

ON SOME SPECIAL FINSLER METRICS
IN PSYCHOMETRY

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Dedicated to the memory of Professor Péter Kiss

Abstract. An expansive use of Finsler metrics can be observed in physics, biology, geology, financial mathematics. It is a great improvement for us dealing with Finsler geometry to know that Finsler metrics can be applied even in psychology. The aim of this present paper is to show some Finsler metrics being important even in applications such as Hilbert metric. These classical Finsler metrics have been formulated since the beginning of 1900 and they are even projects of current research, too. Since the book entitled “An Introduction to Riemann–Finsler Geometry” (Springer-Verlag, 2000) written by D. Bao, S. S. Chern and Z. Shen was published, the previous names of concepts of Finsler geometry and Finsler metric were replaced by Riemann–Finsler geometry and Riemann–Finsler metric.

1. Introduction

First of all let us give the concept of Riemann-Finsler metric.

Definition 1. Let an n -dimensional differentiable manifold M be given with a tangent space $T_x M$ in the point (x^i) ($i = 1, 2, \dots, n$) of M . Let us denote the coordinates of vectors of $T_x M$ by (y^i) . The function $L(x, y): TM (= \bigcup_x T_x M) \rightarrow \mathbf{R}$ is Riemann–Finsler metric, if the following properties hold:

- (1) **Regularity:** $L(x, y)$ is a function C^∞ on the manifold $TM \setminus O$ of nonzero tangent vectors.
- (2) **Positive homogeneity:** $L(x, \lambda y) = \lambda L(x, y)$ for all $\lambda > 0$.
- (3) **Strong convexity:** the $n \times n$ matrix $g_{ij}(x, y) = \frac{\partial^2 L^2}{\partial y^i \partial y^j}(x, y)$ is positive definit at every $y \neq 0$.

Remark. In some situations, the Riemann–Finsler metric $L(x, y)$ satisfies the criterion $L(x, y) = L(x, -y)$. In general, we consider this property to be too restrictive. In Example 1 we present an original Finsler metric, which has not this symmetric property.

This definition mentioned above can be found in the doctoral dissertation of Paul Finsler “Über Kurven und Flächen in allgemeinen Räumen”, 1918, Göttingen.

Essentially the same definition was given by Riemann in his famous habilitation dissertation “Über die Hypothesen, welche der Geometrie zugrund liegen”, 1854.

Since this definition was considered to be too general in determining the tensor of curvature, Riemann chose a well-known special case

$$L^2(x, y) = g_{ij}(x)y^i y^j,$$

and he stated “we will now stick to the case ellipsoids (quadratic forms), because if not, the computation would become very complicated”.

An American–Chinese professor Shiing-Shen Chern who is one of the living geometers with the most significant scientific achievements in differential geometry denies Riemann’s statement. He wrote in his latest two papers where he pointed out:

“In fact, the general case is just as simple and a main point went unnoticed by Riemann and his successors” [1].

“I believe a major part of differential geometry in the 21th century should be Riemann–Finsler geometry” [2].

2. Randers metrics

It is not difficult to construct an non-trivial (i.e. non-Riemannian) Riemann–Finsler metric. G. Randers studied the following metric in 1941:

$$L(x, y) = \alpha(x, y) + \beta(x, y)$$

where $\alpha^2(a, y) = a_{ij}(x)y^i y^j$ is a Riemann metric, $\beta(x, y) = b_i(x)y^i$ is an 1-form [3].

We can illustrate this metric in two-dimensional case in the following way:

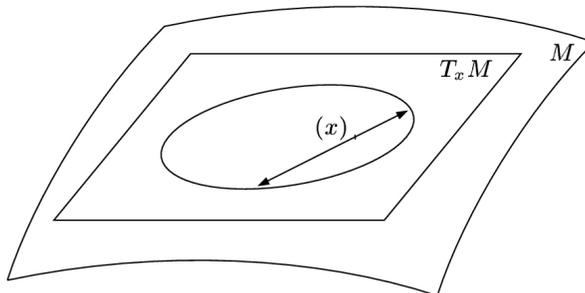


Figure 1

In the tangent space $T_x M$ the indicatrix is an ellipse whose focus is the origin. So we get an original Riemann–Finsler metric, where

$$L(x, y) \neq L(x, -y).$$

We can consider the generalization of a Randers metric as Funk metric from which the Hilbert metric can be derived.

3. Funk distance function

Let \mathbf{E}^n be an n -dimensional Euclidean space, and D be a strictly convex domain in \mathbf{E}^n , and in \mathbf{E}^n let ∂D denote the border of D .

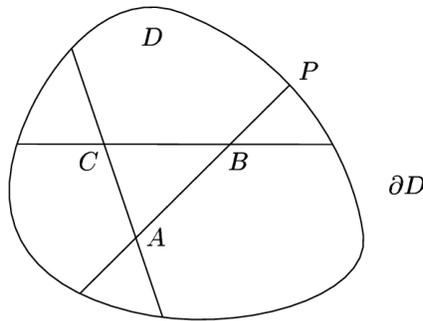


Figure 2

Consider two arbitrary points A and B of D and let the line $|AB|$ meet ∂D in a point P and let the order of the points be A, B, P .

Definition 2. ([4]) Given a positive constant k the **Funk distance function** $f(A, B)$ can be defined as follows:

$$f(A, B) = \frac{1}{k} \log(AP/BP)$$

where AP and BP denote Euclidean distances.

From this definition it follows that the Funk distance function has the properties:

- (1) $f(A, B) \geq 0$ for every two points A and B of D ;
- (2) $f(A, B) = 0$ if and only if $A = B$;
- (3) $f(A, B) + f(B, C) \geq f(A, C)$ holds for every three points A, B, C of D . Equality holds if and only if B is on the line $|AC|$;
- (4) Generally $f(A, B) \neq f(B, A)$, but $f(A, A_n) \rightarrow 0$ if and only if $f(A_n, A) \rightarrow 0$.

4. Hilbert distance function

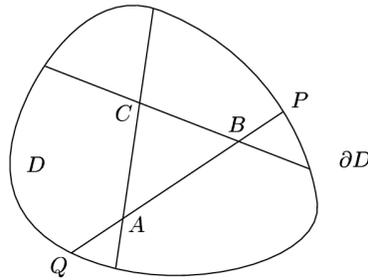


Figure 3

Definition 3. ([5]) The Hilbert distance function is obtained by the symmetrisation of the Funk distance function:

$$h(A, B) = \frac{1}{2} \{f(A, B) + f(B, A)\} = \frac{1}{2k} \log(AP/BP \times BQ/AQ).$$

Here the line $|AB|$ meets the border of D in the points P and Q and the order of the points is Q, A, B, P .

The Hilbert distance function has the following properties:

- (1) $h(A, B) \geq 0$ for every two points A and B of D ;
- (2) $h(A, B) = 0$ if and only if $A = B$;
- (3) $h(A, B) + h(B, C) \geq h(A, C)$ holds for every three points A, B, C of the domain D . Equality holds if and only if the point B is on the line $|AC|$ provided if D is strictly convex;
- (4) $h(A, B) = h(B, A)$.

5. Funk and Hilbert metrics

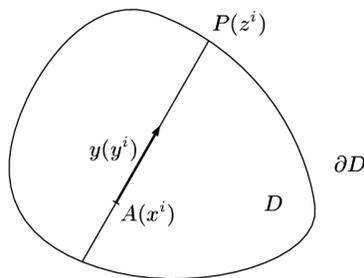


Figure 4

Let (x^i) be the coordinates of the point A , (y^j) be the coordinates of the vector $y \neq 0$. Let us define the function $r(x^i, y^i)$ by the following equality

$$r(x, y) = AP/\|y\|,$$

where AP is an Euclidean distance, $\|y\|$ is the Euclidean norm of the vector y .

The function $r(x, y)$ has the following properties:

- (1) $r(x, y) > 0$ for every pair (x, y) ;
- (2) $r(x, y)$ is of degree (-1) positively homogeneous in y ;
- (3) If $\partial D = \{z^i : \phi(z^i) = 0\}$ then $\phi(x^i + ry^i) = 0$. Namely, if we denote the coordinates of P by (z^i) then $z^i = x^i + r(x, y)y^i$;
- (4) $r(x, y) \in C^\infty$.

Definition 4. [6] $L_f = \frac{1}{kr(x, y)}$ and $L_h = \frac{1}{2k[r(x, y) + r(x, -y)]}$ are Funk metric and Hilbert metric respectively.

Theorem 5. 1. [7] *The Funk metric and the Hilbert metric are original Riemann-Finsler metrics.*

Theorem 5. 2. [8] *The Funk space (D, L_f) and Hilbert space (D, L_h) have constant curvatures with the values $(-k^2/4)$ and $(-k^2)$ respectively.*

An interesting special case follows:

Let the border ∂D of the strictly convex domain D be given by a curve of second order, which is non-degenerated as follows:

$$\partial D : \varphi(z^i) = 0,$$

where

$$\varphi(z^i) = b_{ij}z^iz^j + c_iz^i + d, b_{ij} = b_{ji}.$$

then $L_f = \frac{1}{k}[(a_{ij}(x)y^iy^j)^{d\frac{1}{2}} + b_i(x)y^i]$ and $L_h = \frac{1}{2}[a_{ij}(x)y^iy^j]^{d\frac{1}{2}}$.

So in this case (D, L_f) is a Randers space with a negative constant curvature $(-k^2/4)$, and (D, L_h) is a Riemannian space with a negative constant curvature $(-k^2)$. If D is a unit circle then (D, L_h) gives the well-known Klein model of the hyperbolic space.

6. Some application in physics, in biology and in psychology

Let $\mathbf{R}^n = (M, \alpha)$ be an n -dimensional Riemannian space with a Riemannian metric α , $\alpha^2 = a_{ij}(x)y^iy^j$, and with a differential one-form $\beta = b_i(x)y^i$ on M .

Definition 5. An (α, β) -metric is a Finsler metric $L(\alpha, \beta)$ on M which is a positively homogeneous function of degree one the arguments (α, β) .

The Randers metric $L = \alpha + \beta$ and the Kropina metric $L = \alpha^2/\beta$ have played a central role in the theory of (α, β) -metrics, and have been the bases of various branches of theoretical physics [9].

In biology there are a lot of Finsler metrics which are suitable to describe biological models and now we intend to show only one of them which arises in coral reef ecology:

If we consider the following local coordinate system in two-dimensional case $\underline{x} = (x^1, x^2) = (x, y)$ and $\underline{u} = (y^1, y^2) = (h, v)$, then this metric has the following form

$$L(x, y, h, v) = e^{\phi(x, y)} N(h, v),$$

where N is a special Minkowski metric (the main scalar is constant, which is a very restrictive condition for the metric) [10].

This metric is very similar to the metric, which is used in psychometry when a psychometric function has radial symmetry. Then the applicable Finsler metric is of the following form:

$$(\star) F(\underline{x}, \underline{u}) = \xi(\underline{x})|\underline{u}|,$$

where $\xi(\underline{x}) > 0$ and $|\underline{u}|$ denotes the Minkowski norm [11], [12].

This type of Finsler metrics are called conform Minkowski, or conform flat metrics. Properties of this type of metrics are being worked out presently. Consider the following Finsler metrics with the property (\star) which are defined as in the paper mentioned above:

- (1) $F(\underline{x}, \underline{u}) = e^{ax+by} \sqrt[4]{h^4 + v^4 + h^2v^2}$
- (2) $F(\underline{x}, \underline{u}) = e^{cxy} \sqrt[4]{h^4 + v^4 + h^2v^2}$
- (3) $F(\underline{x}, \underline{u}) = e^{ax+by} \sqrt[4]{(h^2 + v^2 + hv)(h^2 + v^2)}$
- (4) $F(\underline{x}, \underline{u}) = e^{cxy} \sqrt[4]{(h^2 + v^2 + hv)(h^2 + v^2)}$, where $a, b, c \in \mathbf{R}$ are constants.

The Gaussian curvature of the first of these two dimensional Finsler metrics is as follows [13]

$$\begin{aligned} & -72\sqrt{u^4 + v^4 + u^2v^2}(4b^2u^{14} - 4abvu^{13} - 40u^{12}b^2v^2 + u^{12}a^2v^2 + 34av^3u^{11} \\ & - 83u^{10}b^2v^4 - 7u^{10}a^2v^4 + 146av^5bu^9 - 50u^8a^2v^6 - 95u^8b^2v^6 + 188av^7bu^7 \\ & - 50u^6b^2v^38 - 95u^6a^2v^8 + 146av^9bu^5 - 7u^4b^2v^{10} - 83u^4a^2v^{10} + 34av^{11}bu^3 \\ & + u^2b^2v^{12} - 40u^2a^2v^{12} - 4av^{13}bu + 4a^2v^{14})e^{(-2\alpha x - 2by)}/(2u^4 + 11u^2v^2 + 2v^4)^4. \end{aligned}$$

The Gaussian curvature of the others is much more complicated.

It would be interesting studied under what conditions a Randers metric applied in so many fields could be applied in psychometry as Finsler metric. One can even examine under what conditions a Randers metric is conform flat (conform Minkowski). This means a rather complicated examination. A necessary

and sufficient condition is known for a Randers metric to be conform Minkowski. Meanwhile this result is too complicated in respect of applications.

Determining the differential equations of the geodetics of the metrics mentioned above could provide an important problem ($n = 2$).

It may be interesting to examine under what conditions a Finsler metric applied in psychometry is of Douglas type. That is, it occurs if and only if the differential equations of the geodetics ($y = y(x)$) in two dimension is as follows:

$$y'' = \frac{d^2y}{dx^2} = a(x, y)(y')^3 + b(x, y)(y')^2 + c(x, y)y' + d(x, y),$$

that is the differential equation of the geodetics is a polynom of degree three in $y' = \frac{dy}{dx}$ [14], [15], [16].

This result may be of importance because the psychometric metric can be measured along the geodesics.

Remark 2. Certainly Randers metric can only be applied in psychometry if non-symmetrical metrics are also allowed in studies of some psychometric problems. We can find a refrence to this possibility in the paper [11]. We hope that in the near future we can characterize the metric functions which is useful in psychometry and which we described in the present paper.

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