rényi, A., Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.*, **8** (1957), 477–493.

[2] Parry, W., On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.*, **11** (1960), 401–416.

[3] Daróczy, Z., Járai, A. and Kátai, I., Intervalfüllende Folgen und volladditive Funktionen, *Acta Sci. Math. (Szeged)*, **50** (1986), 337–350.

[4] Daróczy, Z. and Kátai, I., Additive functions, *Analysis Math.*, **12**, (1986), 85–96.

[5] Daróczy, Z. and Kátai, I., Interval filling sequences and additive functions, *Acta Sci. Math. (Szeged)*, **52**, (1988), 337–347.

[6] Daróczy, Z., Járai, A. and Kátai, I., Interval filling sequences, *Annales Univ. Eötvös L. Sectio Computatorica*, **6** (1985), 53–63.

[7] Daróczy, Z. and Kátai, I., On differentiable additive funcitons, *Annales Univ. Eötvös L., Sectio Computatorica*, **6** (1985), 53–63.

[8] Daróczy, Z. and Kátai, I., Continuous additive functions and difference equations of infinite order, *Analysis Math.*, **12** (1986), 237–249.

[9] Daróczy, Z. and Kátai, I., On functions additive with respect to interval filling sequences, *Acta Math. Hungar.*, **51** (1988), 185–200.

[10] Daróczy, Z. and Kátai, I., Univoque sequences, *Publ. Math. Debrecen*, **42**, (1993), 397–407.

[11] Daróczy, Z. and Kátai, I., On the structure of univoque numbers, *Publ. Math. Debrecen*, **46** (1995), 385–408.

[12] Daróczy, Z., Kátai, I. and Szabó, T., On completely additive functions related to interval filling sequences, *Archiv Math.*, **54** (1990), 173–179.

[13] Wirsing, E., *A proof is given in a letter to me*, 1984.

[14] Wirsing, E., Tang Yuansheng and Shao Pintsung, On a conjecture of Kátai for additive functions, *J. Number Theory*, **56** (1996), 391–395.

[15] Daróczy, Z. and Kátai, I., On additive arithmetical functions with values in topological groups, *Publ. Math. Debrecen*, **33** (1986), 287–292.

[16] Kátai, I. and Subbarao, M. V., The characterization of $n^{i\tau}$ as a multiplicative function, *Acta Math. Hungar.*

[17] Kátai, I. and Subbarao, M. V., On the multiplicative function $n^{i\tau}$, *Studia Sci. Math.*, **34** (1998), 211–218.

[18] Wirsing, E., *On a problem of Kátai and Subbarao*, (unpublished manuscript).

[19] Indlekofer, K.-H. and Kátai, I., *A note on a theorem of Daboussi*, (manuscript).

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