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A NOTE ON BINOMIAL COEFFICIENTS AND EQUATIONS OF PYTHAGOREAN TYPE

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Dedicated to the memory of Professor Péter Kiss

Abstract. The aim of this paper is to solve three diophantine equations of Pythagorean type.

1. Introduction

In [1] Luca determined all consecutive binomial coefficients satisfying the equation

$$\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^2.$$

His nice work leads to those Fibonacci numbers which are square or twice a square. In this note we apply Luca's method to find all the solutions (n,k) of

$$a\binom{n}{k}^2 + b\binom{n}{k+1}^2 = \binom{n}{k+2}^2, \tag{1}$$

where $(a,b) = (1,2)$ and $(a,b) = (2,1)$. Moreover, the solutions of the diophantine equation

$$\binom{n}{k}^2 + \binom{n+1}{k}^2 = \binom{n+2}{k}^2 \tag{2}$$

are also provided. The results are the following.

Theorem 1. *If $n \in \mathbf{N}$, $n \geq 2$ and $k \in \mathbf{N}$, $k \leq n-2$ satisfy equation (1) with $(a,b) = (1,2)$ then $(n,k) = (14,4)$.*

Theorem 2. *Equation (1) with $(a,b) = (2,1)$ has no solution in $n \in \mathbf{N}$ and $k \in \mathbf{N}$ $(n \geq 2\ ,\ k \leq n-2)$.*

Theorem 3. *If $n \in \mathbf{N}$ and $k \in \mathbf{N}$, $k \leq n$ satisfy equation (2) then $(n,k) = (3,1)$.*

Obviously, one can gain similar type of results as Theorem 1–3 if the symmetry of Pascal triangle is considered. For the proofs we follow paper [1] and go into details in only one case. The general case (1) seems to be more complicated. Even if $a = 1$ or $b = 1$, though the analogous equation to (6) exists, but the corresponding equation (11) or (12) is more difficult, where one should determine special figurate numbers in second order recurrences.

At the end of this paper we summarize some computational results in case $1 \leq a, b \leq 25$.

2. Proofs

Proof of Theorem 1. If $(a, b) = (1, 2)$ then equation (1) in natural numbers n and k leads to

$$(y+1)^2\left(y^2+2x^2\right) = x^2\left(x-1\right)^2, \tag{3}$$

where $y = k + 1$ and $x = n - k$ are positive integers. Equation (3) implies that $y^2 + 2x^2$ is a square. It is well known (see, for example, [3]), that all the solutions of the diophantine equation $y^2 + 2x^2 = z^2$ in positive integers x, y and z can be expressed as

$$y = d\left|u^2-2v^2\right|, \quad x = 2duv, \quad z = d\left(u^2+2v^2\right), \tag{4}$$

with the conditions $d, u, v \in \mathbf{Z}^+$, $\gcd(u, v) = 1$ and $u \equiv 1 \pmod 2$. It is easy to see that $\gcd(u^2 + 2v^2, 2uv) = 1$. Therefore the consequence

$$\left(d|u^2-2v^2|+1\right)\left(u^2+2v^2\right) = 2uv\left(2duv-1\right) \tag{5}$$

of equation (3) together with (4) implies that

$$e = \frac{2duv-1}{u^2+2v^2} = \frac{d|u^2-2v^2|+1}{2uv} \tag{6}$$

is a positive integer. The system of two linear equations

$$\left.\begin{array}{rcrcr} (u^2+2v^2)e & - & (2uv)d & = & -1 \\ (2uv)e & - & |u^2-2v^2|d & = & 1 \end{array}\right\} \tag{7}$$

in variables e and d has a unique solution, namely

$$\left\{ e = \frac{|u^2 - 2v^2| + 2uv}{D},\ d = \frac{u^2 + 2v^2 + 2uv}{D} \right\} \tag{8}$$

with

$$D = -(u^2 + 2v^2)|u^2 + 2v^2| + 4u^2v^2 = \pm\left(u^4 - 4v^4\right) + 4u^2v^2.$$

Obviously, D is odd. If D has an odd prime divisor p then by (8) we conclude that p divides both $|u^2 - 2v^2| + 2uv$ and $u^2 + 2v^2 + 2uv$. But this is impossible because $\gcd(u, v) = 1$. Consequently $|D| = 1$. Here we must distinguish two cases. Depending on the sign of $u^2 - 2v^2$ either

$$4D = \left(2u^2 + 4v^2\right)^2 - 8\left(u^2\right)^2 = \pm 4, \tag{9}$$

or

$$4D = \left(2u^2 + 4v^2\right)^2 - 8\left(2v^2\right)^2 = \pm 4. \tag{10}$$

Both cases are connected with the Pell sequence $\{P_n\}_{n=0}^{\infty}$ defined by $P_s = 2P_{s-1} + P_{s-2}$, $P_0 = 0$, $P_1 = 1$, and its associate sequence $\{R_n\}_{n=0}^{\infty}$ given by the same recurrence relation and having initial values $R_0 = R_1 = 2$. These recurrences provide all the solutions of the equation $X^2 - 8Y^2 = \pm 4$. Therefore, by (9) or (10) it follows that

$$\left\{\ P_s = u^2\ ,\ R_s = 2u^2 + 4v^2 \right\}$$

or, in the second case

$$\left\{\ P_s = 2v^2,\ R_s = 2u^2 + 4v^2\ \right\}.$$

Fortunately, the squares and twice a squares have already been determined in the Pell sequence (see [2] and [4]):

$$\left.\begin{array}{rcccc} P_s = u^2 & \Leftrightarrow & (s, u) & = & (0,0); (1,1); (7,13); \\ P_s = 2v^2 & \Leftrightarrow & (s, v) & = & (0,0); (2,1). \end{array}\right\} \tag{11}$$

Among them only $(s, v) = (2, 1)$ provides a solution of the original problem, namely $(n, k) = (14, 4)$.

Proof of Theorem 2. This proof is very similar to the previous one, therefore we only indicate the crucial point of it. Equation (1) with $(a, b) = (2, 1)$ and later by $y = k + 1 = 2duv$, $x = n - k = d|u^2 - 2v^2|$ implies that

$$D = \left(u^2 + 2v^2 - uv\right)^2 - (3uv)^2 = \pm 1,$$

which contradicts that $u, v \in \mathbf{Z}^+$.

Proof of Theorem 3. Apply again the procedure of Luca. Equation (2) implies that

$$(y+1)^2 \left(y^2 + x^2\right) = x^2 (x+1)^2 \tag{12}$$

with $x = n$, $y = n - k$. The unknowns x and y are two entries of a Pythagorean triple, hence we have two cases. If

$$x = 2duv, \quad y = d(u^2 - v^2)$$

$(d, u, v \in \mathbf{Z}^+$, $\gcd(u, v) = 1$, $u \geq v$ and $u \not\equiv v \pmod{()2}$ then (12) leads to

$$\left(4u^2 + 2v^2\right)^2 - 5\left(2u^2\right)^2 = \pm 4,$$

otherwise, if we interchange the role of x and y in the equation $x^2 + y^2 = z^2$, it follows that

$$\left(2u^2 + 2v^2 - 2uv\right)^2 - 5\,(2uv)^2 = \pm 4.$$

As in [1], we must know the square and twice a square Fibonacci numbers. In the first case $F_s = 2u^2$, $L_s = 4u^2 + 2v^2$ provide the only solution $(n, k) = (3, 1)$. From $F_s = 2uv$, $L_s = 2u^2 + 2v^2 - uv$ we conclude that $F_{s-1} = (u - v)^2$ and it gives no more binomial coefficients satisfying (2) (see [1].

Computational results

If $1 \leq a, b \leq 25$, applying a simple computer search, we found all the solutions of equation (1) in the intervals $2 \leq n \leq 250$, $0 \leq k \leq n - 2$. The results are shown in the following table.

a	1	1	1	1	1	2	3	4	4	4	4	5	7	9	9	9	9
b	1	2	7	14	23	8	24	2	5	12	21	1	2	3	6	7	19
n	62	14	43	98	173	26	64	4	19	44	83	14	7	11	6	14	53
k	26	4	10	18	28	5	9	0	4	8	13	4	1	2	0	2	8

a	10	11	11	13	13	13	16	16	16	16	16	16	16	17	18	19
b	6	4	14	1	4	23	1	3	4	8	12	13	14	13	2	5
n	43	118	23	25	19	34	7	134	76	19	8	13	28	94	11	43
k	10	33	3	7	4	4	1	38	20	3	0	1	4	18	2	10

a	20	20	22	22	23	25	25	25	25	25	25	25
b	1	11	3	9	1	3	6	8	15	20	22	23
n	4	44	19	89	229	5	14	11	29	10	22	46
k	0	8	4	19	68	0	2	1	4	0	2	6

References

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