ON INTERSECTION OF NORMAL FITTING CLASSES OF FINITE GROUPS

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Dedicated to the memory of Professor Péter Kiss

Abstract. Intersections of $\mathfrak{X}$-normal Fitting classes are studied for Fischer class $\mathfrak{X}$ of partially soluble groups.

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Introduction

When describing Fitting classes of finite soluble groups structure and their classification the basic result is Blessenohl–Gaschütz's theorem [1]: the intersection of any set of non-identity normal Fitting classes is non-identity normal Fitting class again.

Remind that a class $\mathfrak{F}$ of finite groups is called a Fitting class if the following two conditions hold:

(i) if $G \in \mathfrak{F}$ and $N \lhd G$, then $N \in \mathfrak{F}$;

(ii) if M, $N \lhd G = MN$ with M and N in $\mathfrak{F}$, then $G \in \mathfrak{F}$.

We note from the definition of a Fitting class it follows that every finite group G has a unique maximal normal $\mathfrak{F}$-subgroup called the $\mathfrak{F}$-radical of G denoting $G_{\mathfrak{F}}$.

A Fitting class $\mathfrak{F}$ is called normal in a class of finite groups $\mathfrak{X}$ or $\mathfrak{X}$-normal [4] if $\mathfrak{F} \subseteq \mathfrak{X}$ and $G_{\mathfrak{F}}$ is maximal among subgroups of G belonging to $\mathfrak{F}$ for all groups $G \in \mathfrak{X}$. In the case when $\mathfrak{X} = \mathfrak{S}$ ($\mathfrak{S}$ is the class of all finite soluble groups) $\mathfrak{F}$ is called $\mathfrak{S}$-normal or simply normal Fitting class.

In this paper we develop and extend the above-mentioned result by Blessenohl–Gaschütz in two directions. In the first place, we prove an analog of Blessenohl–Gaschütz's theorem for $\mathfrak{X}$-normal Fitting classes where $\mathfrak{X}$ is a Fischer class (in particular $\mathfrak{X} \subseteq \mathfrak{S}$). In the second place, we replace a solvability condition for the groups of the class $\mathfrak{X}$ with a partially solvability condition. In the course of this paper we consider only finite groups. We use the terminology and notations of [2].

1. Some notations and lemmas

Let $\mathfrak{F}$ be a Fitting class. A subgroup V of a group G is called an $\mathfrak{F}$-injector of G if $V \bigcap N$ is maximal in N from the subgroups in $\mathfrak{F}$ for any subnormal subgroup

N of G. A famous Fischer–Gaschütz–Hartley's theorem [3] that every group $G \in \mathfrak{S}$ has a unique class of conjugate $\mathfrak{F}$-injectors is a synthesis of well-known Sylow's and Hall's theorems.

We note that if $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes then their product $\mathfrak{F}\mathfrak{H}$ is the class of groups $(G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$ which is a Fitting class. In particular $\mathfrak{F}\mathfrak{S}$ is the class of all groups G such that factor group by the $\mathfrak{F}$-radical of G is soluble.

The following lemma extends Fischer–Gaschütz–Hartley's theorem.

Lemma 1.1. (V. Sementovskii [5]) *If $G \in \mathfrak{F}\mathfrak{S}$ then*
(a) G has a unique class of conjugate $\mathfrak{F}$-injectors;
(b) if V is an $\mathfrak{F}$-injector of G and $V \subseteq H \subseteq G$, then V is also an $\mathfrak{F}$-injector of H.

We shall use the definition of $\mathfrak{X}$-normal Fitting class which is equivalent to above-mentioned (in introduction) Laue's definition [4].

Definition 1.2. Let $\mathfrak{F}$ and $\mathfrak{X}$ be Fitting classes such that $\mathfrak{F} \subseteq \mathfrak{X} \subseteq \mathfrak{F}\mathfrak{S}$. We call the Fitting class $\mathfrak{F}$ an $\mathfrak{X}$-normal or normal in $\mathfrak{X}$ if for any group $G \in \mathfrak{X}$ its $\mathfrak{F}$-injector is a normal subgroup of G. We denote this by $\mathfrak{F} \triangleleft \mathfrak{X}$.

The following example gives a construction procedure of wide family of $\mathfrak{X}$-normal Fitting classes.

Example 1.3. Let $\mathfrak{F}$ be any non-empty Fitting class and $\mathfrak{X} = \mathfrak{F}\mathfrak{N}$ where $\mathfrak{N}$ is the class of all nilpotent groups. Then for any group $G \in \mathfrak{X}$ its $\mathfrak{F}$-injector $V = G_{\mathfrak{F}}$. In fact since $G/G_{\mathfrak{F}}$ is nilpotent, then $V/G_{\mathfrak{F}}$ is a subnormal subgroup of $G/G_{\mathfrak{F}}$. Therefore V is subnormal in G and $V = G_{\mathfrak{F}}$.

We shall use the result by J. Tits.

Lemma 1.4. ([2, Lemma A 1.2]) *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*
(a) $U \bigcap VW = (U \bigcap V)(U \bigcap W)$;
(b) $U \bigcap UW = U(V \bigcap W)$.

2. The main result

Remind that a Fitting class $\mathfrak{F}$ is called a Fischer class if $G \in \mathfrak{F}$, $K \triangleleft G$, $K \subseteq H \subseteq G$ and H/K is a p-group (p is a prime number) implies $H \in \mathfrak{F}$.

Theorem 2.1. *Let $\mathfrak{X}$ be a Fisher class and $\{\mathfrak{F}_i \mid i \in I\}$ be the set of $\mathfrak{X}$-normal Fitting classes. If $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$ and $\mathfrak{F} \subseteq \mathfrak{X} \subseteq \mathfrak{F}\mathfrak{S}$, then $\mathfrak{F}$ is an $\mathfrak{X}$-normal Fitting class.*

Proof. We proceed by induction on the order of groups in $\mathfrak{X}$. Suppose that the theorem fails to hold. Let $G \in \mathfrak{X}$ be a counter example of minimal order. Since

$G/G_{\mathfrak{F}}$ is soluble by hypothesis then by Lemma 1.1 there exist $\mathfrak{F}$-injectors in G. Let V be an $\mathfrak{F}$-injector of G, such that V is not normal in G. Since $\mathfrak{F} \subseteq \mathfrak{F}_i$ for all $i \in I$, then $G_{\mathfrak{F}} \subseteq G_{\mathfrak{F}_i}$ and by the isomorphism $G/G_{\mathfrak{F}}/G_{\mathfrak{F}_i}/G_{\mathfrak{F}} \cong G/G_{\mathfrak{F}_i}$, we have $G/G_{\mathfrak{F}_i}$ is soluble.

Consequently by Lemma 1.1 there exists an $\mathfrak{F}_i$-injector V_i in G. By hypothesis $V_i \lhd G$ for all $i \in I$. Therefore $\bigcap_{i\in I} V_i \lhd G$. Evidently $\bigcap_{i\in I} V_i \in \mathfrak{F}$. Hence $\bigcap_{i\in I} V_i \subseteq G_{\mathfrak{F}}$.

On the other hand for every $i \in I$ we have the inclusion

$$G_{\mathfrak{F}} \subseteq G_{\mathfrak{F}_i} = V_i.$$

Consequently $G_{\mathfrak{F}} = \bigcap_{i\in I} V_i$ and $\bigcap_{i\in I} V_i \subset V$.

Let M be an arbitrary maximal normal subgroup of G. Since V is an $\mathfrak{F}$-injector of G then the subgroup $V \bigcap M$ is an $\mathfrak{F}$-injector of the group M. Then since $M \in \mathfrak{X}$ it follows that $V \bigcap M \lhd M$ by induction.
We obtain $V \bigcap M = M_{\mathfrak{F}} = G_{\mathfrak{F}} \bigcap M$. Hence for any maximal normal subgroup M of G we have

$$V \bigcap M = \left(\bigcap_{i\in I} V_i\right) \bigcap M. \tag{1}$$

We note that V is not contained in any subnormal subgroup N of G. If this assertion fails to hold i.e. $V \subseteq N \lhd\lhd G$ then there exists an $\mathfrak{F}$-injector in N. By Lemma 1.1 the subgroup V is an $\mathfrak{F}$-injector of N. Then by induction $V \lhd N$. Therefore $V \lhd\lhd G$ and $V = G_{\mathfrak{F}}$. A contradiction because V is not normal subgroup of G.

We show that $G = RV$ for any normal subgroup R of G such that G/R is nilpotent. Let $RV \neq G$. Then a subgroup RV/R is subnormal in G/R. Hence RV is subnormal in G. Consequently V is contained in the subgroup $H = RV$ and $H \lhd\lhd G$, a contradiction.

Now we prove that G is comonolithic. Let G be not comonolithic and M_1 and M_2 be maximal normal subgroups of G. Without loss of generality we consider $M_1 \supseteq G_{\mathfrak{F}}$ and $M_2 \not\supseteq G_{\mathfrak{F}}$. Then $G = M_2 G_{\mathfrak{F}}$. Besides $M_1 \supseteq G_{\mathfrak{F}}$ and $G \in \mathfrak{F}\mathfrak{S}$. It follows that G/M_1 is nilpotent. From above $G = VM_1$. Consequently by the isomorphisms $G/M_1 \cong V/(V \bigcap M_1)$ and $G/M_2 \cong V/(V \bigcap M_2)$ the subgroups $V \bigcap M_1$ and $V \bigcap M_2$ are maximal normal of V.

Suppose $V \bigcap M_1 \neq V \bigcap M_2$. Then $V = (V \bigcap M_1)(V \bigcap M_2)$. Hence by (1)

$$V = \left(\left(\bigcap_{i\in I} V_i\right) \bigcap M_1\right) \left(\left(\bigcap_{i\in I} V_i\right) \bigcap M_2\right)$$

and $V \subseteq G_{\mathfrak{F}}$. Consequently $V = G_{\mathfrak{F}}$. A contradiction because the subgroup V is not normal in G. Therefore $V \bigcap M_1 = V \bigcap M_2$. Then

$$G/M_1 \cong V/V \bigcap M_1 = V/V \bigcap M_2 \cong G/M_2$$

and $G/M_2 \in \mathfrak{N}$. Thus the group $G/(M_1 \bigcap M_2)$ is nilpotent. Hence $G = V(M_1 \bigcap M_2)$. From the other hand $G = VM_1 \bigcap VM_2$. Consequently

$$V\left(M_1 \bigcap M_2\right) = VM_1 \bigcap VM_2.$$

By Lemma 1.2 we have the equality

$$V = \left(V \bigcap M_1\right)\Big) V \bigcap M_2\Big) = V \bigcap M_1.$$

It follows that $V \subseteq M_1$. A contradiction because V is not contained in any subnormal subgroup of G. Thus $M_1 = M_2 = M$ and G is comonolithic. Consequently for every $i \in I$ we have $V_i \subseteq M$. Hence by (1)

$$V \bigcap M = \bigcap_{i \in I} V_i. \tag{2}$$

Then by the isomorphism

$$G/M \cong V/\left(\bigcap_{i \in I} V_i\right)$$

the group $V/(\bigcap V_i)$ is cyclic of prime order p.

Now we show $V_iV \neq G$ for some $i \in I$. Suppose for any $i \in I$ the equality holds $V_iV = G$. If for all $j \in I$ we have $V_j = G$ then $G \in \mathfrak{F}$ and G is an $\mathfrak{F}$-injector for itself. Hence $G = V \triangleleft G$. A contradiction because V is not normal in G. Consequently $V_j \neq G$ for some $j \in I$. Since by hypothesis $V_j \triangleleft G$ then

$$G/V_j \cong V/V \bigcap V_j.$$

By the equality (2)

$$V \bigcap V_j \subseteq V \bigcap M = \bigcap_{i \in I} V_i \subseteq V_j \bigcap V.$$

Then $V_j \bigcap V = \bigcap_{i \in I} V_i$. Since $V/(\bigcap_{i \in I} V_i) \cong G/V_j$ it follows that G/V_j is a cyclic group of prime order p. Hence V_j is a maximal normal subgroup of G. Therefore $V_j = M$.

It is easily seen that $V_j \in \mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$. In fact if for $i \neq j$ $(i \in I)$ we have $V_i \neq G$ then we analogously conclude $V_i = M = V_j$ and $V_j \in \mathfrak{F}_i$.

If $V_i = G$ then $V_j \lhd V_i \in \mathfrak{F}_i$ and $V_j \in \mathfrak{F}_i$. Consequently $V_j \in \mathfrak{F}_i$ for all $i \neq j$. Therefore $V_j \in \mathfrak{F}$. Hence $V_j \subseteq G_{\mathfrak{F}} \subseteq V$. By hypothesis $V_j V = G$ and we obtain $V = G$ and $G \in \mathfrak{F}$. A contradiction because V is not normal in G.

Thus there exists $i \in I$ such that $V_i V \neq G$. We prove that $V_i V \in \mathfrak{X}$. In fact since a group $\overline{V} = V/(\bigcap_{i\in I} V_i)$ is simple then its normal subgroup $(V \bigcap V_i)/(\bigcap_{i\in I} V_i)$ either coincides with $\overline{V}$ or $(V \bigcap V_i)/(\bigcap_{i\in I} V_i)$ is the identity group. In the first case we have $V = V \bigcap V_i \subseteq V_i$. A contradiction because V is not contained in any subnormal subgroup of G. Thus we conclude $V \bigcap V_i = \bigcap_{i\in I} V_i$. Then by the isomorphism $V_i V/V_i \cong V/V \bigcap V_i$ the group $V_i V/V_i$ is a p-group.

Consequently since $G \in \mathfrak{X}$ and $\mathfrak{X}$ is a Fischer class if follows that $V_i V \in \mathfrak{X}$. We have $|V_i V| < |G|$ and by Lemma 1.1 V is an $\mathfrak{F}$-injector of $V_i V$. Hence by induction $V \lhd V_i V$. Since $V \in \mathfrak{F}_i$ then $V \subseteq (V_i V)_{\mathfrak{F}_i}$. By Lemma 1.1 V_i is an $\mathfrak{F}_i$-injector of $V_i V$. Hence $(V_i V)_{\mathfrak{F}_i} = V_i$. Thus $V \subseteq V_i$. A contradiction because V is not contained in any subnormal subgroup of G. The contradiction indicates that $\mathfrak{F}$ is an $\mathfrak{X}$-normal Fitting class. The theorem is proved.

We note by [1, Theorem 5.1] that every non-identity $\mathfrak{S}$-normal Fitting class contains the class of all nilpotent groups $\mathfrak{N}$. Therefore in the case $\mathfrak{X} = \mathfrak{S}$ we have the Blessenohl–Gaschütz's result.

Corollary 2.2. [1, Theorem 6. 2] *In the set of all non-identity normal Fitting classes there exists a unique minimal element by inclusion.*

References

[1] Blessenohl, D. and Gaschütz, W., Über normale Schunk und Fittingklassen, *Math. Z.*, **148** (1970), 1–8.

[2] Doerk, K. and Hawkes, T. O., *Finite Soluble Groups*, Berlin–New York, Walter de Gruyter, 1992.

[3] Fischer, B., Gaschütz, W. and Hartley, B., Injectoren endlicher auflösbaren Gruppen, *Math. Z.*, **102** (1967), 337–339.

[4] Laue, H., Über nichtauflösbare normalen Fittingklassen, *J. Algebra*, **45** (1977), 274–283.

[5] Sementovskii, V. G., *Injectors of finite groups*, Minsk, Nauka i Tekhnika, 1984, 166–170 (in Russian).

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