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STRUCTURE OF THE GROUP OF QUASI MULTIPLICATIVE ARITHMETICAL FUNCTIONS

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Dedicated to the memory of Professor Péter Kiss

Abstract. The structure of the group of quasi multiplicative arithmetical functions such that $f(1)\neq 0$ with respect to Dirichlet and the more general Davison convolution via an isomorphism to a subgroup of upper triangular and Toeplitz matrices will be described.

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1. Introduction

In what follows unless contrary is stated $\mathbf{K}$ denotes a field between the field of complex $\mathbf{C}$ and the field of rational numbers $\mathbf{Q}$. Let Arit($\mathbf{K}$) denote the set of all $\mathbf{K}$-valued arithmetical functions (i.e. functions defined on the set $\mathbf{N}$ of positive integers with values in $\mathbf{K}$), and Mult($\mathbf{K}$) the set of *nonzero* (i.e. non identically vanishing) **multiplicative** arithmetical functions f, that is functions such that $f(nm) = f(n)f(m)$ whenever $(m, n) = 1$. The sets Arit($\mathbf{K}$) and Mult($\mathbf{K}$) endowed with the **Dirichlet convolution**

$$(f \star_D g)(n) = \sum_{d_1 d_2 = n} f(d_1)g(d_2)$$

are of basic importance in various number-theoretical considerations.

Given an $f \in \text{Arit}(\mathbf{K})$ we can assign it the formal Dirichlet series

$$f \mapsto T(f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{1}$$

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If we define the multiplication of formal Dirichlet series by

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\sum_{n=1}^{\infty}\frac{(f\star_D g)(n)}{n^s},$$

then for the set of all formal Dirichlet series

$$\mathcal{D}(\mathbf{K})=\left\{\sum_{n=1}^{\infty}\frac{a_n}{n^s}\ :\ a_n\in\mathbf{K}\right\}$$

with multiplication defined above we have:

Lemma 1. ([13, Theorem 4.6.1]) *The map T defined by (1) gives an isomorphism between the semigroups* $(\mathrm{Arit}(\mathbf{K}),\star_D)$ *and* $(\mathcal{D}(\mathbf{K}),.)$.

The underlying property for the investigation that follows is the following result due to Bell:

Lemma 2. *(a) The set of arithmetical functions $f\in\mathrm{Arit}(\mathbf{K})$ for which $f(1)\neq 0$ forms a commutative group with respect to Dirichlet convolution $\star_D$.*
(b) The set $(\mathrm{Mult}(\mathbf{K}),\star_D)$ *forms a subgroup of the group* $(\mathrm{Arit}(\mathbf{K}),\star_D)$.

Dehaye [5] analyzed the structure of the group $(\mathrm{Mult}(\mathbf{R}),\star_D)$ of real valued non-zero multiplicative functions with respect to the Dirichlet convolution $\star_D$. He proved (among other) that $(\mathrm{Mult}(\mathbf{R}),\star_D)$ is isomorphic to the complete direct[1] product $\widetilde{\prod}_{i\in\mathbf{N}}\mathrm{D}^1_{\mathbf{R}}$ of countably many copies of $\mathrm{D}^1_{\mathbf{R}}$, where $\mathrm{D}^1_{\mathbf{R}}$ is the set of all matrices

$$\begin{pmatrix}1&a&b&c&d&\cdots\\0&1&a&b&c&\cdots\\0&0&1&a&b&\cdots\\0&0&0&1&a&\cdots\\0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}$$

which all entries on descending diagonals are equal real numbers while the main diagonal entries are equal to 1. In what follows we show using more number theoretical arguments that his results can be extended to more general types of arithmetical functions and convolutions.

[1] For the definition of the complete (or Cartesian) direct product the reader is referred to [6] or [5] or sources quoted [5], if necessary.

2. Quasi multiplicative functions

If $f \in \mathrm{Arit}(\mathbf{K})$ is a multiplicative arithmetical function, then $f(m)f(n) = f((m,n))f\Big(\frac{m.n}{(m,n)}\Big)$ for all $m, n \in \mathbf{N}$. An arithmetical function f is called **quasi multiplicative** ([11,14]) if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \quad \text{whenever} \quad (m,n) = 1. \tag{2}$$

The set of nonzero $\mathbf{K}$-valued quasi multiplicative functions will be denoted by $\mathrm{Quas}(\mathbf{K})$. The analogue of the second part of Theorem 2 for nonzero quasi multiplicative can be verified by a direct computation:

Lemma 3. *The set* $\mathrm{Quas}(\mathbf{K})$ *forms a commutative group with respect to Dirichlet convolution* $\star_D$.

Note that an f with $f(1) \neq 0$ is quasi multiplicative if, and only if, $f^- = \frac{1}{f(1)}f$ is multiplicative.[2] There follows from this observation (or directly from (2)) that

$$f^-(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \prod_{i=1}^{k} \frac{f(p_i^{\alpha_i})}{f(1)} = \prod_{i=1}^{k} f^-(p_i^{\alpha_i}), \tag{3}$$

or

$$f(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = f(1)^{1-k} \prod_{i=1}^{k} f(p_i^{\alpha_i}) = f(1) \prod_{i=1}^{k} f^-(p_i^{\alpha_i}), \tag{4}$$

whenever $p_1, \ldots, p_k$ are distinct primes and $\alpha_i \in \mathbf{N}$. The next two results follow from well known properties of multiplicative functions:

Lemma 4. *If* $f \in \mathrm{Arit}(\mathbf{K})$ *with* $f(1) \neq 0$, *then* f *is quasi multiplicative if and only if (3) or (4) holds for all* k *tuples* $p_1, \ldots, p_k$ *of distinct primes and all* $\alpha_i \in \mathbf{N}$.

If f is multiplicative then under the isomorphism of Lemma 1 the image $T(f)$ is a Dirichlet series admitting the so called Euler factorization. Therefore if $f \in \mathrm{Quas}(\mathbf{K})$, then applying this fact to the multiplicative function f^- we get:

Lemma 5. *If* $f \in \mathrm{Quas}(\mathbf{K})$ *then* $T(f)/f(1)$ *is the formal product of the series*

$$1 + \frac{f(p)}{f(1)p^s} + \frac{f(p^2)}{f(1)p^{2s}} + \frac{f(p^3)}{f(1)p^{3s}} + \cdots, \tag{5}$$

where the product runs over all primes p.

[2] If f is multiplicative, so is $f(Mn)/f(M)$, where M is any positive integer.

The series (5) is in a one-to-one relation to a formal power series called **Bell series** $f_p(x)$ of an $f \in \mathrm{Arit}(\mathbf{K})$ with $f(1) \neq 0$ modulo the prime p

$$f_p(x) = f(1) + f(p)x + f(p^2)x^2 + \cdots = \sum_{n=0}^{\infty} f(p^n)x^n.$$

In terms of Bell series we can characterize the quasi multiplicative functions as follows:

Lemma 6. *Let $f, g \in \mathrm{Quas}(\mathbf{K})$. Then $f = g$ if, and only if,*

$$f_p(x) = g_p(x) \quad \text{for all primes } \ p,$$

or equivalently

$$f(p^\alpha) = g(p^\alpha) \quad \text{for all primes } p \text{ and integers } \alpha \geq 0.$$

The next result shows a close relation between Bell series and Dirichlet multiplication:

Lemma 7. ([1, Theorem 2.25]) *For any two arithmetical functions f and g let $h = f \star_D g$. Then for every prime p we have*

$$h_p(x) = f_p(x)g_p(x).$$

Perhaps a most natural proof of this result can be modelled using matrix multiplication of infinite upper triangular matrices of the type

$$f_p(x) \mapsto m_{\mathbf{K},D}(f_p) = \begin{pmatrix} f(1) & f(p) & f(p^2) & f(p^3) & \cdots \\ 0 & f(1) & f(p) & f(p^2) & \cdots \\ 0 & 0 & f(1) & f(p) & \cdots \\ 0 & 0 & 0 & f(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{6}$$

Let $\mathcal{P}$ denote the set of all (rational) primes.

Theorem 8. *Let $\mathrm{D}_{\mathbf{K}}$ be the set of matrices of the type*[3]

$$T(a,b,c,d,e,\ldots) = \begin{pmatrix} a & b & c & d & e & \cdots \\ 0 & a & b & c & d & \cdots \\ 0 & 0 & a & b & c & \cdots \\ 0 & 0 & 0 & a & b & \cdots \\ 0 & 0 & 0 & 0 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with } a \neq 0, b, c, d, e, \ldots \in \mathbf{K}.$$

[3] *That is upper triangular (semi-definite) matrices that are constant along all diagonals parallel to the principal diagonal. Matrices possessing the later property are also called* **Toeplitz matrices**.

Then
(a) $\mathrm{D}_{\mathbf{K}}$ *is a group with respect to the matrix multiplication,*
(b) The group $(\mathrm{Quas}(\mathbf{K}), \star_D)$ *is isomorphic to a subgroup of the complete direct product* $\widetilde{\prod}_{p\in\mathcal{P}}\mathrm{D}_{\mathbf{K}}$*, defined by the condition that the diagonal value* a *is a common number in all components of an element of the direct product.*

Proof. (a) The proof can be based either on standard tools from matrix algebra or using our arithmetical background. Using the matrix algebra language let $A(m,n)$, $m,n \in \{1,2,3,\ldots\}$ be the (m,n)th entry of a matrix A. Then $A \in \mathrm{D}_{\mathbf{K}}$ if and only if

(1) if $n > m$ then $A(m,n) = 0$, i.e. A is upper triangular,
(2) $A(m+k, n+k) = A(m,n)$ for all indices $m,n \in \mathbf{N}, k \in \mathbf{Z}$ such that $\min\{m,n,n+k,m+k\} \geq 1$, i.e. A is Toeplitz.

Let $A_i \in \mathrm{D}_{\mathbf{K}}$, $i = 1,2$ and $A = A_1A_2$. Then $A(m,n) = \sum_{t=1}^{\infty} A_1(m,t)A_2(t,n)$. That A is upper triangular is easy to see. What concerns the second property it suffices to prove it for $k = 1$ only. Let $n \leq m$. Then[4]

$$A(m+1,n+1) = \sum_{t=1}^{\infty} A_1(m+1,t)A_2(t,n+1) = \sum_{t=n+1}^{m+1} A_1(m+1,t)A_2(t,n+1)$$
$$= \sum_{t=n}^{m} A_1(m,t)A_2(t,n) = A(m,n),$$

where in the second equality we used the fact that the matrices under consideration are upper triangular. The case $n > m$ is even easier to verify, for in this case at least one of the factors in the first sum vanishes. This shows that $D_{\mathbf{K}}$ is closed under the multiplication of matrices.

The presence of the identity element in $D_{\mathbf{K}}$ is clear. To prove the existence of inverse elements we switch to our arithmetical background.[5]

If $f \in \mathrm{Mult}(\mathbf{K})$, then also $f^{-1} \in \mathrm{Mult}(\mathbf{K})$. Lemma 7 implies that the Bell series modulo p of f^{-1} is given by

$$f_p^{-1}(x) = \frac{1}{f_p(x)}.$$

[4] Another form of the following rearrangement of the summands gives [3, p. 96–97] the **product matrix formula** for an $(m+1,n+1)$ entry of the product of two general Toeplitz matrices saying that $A(m+1,n+1) = A_1(m+1,1)A_2(1,n+1) + A(m,n)$.

[5] For another proof we refer to [3, Corollary of Theorem 2] where it is proved that *the only Toeplitz matrices with Toeplitz inverses are the triangular ones.*

Consequently, $f_p^{-1}(x)$ can be found by formal power series inversion and the corresponding element in $D_{\mathbf{K}}$ can be found for $\ell = f^{-1}$ noting that if

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and E is the infinite identity matrix, then

$$\ell_p(H) = \ell(1)E + \ell(p)H + \ell(p^2)H^2 + \cdots = \begin{pmatrix} \ell(1) & \ell(p) & \ell(p^2) & \ell(p^3) & \cdots \\ 0 & \ell(1) & \ell(p) & \ell(p^2) & \cdots \\ 0 & 0 & \ell(1) & \ell(p) & \cdots \\ 0 & 0 & 0 & \ell(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This proves (a) and simultaneously that if $h = f \star_D g$, then

$$m_{\mathbf{K},D}(h_p) = h_p(H) = f_p(H)g_p(H) = m_{\mathbf{K},D}(f_p)m_{\mathbf{K},D}(g_p),$$

that is that the mapping (6) is a homomorphism. That this mapping is also a bijection follows from the fact that the (quasi) multiplicative functions are uniquely determined by its values at all prime powers arguments (including 1). Since the product of two Bell series modulo p of two quasi multiplicative functions is a Bell series modulo p of a quasi multiplicative function modulo p, (b) follows using the isomorphism which is the composition of the isomorphism described in part (a) and that of of Lemma 1taking into account the Euler factorization from Lemma 5.

If f is a nonzero multiplicative function then $f(1) = 1$ and Dehaye's result mentioned in the introduction for $\mathbf{K} = \mathbf{R}$ follows immediately. Dehaye proved this result via subsets

$$\mathbf{F}^p = \{f \in \mathrm{Mult}(\mathbf{R}) : f(n) = 0 \text{ for every } n > 1 \text{ not divisible by } p\}$$

for each $p \in \mathcal{P}$. However, the multiplicativity of f implies that

$$\mathbf{F}^p = \{f \in \mathrm{Mult}(\mathbf{R}) : f(n) = 0 \text{ for every } n > 1 \text{ which is not a power of } p\}.$$

Consequently, the Euler factorization of an $f \in \mathbf{F}^p$ reduces to one factor only, namely

$$T(f) = 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots,$$

This observation immediately implies, first of all, the result extending [5, Theorem 2.2, Theorem 5.2] to an arbitrary $\mathbf{K}$:

Lemma 9. *For any prime p, $\mathbf{F}^p$ is a group which is isomorphic to $\mathrm{D}^1_{\mathbf{K}}$.*

Secondly, we have more generally:

Theorem 10. *If $P \subset \mathcal{P}$ is any set of primes then the set*

$$\{f \in \mathrm{Quas}(\mathbf{K}) : f(n) = 0 \text{ for every } n > 1 \text{ not divisible by a } p \in P\}.$$

is a group which is isomorphic to a subgroup of the complete direct product $\widetilde{\prod}_{p\in\mathcal{P}}\mathrm{D}_{\mathbf{K}}$, defined by the condition that the diagonal value a is a common number in all components of an element of the direct product. Its subset

$$\mathbf{F}^P = \{f \in \mathrm{Mult}(\mathbf{K}) : f(n) = 0 \text{ for every } n > 1 \text{ not divisible by a } p \in P\}$$

forms a subgroup which is isomorphic to $\widetilde{\prod}_{p\in P}\mathrm{D}^1_{\mathbf{K}}$.

Other results proved by Dehaye state that the group $\mathrm{Mult}(\mathbf{R})$ is torsion-free [5, Theorem 2.1] and divisible [5, Theorem 7.1]. To extend these results the following simple result will be useful:

Lemma 11. *Let $g \in \mathrm{Arit}(\mathbf{C})$ such that $g(1) \neq 0$. Then the equation $f^{(n)} = g$, where $f^{(n)} = \overbrace{f \star_D \ldots \star_D f}^{n \text{ times}}$, is soluble in $\mathrm{Arit}(\mathbf{C})$ and has n solutions here.*

Proof. The equation of the theorem can be solved inductively either by starting with the equation $(T(f))^n = T(g)$, or equivalently setting

$$\begin{aligned} &f(1) = \sqrt[n]{g(1)}, \quad \text{and} \\ &f(k) = \frac{1}{n(f(1))^{n-1}}\left(g(k) - \sum_{\substack{d_1\ldots d_n=k \\ d_1,\ldots,d_n\neq k}} f(d_1)\ldots f(d_n)\right), \quad \text{for } k > 1. \end{aligned} \tag{7}$$

Clearly if f is one solution of our equation, then all solutions of this equation are given by $\omega_i f$, where ω_i runs over all nth roots of unity.

A group $(G, .)$ is called **divisible** if the equation $x^n = a$ has a solution in G for every $a \in G$.

Theorem 12. *(a) The group $\{f \in \mathrm{Arit}(\mathbf{C}) : f(1) \neq 0\}$ is divisible and has torsion. Its torsion part is isomorphic to the group of all complex roots of unity, that is to group $\mathbf{Q}/\mathbf{Z}$.*
(b) If $\mathbf{C} \supset \mathbf{K} \supset \mathbf{Q}$, then $\mathrm{Mult}(\mathbf{K})$ is divisible and torsion-free.
(c) The groups $\mathrm{Arit}^+(\mathbf{R}) = \{f \in \mathrm{Arit}(\mathbf{R}) : f(1) > 0\}$ and $\mathrm{Quas}^+(\mathbf{R}) = \{f \in \mathrm{Quas}(\mathbf{R}) : f(1) > 0\}$ are divisible and torsion-free.

Proof. The proof follows easily from the previous Lemma. The verification that the solution given by (7) is (quasi) multiplicative can be proved directly.

The groups like Arit($\mathbf{K}$), Quas($\mathbf{K}$) or Mult($\mathbf{K}$) are not the only groups of arithmetical functions. In [14] infinite chains of subgroups of Arit($\mathbf{C}$) are constructed. The solvability of the equation $f^{(n)} = g$ which was investigated in many papers, cf. [8] and the papers quoted here, has an interesting grouptheoretic consequence ([6, §20]):

Corollary 13. *The groups* Arit($\mathbf{C}$), Arit$^+$($\mathbf{R}$), Quas$^+$($\mathbf{R}$) *and* Mult($\mathbf{K}$) *with* $\mathbf{C} \supset \mathbf{K} \supset \mathbf{Q}$ *have no maximal proper subgroup.*

Another consequence is the solvability of more general systems of compatible equations

$$\prod_{j\in J} x_j^{n_{ij}} = g_i, \qquad g_i \in G, i \in I$$

where among the integers n_{ij} only finitely many are nonzero for every j (cf. [6, §22]).

3. Davison convolution

The Dirichlet convolution has many possible generalizations. The following one was introduced in [4]. Let K be a $\mathbf{K}$-valued function defined on the set of the all ordered couples (n, d) of positive integers n, d satisfying $d|n$. Let $f, h \in$ Arit($\mathbf{K}$) be two arithmetical functions. By **(Davison) K-convolution** $f \star_K g$ we shall mean the operation

$$(f \star_K g)(n) = \sum_{d|n} K(n,d) f(d) g\left(\frac{n}{d}\right) = \sum_{d_1 d_2 = n} K(d_1 d_2, d_1) f(d_1) g(d_2).$$

The function K is called **kernel (of the convolution)**.

As already mentioned the set of non-zero multiplicative functions f endowed with Dirichlet' s convolution $\star_D$ forms a commutative group (cf. [1, Chapt. 2] or [12, Theorem 4.12]). To ensure a similar property with respect to the Davison convolution $\star_K$ some properties should be imposed on the kernel function K (cf. [4]):

(i) *The Davison convolution* $\star_K$ *is associative if and only if we have*

$$K(abc, bc)K(bc, c) = K(abc, c)K(ab, b) \text{ for every } a, b, c \in \mathbf{N},$$

or equivalently,

$$K(n,d)K(d,e) = K(n,e)K\left(\frac{n}{e}, \frac{d}{e}\right) \text{ for every } n, d, e \in \mathbf{N} \text{ with } d|n \text{ and } e|d.$$

(ii) *The Davison convolution* $\star_K$ *is commutative if and only if for every couple of elements* $a,\, b \in \mathbf{N}$ *there holds*

$$K(ab,a) = K(ab,b) \text{ for every } a,b \in \mathbf{N},$$

or equivalently,

$$K(n,d) = K\left(n, \frac{n}{d}\right) \text{ for every } n,\, d \in \mathbf{N} \text{ with } d|n.$$

The Davison convolution as operation does not possess the neutral element in general.

(iii) *The identity function* δ_1 *defined by* $\delta_1(n) = \delta_{1n}$, *where* δ_{ij} *is the Kronecker delta, is the unit element with respect to* $\star_K$ *if and only if*

$$K(n,n) = K(n,1) = 1 \text{ for every } n \in \mathbf{N}.$$

The next important question is the keeping up of the multiplicativity of arithmetical functions under the influence of the Davison convolution.

(iv) *The Davison convolution* $f \star_K g$ *of two multiplicative functions* f, g *is a multiplicative function if and only if*

$$K(abcd,ac) = K(ab,a)K(cd,c) \text{ for every } a,b,c,d \in \mathbf{N} \text{ with } (ab,cd) = 1.$$

The question about the existence of the inverse function f^{-1} to a given $f \in \mathrm{Arit}(\mathbf{K})$ with respect to the Davison convolution can be solved surprisingly quickly:

(v) *the inverse function* f^{-1} *of* f *with respect to* $\star_K$ *exists if and only if* $f(1) \neq 0$.
When this condition is fulfilled then f^{-1} can be defined recursively by

(j) If $n = 1$ then $f^{-1}(1) = \frac{1}{K(1,1)f(1)} = \frac{1}{f(1)}$.

(jj) Let $n > 1$ and suppose that $f^{-1}(m)$ is already defined for the all $m < n$. Then put

$$\begin{aligned} f^{-1}(n) &= \frac{-1}{K(n,n)f(1)} \sum_{\substack{bc=n \\ c\neq n}} f(b)f^{-1}(c)K(n,b) \\ &= \frac{-1}{f(1)} \sum_{\substack{bc=n \\ c\neq n}} f(b)f^{-1}(c)K(n,b). \end{aligned}$$

The first part of Lemma 2 can be now reproved using (v) in the following form:

Lemma 14. *The set of $f \in \mathrm{Arit}(\mathbf{K})$ for which $f(1) \neq 0$ forms a commutative group with respect to a K-convolution satisfying conditions (i)–(iii).*

Given a prime $p \in \mathcal{P}$ and a $\mathbf{K}$–valued function K defined on the set of all ordered couples (n, d) of positive integers n, d satisfying $d|n$, define $\mathrm{D}_{\mathbf{K},K,p}$ as the set of matrices of the type

$$(8) \qquad \begin{pmatrix} aK(1,1) & bK(p,p) & cK(p^2,p^2) & dK(p^3,p^3) & eK(p^4,p^4) & \cdots \\ 0 & aK(p,1) & bK(p^2,p) & cK(p^3,p^2) & dK(p^4,p^3) & \cdots \\ 0 & 0 & aK(p^2,1) & bK(p^3,p) & cK(p^4,p^2) & \cdots \\ 0 & 0 & 0 & aK(p^3,1) & bK(p^4,p) & \cdots \\ 0 & 0 & 0 & 0 & aK(p^4,1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a \neq 0$ and $a, b, c, d, e, \ldots \in \mathbf{K}$. If we put $a = 1$ in elements of $\mathrm{D}_{\mathbf{K},K,p}$ we get a subset, say $\mathrm{D}^1_{\mathbf{K},K,p}$. As a generalization of Theorem 8 we get:

Theorem 15. *Let $\star_K$ be a Davison convolutions satisfying properties (i)–(iv). Then*

(a) $\mathrm{D}_{\mathbf{K},K,p}$ is a group with respect to the matrix multiplication,

(b) The group $(\mathrm{Quas}(\mathbf{K}), \star_K)$ is isomorphic to the subgroup of $\widetilde{\prod}_{p\in\mathcal{P}} \mathrm{D}_{\mathbf{K},K,p}$ defined by the condition that the diagonal value a is a common number in all components (8) of an element of the direct product.

(c) The group $(\mathrm{Mult}(\mathbf{K}), \star_K)$ is isomorphic to the group $\widetilde{\prod}_{p\in\mathcal{P}} \mathrm{D}^1_{\mathbf{K},K,p}$.

Proof. Since a quasi multiplicative function f is uniquely determined by values $f(1) \neq 0$, $f(p)$, $f(p^2)$, ... for every $p \in \mathcal{P}$, instead of working with indices of entries of matrices we can suppose without loss of generality that the elements of $\mathrm{D}_{\mathbf{K},K,p}$ are of the form

$$m_{\mathbf{K},K,}(f_p) = \begin{pmatrix} f(1)K(1,1) & f(p)K(p,p) & f(p^2)K(p^2,p^2) & f(p^3)K(p^3,p^3) & f(p^4)K(p^4,p^4)\cdots \\ 0 & f(1)K(p,1) & f(p)K(p^2,p) & f(p^2)K(p^3,p^2) & f(p^3)K(p^4,p^3)\cdots \\ 0 & 0 & f(1)K(p^2,1) & f(p)K(p^3,p) & f(p^2)K(p^4,p^2)\cdots \\ 0 & 0 & 0 & f(1)K(p^3,1) & f(p)K(p^4,p) \ \cdots \\ 0 & 0 & 0 & 0 & f(1)K(p^4,1) \ \cdots \\ \vdots & \vdots & \vdots & 0 & \vdots \quad \ddots \end{pmatrix}$$

where $f \in \mathrm{Quas}(\mathbf{K})$. Then the (i, j), $j \geq i$, entry of the product $m_{\mathbf{K},K,}(g_p)m_{\mathbf{K},K,}(f_p)$ is

$$\sum_{k=i}^{j} g(p^{k-i})f(p^{j-k})K(p^{k-1},p^{k-i})K(p^{j-1},p^{j-k}) =$$
$$\sum_{k=0}^{j-i} g(p^{k})f(p^{j-i-k})K(p^{i+k-1},p^{k})K(p^{j-1},p^{j-i-k}),$$

while its expected value is

$$(g \star_K f)(p^{j-i})K(p^{j-1},p^{j-i}) = \left(\sum_{k=0}^{j-i} g(p^k)f(p^{j-i-k})K(p^{j-i},p^k)\right) K(p^{j-1},p^{j-i}).$$

To prove that our multiplication is well defined we have to prove that

$$K(p^{j-1},p^{j-i})K(p^{j-i},p^k) = K(p^{j-1},p^{j-i-k})K(p^{i+k-1},p^k). \tag{9}$$

There follows from (i) that

$$K(p^{a+b+c},p^{b+c})K(p^{b+c},p^c) = K(p^{a+b+c},p^c)K(p^{a+b},p^b), \quad a,b,c \in \{0,1,2,\ldots\}.$$

Taking $a = i-1$, $b = k$, and $c = j-i-k$ we get

$$K(p^{j-1},p^{j-i})K(p^{j-i},p^{j-i-k}) = K(p^{j-1},p^{j-i-k})K(p^{i-1+k},p^k),$$

but (ii) implies $K(p^{j-i},p^{j-i-k}) = K(p^{j-i},p^k)$ and (9) follows.

The existence of the identity element and the inverse one in $D_{\mathbf{K},k,p}$ follows now from the fact that such elements exist in the set of quasi multiplicative functions.

There follows from the above lines that the mapping

$$f \in \mathrm{Quas}(\mathbf{K}) \mapsto \prod_{p\in\mathcal{P}} m_{\mathbf{K},K}(f_p)$$

is the desired isomorphism from $(\mathrm{Quas}(\mathbf{K}),\star_K)$ onto the subgroup of $\widetilde{\prod}_{p\in\mathcal{P}}\mathrm{D}_{\mathbf{K},K,p}$ defined by the condition that the diagonal value a is a common number in all components of an element of the direct product, thereby proving statement (b). The statement (c) follows in turn.

The definition of the quasi multiplicativness depends only on the ordinary multiplication between positive integers and the elements of $\mathbf{K}$, therefore the next corollary might be surprising at the first sight:

Corollary 16. [(81, p.191)] *Let $\star_L$ and $\star_K$ be two Davison convolutions satisfying properties (i)–(iv). Then the couples groups $(\mathrm{Quas}(\mathbf{K}),\star_L)$ and $(\mathrm{Quas}(\mathbf{K}),\star_K)$, and $(\mathrm{Mult}(\mathbf{K}),\star_K)$ and $(\mathrm{Mult}(\mathbf{K}),\star_L)$ are isomorphic.*

Proof. Using the above ideas an alternative proof (to that given in 81) can be given as follows.

If $f, g \in \mathrm{Quas}(\mathbf{K})$ then the mapping

$$m_{\mathbf{K},K}(f_p) \mapsto m_{\mathbf{K},L}(f_p)$$

is one–to–one and maps $D_{\mathbf{K},K,p}$ onto $D_{\mathbf{K},L,p}$ while

$$m_{\mathbf{K},K}(f_p \star_K g_p) \mapsto m_{\mathbf{K},L}(f_p \star_L g_p).$$

This induces an isomorphism between the subgroups of $\widetilde{\prod}_{p\in\mathcal{P}} \mathrm{D}_{\mathbf{K},K,p}$ and $\widetilde{\prod}_{p\in\mathcal{P}} \mathrm{D}_{\mathbf{K},L,p}$ defined by the condition that the diagonal value a is a common number in all components of an element of the direct product.

The reformulation of the remaining results of previous section for Davison convolutions due to the above isomorphism is left to the reader.

4. Concluding generalization

In the previous reasoning we used from the properties of positive integer only the unique factorization property. Thus all previous results can be lifted to arithmetical functions defined on the so called arithmetical semigroups.

Let G denote a free commutative semigroup relative to a multiplication operation denoted by juxtaposition, with identity element 1_G and with at most countably many generators. Such a semigroup will be called **arithmetical semigroup** if in addition a real-valued **norm** $|\cdot|$ is defined on G such that

(1) $|1_G| = 1, |a| > 1$ for all $a \in G$,

(2) $|ab| = |a|.|b|$ for all $a, n \in G$,

(3) the total number

$$N_G(x) = \sum_{\substack{|a|\le x \\ a\in G}} 1$$

of elements $a \in G$ of norm not exceeding x is finite for each real x.

The role of primes take over the generators of G.

More details on abstract approach to the theory of arithmetical functions via the notion of arithmetical semigroup can be found in [9] or [10], where the interested reader may also find many instances of arithmetical semigroups.

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References

[1] Apostol, T. M., *Introduction to Analytic number Theory*, Springer-Verlag, New York – Heidelberg – Berlin, 1976.

[2] Bell, E. T., *Arithmetical Theory of Numerical Functions*, University of Washington 1915.

[3] Brown, A., Halmos, P. R., Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.*, **213** 1963, 89–102.

[4] Davison, T. M. K., On arithmetical convolution, *Canad. Math. Bull.*, **9** (1966), 287–296.

[5] Dehaye, P. O., On the structure of the group of multiplicative arithmetical functions, *Bull. Belg. Math. Soc.*, **9** (2002), 15–21.

[6] Fuchs, L., *Infinite Abelian Groups*, Vol. I, Academic Press, New York and London, 1970.

[7] Haukkanen, P., Some classes of quasi–fields having isomorphic additive and multiplicative groups, *Rend. Mat. Appl.*, (7) **7** (1987), 181–191.

[8] Haukkanen, P., On the Davison convolution of arithmetical functions, *Canad. Math. Bull.*, **32** (4) (1989), 467–473.

[9] Knopfmacher, J., *Abstract Analytic Number Theory*, North-Holland Mathematical Library Vol.12, North-Holland & American Elsevier, Amsterdam – Oxford – New York, 1975.

[10] Knopfmacher, J., Recents developments and applications of abstract analytic number theory, *Quaest. Math.*, **24**, (2001), 291–3070.

[11] Lahiri, D. B., Hypo-multiplicative number-theoretic functions, *Aequationes Math.*, **9** (1973), 184–192.

[12] Niven, I., Zuckerman, H. S., *An Introduction to the Theory of Numbers* (Third Edition), J. Wiley & Sons, New York – London – Sydney – Toronto, 1972.

[13] Shapiro, H. N., *Introduction to the Theory of Numbers*, Wiley, 1983.

[14] Succi, F., Sul gruppo moltiplicativo delle funzioni aritmetiche regolari, *Rend. Mat. Appl.*, (5) **19** (1960), 458–472.

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