



Professor Péter Kiss (1937-2002)

PÉTER KISS AND THE LINEAR RECURSIVE SEQUENCES

Kálmán Liptai, Ferenc Mátyás (Eger, Hungary)

Dedicated to the memory of Professor Péter Kiss

Péter Kiss was born in Nagyréde in 1937. He attended secondary school in Gyöngyös and in 1955 he entered the Eötvös Lóránd University Faculty of Science in Budapest. He took his teacher's diploma in mathematics and physics. After finishing university, he taught at the Gárdonyi Géza Secondary School in Eger for 12 years.

He began to teach at what is now called the Eszterházy Károly College at the Department of Mathematics in 1972 and taught there until his death in 2002. He took a special interest in Number Theory. His doctoral thesis "Second order linear recurrence and pseudoprime numbers" was submitted in 1977. He obtained the candidate's degree in 1980, the title of his dissertation was "Second order linear recursive sequences and their applications in diophantine problems". In 1995 Péter Kiss habilitated at the Kossuth Lajos University of Debrecen and he was inaugurated as professor. He got the Szent-Györgyi Albert prize in 1997. He got the title of doctor of mathematical science of Hungarian Academy of Sciences in 1999.

His lectures were lucid and meticulously crafted and through him many of his students grew to like mathematics and research. He brought into existence a research group in Number Theory and supported the work of his inquiring students and colleagues. One of his students, Bui Minh Phong, was awarded the Rényi Kató prize in 1976. He was the supervisor of the doctoral theses of the following colleagues: Ferenc Mátyás, Sándor Molnár, Béla Zay, Kálmán Liptai, László Szalay, and helped Bui Minh Phong, László Gerőcs and Pham Van Chung in writing of their theses.

He took an enthusiastic part in the everyday world of mathematics. He held several county and national posts in the János Bolyai Mathematical Society. He was a contributor to the abstracting journals *Mathematical Reviews* and *Zentralblatt für Mathematik* and he was also a permanent member of organizing committee of the Fibonacci Conference. He was a highly respected member of the community of mathematicians. This was proved by many joint papers, invitations to conferences and friends all over the world.

This paper is devoted to the summary of his academic achievements.

1. Introduction

In 1202 Leonardo Pisano, or Fibonacci, employed the recurring sequence $1, 2, 3, 5, 8, 13, \dots$ in a problem on the number of offspring of a pair of rabbits. Let's denote by F_n and F_{n+1} the n -th and $(n+1)$ -th term of this sequence, respectively. In this case $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$. Simple generalizations of the Fibonacci sequence are the second order linear recurrences. The sequence $\{R_n\}_{n=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence if the recurrence relation

$$R_n = AR_{n-1} + BR_{n-2} \quad (n > 1)$$

holds for its terms, where $A, B \neq 0, R_0$ and R_1 are fixed rational integers and $|R_0| + |R_1| > 0$. The sequence $R(A, B, 2, A)$ is called the associate sequence of the sequence $R(A, B, 0, 1)$.

The polynomial $x^2 - Ax - B$ is called the companion polynomial of the second order linear recurrence $R = R(A, B, R_0, R_1)$. The zeros of the companion polynomial will be denoted by α and β . In the sequel we assume that the sequence is not degenerate, i.e. α/β is not a root of unity, and we order α and β so that $|\alpha| \geq |\beta|$. Using this notation, we get that

$$R_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta},$$

where $a = R_1 - R_0\beta$ and $b = R_1 - R_0\alpha$.

Consider now a generalization of second order linear recurrences.

The sequence $G(A_1, A_2, \dots, A_k, G_0, G_1, \dots, G_{k-1}) = \{G_n\}_{n=0}^{\infty}$ is called a k -th order linear recursive sequence of rational integers if

$$G_n = A_1G_{n-1} + A_2G_{n-2} + \dots + A_kG_{n-k} \quad (n > k - 1),$$

for certain fixed rational integers A_1, A_2, \dots, A_k with $A_k \neq 0$ and G_0, G_1, \dots, G_{k-1} not all zero. The companion polynomial of a recurrence with coefficients A_1, A_2, \dots, A_k is given by $x^k - A_1x^{k-1} - A_2x^{k-2} - \dots - A_k$. Denote by $\alpha = \alpha_1, \alpha_2, \dots, \alpha_s$ the distinct zeros of the companion polynomial. Assume that $\alpha, \alpha_2, \dots, \alpha_s$ has multiplicity $1, m_2, \dots, m_s$ respectively and that $|\alpha| > |\alpha_i|$ for $i = 2, \dots, s$. The zero α is called the dominating root of the polynomial. It is known that in this case the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \dots + r_s(n)\alpha_s^n \quad (n \geq 0),$$

where the r_i 's ($i = 2, \dots, s$) are polynomials of degree $m_i - 1$ and the coefficients of these polynomials as well as a are elements of the algebraic number field $\mathbb{Q}(\alpha, \alpha_2, \dots, \alpha_s)$.

2. Common terms and difference of the terms of linear recurrences

Let $G(A_1, \dots, A_k, G_0, \dots, G_{k-1})$ and $H(B_1, \dots, B_r, H_0, \dots, H_{r-1})$ be linear recurrence sequences having dominating roots. Let $p_1 < p_2 < \dots < p_s$ be different primes and denote by S the set of rational integers which have only these primes as prime factors. We suppose that $1 \in S$.

M. Mignotte (1978) studied the common terms of linear recurrences, that is, the equation

$$G_x = H_y.$$

P. Kiss proved the following theorem in [19].

Theorem 2.1. *Let G and H be linear recurrence sequences with dominating roots α and β , respectively. In this case*

$$G_n = a\alpha^n + g_2(n)\alpha_2^n + \dots + g_s(n)\alpha_s^n,$$

and

$$H_n = b\beta^n + q_2(n)\beta_2^n + \dots + q_t(n)\beta_t^n.$$

We suppose that $G_i \neq a\alpha^i$, $H_j \neq b\beta^j$ and $s_1a\alpha^i \neq s_2b\beta^j$ for any $s_1, s_2 \in S$ if $\max(i, j) > n_0$. If

$$s_1G_x = s_2H_y$$

for some $s_1, s_2 \in S$, then $\max(x, y) < n_1$, where n_1 is effectively computable and depends on S, n_0 and the parameters of the sequences G and H .

P. Erdős asked whether the terms of the recurrence sequences could be close to each other. P. Kiss answered this question in [30].

Theorem 2.2. *Suppose that G and H are linear recurrences satisfying the conditions of Theorem 2.1. Then for any integers $s_1, s_2 \in S$*

$$||s_1G_x| - |s_2H_y|| > \exp\{c \cdot \max(x, y)\}$$

for all integers $x, y > n_2$, where c and n_2 are effectively computable positive numbers depending only on S, n_0 and the parameters of G and H .

P. Kiss generalized a result of Shorey and Stewart in [30].

Theorem 2.3. *Let G be a linear recurrence sequence satisfying the conditions of Theorem 2.1. If*

$$sx^q = G_n$$

for some positive integers $s \in S, q, n$ and $x > 1$, then $q < n_3$, where n_3 is an effectively computable positive number depending only on S, n_0 and the parameters of G .

A similar result was proved in the same paper.

Theorem 2.4. *Let G be a linear recurrence sequence as in Theorem 2.1. Furthermore assume that $k > 2$, $|\alpha_2| \neq 1$, $|\alpha_2| > |\alpha_3| \geq |\alpha_j|$ ($j > 3$) and $g_2(i) \neq 0$, if $i > n_0$. Then*

$$|sx^q - G_n| > e^{cn}$$

for all positive integers $s \in S, x, q, n$ and with $q, n > n_4$, where n_4 is an effectively computable positive number depending only on S, n_0 and the parameters of G .

3. Prime divisors of second order linear recurrences

Let $R(A, B, 0, 1)$ be a non-degenerate second order linear recurrence sequence where $R_0 = 0, R_1 = 1$ and $(A, B) = 1$. If p is a prime with $p|B$, then there are terms R_n of R (different from $R_0 = 0$) which are divisible by p . The least index of these terms is called the rank of apparition of p in the sequence R and is denoted by $r(p)$. Thus $p | R_{r(p)}$, but $p \nmid R_m$ if $0 < m < r(p)$. If $r(p) = n$, then we say that p is a primitive divisor of R_n . If p is a primitive divisor of R_n and $p^k | R_n$ ($k \geq 1$), but $p^{k+1} \nmid R_n$, then we say p^k is a primitive prime power divisor of R_n . P. Kiss proved the following theorem in [36].

Theorem 3.1. *Let \mathcal{R}_n be the product of primitive prime power divisors of R_n . Then*

$$\sum_{n \leq x} \log \mathcal{R}_n = \frac{3 \cdot \log |\alpha|}{\pi^2} x^2 + O(x \log x),$$

provided that x sufficiently large. (The constant involved in $O()$ depends on the parameters of the sequence.)

In the joint paper [45] P. Kiss and B. M. Phong studied the reciprocal sum of primitive prime divisors of the terms of second order linear recurrences. To formulate this, let $R(A, B, 0, 1)$ be a second order linear recurrence and

$$p(n) = \sum_{r(p)=n} \frac{1}{p}$$

the reciprocal sum of primitive prime divisors of R_n ($n > 0$), ($p(n) = 0$, if there is no primitive prime divisor of R_n). Furthermore let

$$f(n) = \sum_{p|R_n} \frac{1}{p}$$

be the reciprocal sum of all prime divisors of the term R_n , ($f(n) = 0$, if there is no prime divisor). Using this notation they proved that

$$f(n) < \log \log \log n + c$$

for sufficiently large n . This is the best possible result apart from the constant c .

The average of the previous functions was studied in Kiss [47]. The main results are the following.

Theorem 3.2. *There exists a constant $c > 0$ depends on the sequence R such that*

$$\sum_{n \leq x} f(n) = cx + O(\log \log x)$$

for sufficiently large x .

Theorem 3.3. *There exists an absolute constant $c > 0$ such that*

$$p(n) < c \frac{(\log \log n)^2}{n}$$

for sufficiently large n . Furthermore

$$\sum_{n \leq x} p(n) = \sum_{r(p) \leq x} \frac{1}{p} = \log \log x + O(1).$$

4. Approximation problems

Let $G(A, B, R_0, R_1)$ be a nondegenerate second order linear recurrence, and $D = A^2 + 4B$ denote the discriminant of its companion polynomial. If $D > 0$ then the quotient R_{n+1}/R_n is a convergent of the irrational number α . The sharpness of the convergent was studied in Kiss [16].

Theorem 4.1. *Suppose that $D > 0$, $G_0 = 0$, $G_1 = 1$ and that α is an irrational number. Then the inequality*

$$\left| \alpha - \frac{G_{n+1}}{G_n} \right| < \frac{1}{c \cdot G_n^2}$$

holds for some $c > 0$ and infinitely many n if and only if $|B| = 1$ and $c \leq \sqrt{D}$. Moreover if $|B| = 1$ and the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{D}q^2}$$

holds for some rational number p/q then $p/q = G_{n+1}/G_n$ for some positive integer n .

In general G_{n+1}/G_n is a weaker convergent of α . In the joint paper [55] P. Kiss and Zs. Sinka proved the following theorem.

Theorem 4.2. *Let G be a non-degenerate second order linear recurrence with $D > 0$. Define the numbers k_0 and c_0 by*

$$k_0 = 2 - \frac{\log |B|}{\log |\alpha|} \quad \text{and} \quad c_0 = \frac{\sqrt{D}^{k_0-1}}{|a^{k_0-1}b|}$$

and let k and c positive real numbers (a and b were defined in the introduction). Then

$$\left| \alpha - \frac{G_{n+1}}{G_n} \right| < \frac{1}{cG_n^k}$$

holds for infinitely many integer n if and only if $k < k_0$ and c is arbitrary, or $k = k_0$ and $c < c_0$, or $k = k_0$, $c = c_0$ and $B > 0$, or $k = k_0$, $c = c_0$, $B < 0$ and $b/a > 0$.

P. Kiss and R. F. Tichy [39], [40] have dealt with the convergent of $|\alpha|$ by rational numbers of the forms $\left| \frac{G_{n+1}}{G_n} \right|$.

Theorem 4.3. *Let G be a non-degenerate second order linear recurrence. If $D < 0$ then there is a positive number c , depending only on the parameters of the sequence G , such that*

$$\left| |\alpha| - \left| \frac{G_{n+1}}{G_n} \right| \right| < \frac{1}{n^c}$$

for infinitely many n .

Furthermore they showed that apart from the constant c , it is the best possible approximation.

Theorem 4.4. *Let G be a non-degenerate second order linear recurrence. If $D < 0$ then there is a positive number c' , such that*

$$\left| |\alpha| - \left| \frac{G_{n+1}}{G_n} \right| \right| > \frac{1}{n^{c'}}$$

for any sufficiently large n .

For the Fibonacci sequence Y. V. Matijasevich and R. K. Guy proved that

$$\lim_{n \rightarrow \infty} \sqrt{\frac{6 \cdot \log(F_1 \cdot F_2 \cdots F_n)}{\log[F_1, F_2, \dots, F_n]}} = \pi.$$

In the joint paper [38] P. Kiss and F. Mátyás generalized this result. They showed that the Fibonacci sequence can be replaced by any non-degenerate second order linear recurrence sequence G with $G_0 = 0$, $G_1 = 1$ and $(A, B) = 1$. Using a Baker type result, they also gave an error term of the form $O(1/\log n)$.

5. Recursive sequences and diophantine equations

The equation

$$x^2 - Dy^2 = N,$$

with given integers D and N and variables x and y , is called Pell's equation. If D is negative, it can have only a finite number of rational integer solutions. If D is a perfect square, say $D = a^2$, the equation reduces to

$$(x - ay)(x + ay) = N$$

and again there are only a finite number of solutions. The most interesting case arises when D is a positive integer and not a perfect square.

In [8] P. Kiss and F. Várnai proved that the solutions (x, y) of the equation

$$x^2 - 2y^2 = N$$

can be given with the help of terms of finitely many second order linear recurrences $P(2, 1, P_0, P_1)$, such that

$$(x, y) = (\pm(P_{2n} + P_{2n+1}), \pm P_{2n+1}).$$

P. Kiss [25] generalized this result in the following form.

Theorem 5.1. *If the equation*

$$x^2 - (a^2 + 1)y^2 = N$$

has a solution for a fixed integer $a > 0$, then all solutions (x, y) can be given with the help of finitely many linear recurring sequences $G(2a, -1, G_0, G_1)$ such that

$$(x, y) = (\pm(G_{2n} + aG_{2n+1}) \pm G_{2n+1}),$$

where

$$0 \leq G_1 < 2a\sqrt{N} \quad \text{for } N > 0$$

and

$$0 \leq G_1 < (2a^2 + 1)\sqrt{\frac{-N}{a^2 + 1}} \quad \text{for } N < 0.$$

In the same paper P. Kiss proved the following theorem.

Theorem 5.2. *If the equation*

$$x^2 - (a^2 - 4)y^2 = 4N$$

has a solution for a fixed integer $a > 0$, then all solutions (x, y) can be given with the help of finitely many second order linear recurring sequences $G(a, -1, G_0, G_1)$ such that

$$(x, y) = (\pm H_{2n}, \pm G_{2n}),$$

where H is the associate sequence of G and

$$0 \leq G_1 < \sqrt{N} \quad \text{for } N > 0$$

and

$$0 \leq G_1 < a\sqrt{\frac{-N}{a^2 - 4}} \quad \text{for } N < 0.$$

In their joint paper [77] P. Kiss and K. Liptai found relationships between Fibonacci numbers and solutions of special diophantine equations.

Theorem 5.3. All positive integer solutions of the equation

$$x^2 + x(y - 1) - y^2 = 0$$

are of the form

$$(x, y) = (F_{2h+1}^2, F_{2h+1}F_{2h+2}),$$

where F_i is the i -th Fibonacci number.

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Kálmán Liptai

Department of Applied Mathematics
Károly Eszterházy College
H-3301, Eger, P.O. Box 43.
Hungary
e-mail: liptaik@ektf.hu

Ferenc Mátyás

Department of Mathematics
Károly Eszterházy College
H-3301, Eger, P.O. Box 43.
Hungary
e-mail: matyas@ektf.hu