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# ON ADDITIVE FUNCTIONS SATISFYING CONGRUENCE PROPERTIES

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*Dedicated to the memory of Professor Péter Kiss*

**Abstract.** In this paper, we consider those integer-valued additive functions $f_1$ and $f_2$ for which the congruence $f_1(an+b) \equiv f_2(cn)+d \pmod{n}$ is satisfied for all positive integers $n$ and for some fixed integers $a\geq 1$, $b\geq 1$, $c\geq 1$ and $d$. Our result improve some earlier results of K. Kovács, I. Joó, I. Joó & B. M. Phong and P. V. Chung concerning the above congruence.

## 1. Introduction

The problem concerning the characterization of some arithmetical functions by congruence properties initiated by Subbarao [10] was studied later by several authors. M. V. Subbarao proved that if an integer-valued multiplicative function $g(n)$ satisfies the congruence

$$g(n+m) \equiv g(m) \pmod{n}$$

for all positive integers $n$ and $m$, then there is a non-negative integer $\alpha$ such that

$$g(n) = n^{\alpha}$$

holds for all positive integers $n$. Recently some authors generalized and improved this result in a variety of ways. A. Iványi [3] obtained that the same result holds when $m$ is a fixed positive integer and $g$ is an integer-valued completely multiplicative function. For further results and generalizations of this problem we refer to the works of B. M. Phong [7]–[8], B. M. Phong & J. Fehér [9], I. Joó [4] and I. Joó & B. M. Phong [5]. For example, it follows from [8] that if an integer-valued multiplicative function $g(n)$ satisfies the congruence

$$g(An+B) \equiv C \pmod{n}$$

for all positive integers $n$ and for some fixed integers $A \geq 1$, $B \geq 1$ and $C \neq 0$ with $(A,B) = 1$, then there are a non-negative integer $\alpha$ and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that

$$g(n) = \chi_A(n)n^{\alpha}$$

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holds for all positive integers $n$ which are prime to $A$.

In the following let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all integer-valued additive and completely additive functions, respectively. Let $I\!N$ denote the set of all positive integers. A similar problem concerning the characterization of a zero-function as an integer-valued additive function satisfying a congruence property have been studied by K. Kovács [6], P. V. Chung [1]–[2], I. Joó [4] and I. Joó & B. M. Phong [5]. It was proved by K. Kovács [6] that if $f \in \mathcal{A}^*$ satisfies the congruence

$$f(An+B) \equiv C \qquad (\mathrm{mod}\ n)$$

for some integers $A \geq 1$, $B \geq 1$, $C$ and for all $n \in I\!N$, then

$$f(n) = 0$$

holds for all $n \in I\!N$ which are prime to $A$. This result was extended in [1], [2], [4] and [5] for integer-valued additive functions $f$. It follows from the results of [2] and [4] that for integers $A \geq 1$, $B \geq 1$, $C$ and functions $f_1 \in \mathcal{A}$, $f_2 \in \mathcal{A}^*$ the congruence

$$f_1(An+B) \equiv f_2(n) + C \qquad (\mathrm{mod}\ n) \quad (\forall n \in I\!N)$$

implies that $f_2(n) = 0$ for all $n \in I\!N$ and $f_1(n) = 0$ for all $n \in I\!N$ which are prime to $A$.

Our purpose in this paper is to improve the above results by showing the following

**Theorem 1.** *Assume that $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$ are fixed integers and the functions $f_1$, $f_2$ are additive. Then the congruence*

$$f_1(an+b) \equiv f_2(cn) + d \qquad (\mathrm{mod}\ n) \tag{1}$$

*is satisfied for all $n \in I\!N$ if and only if the equation*

$$f_1(an+b) = f_2(cn) + d \tag{2}$$

*holds for all $n \in I\!N$.*

**Theorem 2.** *Assume that $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$ are fixed integers. Let $a_1 = \frac{a}{(a,\, b)}$, $b_1 = \frac{b}{(a,\, b)}$ and*

$$\mu := \begin{cases} 1 & \text{if } 2 \mid a_1 b_1 \\ 2 & \text{if } 2 \nmid a_1 b_1. \end{cases}$$

*If the additive functions $f_1$ and $f_2$ satisfy the equation (2) for all $n \in I\!N$, then*

$$f_1(n) = 0 \quad \text{for all} \quad n \in I\!N,\ (n, \mu a b_1) = 1$$

*and*

$$f_2(n) = 0 \quad \textit{for all} \quad n \in I\!N,\ (n, \mu c b_1) = 1.$$

## 2. Lemmas

**Lemma 1.** *Assume that $f^* \in \mathcal{A}^*$ satisfies the congruence*

$$f^*(An + B) \equiv f^*(n) + D \pmod{n}$$

*for some fixed integers $A \geq 1$, $B \geq 1$ and $D$. Then $f^*(n) = 0$ holds for all $n \in I\!N$.*

**Proof.** Lemma 1 follows from Theorem 2 of [4].

**Lemma 2.** *Assume that $f \in \mathcal{A}$ satisfies the congruence*

$$f(An + B) \equiv D \pmod{n}$$

*for some fixed integers $A \geq 1$, $B \geq 1$ and $D$. Then $f(n) = 0$ holds for all $n \in I\!N$ which are prime to $A$.*

**Proof.** This is the result of [1].

**Lemma 3.** *Assume that $f_1$, $f \in \mathcal{A}$ satisfy the congruence*

$$f_1(An + 1) \equiv f(Cn) + D \pmod{n} \tag{3}$$

*holds for all $n \in I\!N$ with some integers $A \geq 1$, $C \geq 1$ and $D$. Then*

$$f(n) = f[(n, 6C^2)] \quad \textit{for all} \quad n \in I\!N$$

*and $f_1(m) = 0$ holds for all $m \in I\!N$, which are prime to $6AC$. Here $(x, y)$ denotes the greatest common divisor of the integers $x$ and $y$.*

**Proof.** In the following we shall denote by $n^*$ the product of all distinct prime divisors of positive integer $n$.

For each positive integer $M$ let $P = P(M)$ be a positive integer for which

$$(M^2 - 1)^* | ACP. \tag{4}$$

It is obvious from (4) that

$$(ACM(M + 1)Pn + 1, AC(M + 1)Pn + 1) = 1,$$

$$\big(C^2(M + 1)^2 Pn, ACMPn + 1\big) = 1$$

and

$$(ACM(M+1)Pn+1)\,(AC(M+1)Pn+1) = AC(M+1)^2Pn[ACMPn+1]+1$$

hold for all $n \in I\!N$. Using these relations and appealing to the additive nature of the functions $f_1$ and $f$, we can deduce from (3) that

$$f(ACMPn+1) \tag{5}$$

$$\equiv -f\left(C^2(M+1)^2Pn\right) + f\left(C^2M(M+1)Pn\right) + f\left(C^2(M+1)Pn\right) + D \pmod n$$

is satisfied for all $n$, $M \in I\!N$, where $P = P(M)$ satisfies the condition (4).

Let $M = 2$, $P(2) = 3$ and $M = 3$, $P(3) = 2$. In these cases (4) is true and so it follows from (5) that

$$f(6ACn+1) \equiv -f(27C^2n) + f(18C^2n) + f(9C^2n) + D \pmod n \tag{6}$$

and

$$f(6ACn+1) \equiv -f(32C^2n) + f(24C^2n) + f(8C^2n) + D \pmod n \tag{7}$$

are satisfied for all $n \in I\!N$. Let $N$ and $n$ be positive integers with the condition

$$(N(N+1), 6ACn+1) = 1. \tag{8}$$

By using the relation

$$(6ACn+1)(6^2A^2C^2Nn^2+1) = 6ACn\,[6ACNn(6ACn+1)+1]+1$$

and that

$$\left(6ACn+1, 6^2A^2C^2Nn^2+1\right) = (6ACn+1, N+1) = 1,$$

$$(6ACNn, 6ACn+1) = (6ACn+1, N) = 1,$$

it follows from (6) and (7) that

$$-f\left(162AC^3Nn^2\right) + f\left(108AC^3Nn^2\right) + f\left(54AC^3Nn^2\right) \equiv -f\left(27C^2Nn\right) \tag{9}$$

$$+f\left(18C^2Nn\right) + f\left(9C^2Nn\right) - f\left(27C^2n\right) + f\left(18C^2n\right) + f\left(9C^2n\right) + D \pmod n$$

and

$$-f\left(192AC^3Nn^2\right) + f\left(144AC^3Nn^2\right) + f\left(48AC^3Nn^2\right) \equiv -f\left(32C^2Nn\right) \tag{10}$$

$$+f\left(24C^2Nn\right) + f\left(8C^2Nn\right) - f\left(32C^2n\right) + f\left(24C^2n\right) + f\left(8C^2n\right) + D \pmod n$$

hold for all $n$, $N \in I\!N$ satisfying (8).

Let $Q$ be a fixed positive integer. First we apply (9) when $N=1$, $n=Qm$, $(m,Q)=1$ and $m\to\infty$. It is obvious that (8) holds, and so by (9) we have

$$f\left(Q^2\right)=2f(Q)\quad\text{for}\quad Q\in I\!N,(Q,6AC)=1. \tag{11}$$

Now let $N=Q$ and $n=Q^k(6CQm+1)$ with $k$, $m\in I\!N$. It is obvious that (8) holds for infinity many integers $m$, because $\left(36AC^2Q^{k+1},6ACQ^k+1\right)=1$. These with (9) show that

$$f\left(Q^{2k+1}\right)=f\left(Q^k\right)+f\left(Q^{k+1}\right)\quad\text{for all}\quad Q\in I\!N,(Q,6AC)=1. \tag{12}$$

From (11) and (12) we obtain that

$$f\left(Q^k\right)=kf(Q)\quad\text{for all}\quad Q\in I\!N,(Q,6AC)=1. \tag{13}$$

Thus, by using the additivity of $f$ it follows from (8) and (13) that (9) and (10) hold for all $N$, $n\in I\!N$, and they with $n=Qm$, $(m,6ACNQ)=1$, $m\to\infty$ imply that

$$-f\left(162AC^3NQ^2\right)+f\left(108AC^3NQ^2\right)+f\left(54AC^3NQ^2\right)=-f\left(27C^2NQ\right)$$
$$+f\left(18C^2NQ\right)+f\left(9C^2NQ\right)-f\left(27C^2Q\right)+f\left(18C^2Q\right)+f\left(9C^2Q\right)D$$

and

$$-f\left(192AC^3NQ^2\right)+f\left(144AC^3NQ^2\right)+f\left(48AC^3NQ^2\right)=-f\left(32C^2NQ\right)$$
$$+f\left(24C^2NQ\right)+f\left(8C^2NQ\right)-f\left(32C^2Q\right)+f\left(24C^2Q\right)+f\left(8C^2Q\right)+D$$

hold for all $N$, $Q\in I\!N$. Consequently

(14)
$$-f\left(27C^2NQ\right)+f\left(18C^2NQ\right)+f\left(9C^2NQ\right)-f\left(27C^2Q\right)+f\left(18C^2Q\right)+f\left(9C^2Q\right)$$
$$-f\left(27C^2NQ^2\right)+f\left(18C^2NQ^2\right)+f\left(9C^2NQ^2\right)-f\left(27C^2\right)+f\left(18C^2\right)+f\left(9C^2\right)$$

and

(15)
$$-f\left(32C^2NQ\right)+f\left(24C^2NQ\right)+f\left(8C^2NQ\right)-f\left(32C^2Q\right)+f\left(24C^2Q\right)+f\left(8C^2Q\right)$$
$$=-f\left(32C^2NQ^2\right)+f\left(24C^2NQ^2\right)+f\left(8C^2NQ^2\right)-f\left(32C^2\right)+f\left(24C^2\right)+f\left(8C^2\right)$$

are satisfied for all $N$, $Q\in I\!N$.

For each prime $p$ let $e=e(p)$ be a non-negative integer for which $p^e\parallel C^2$.

First we consider the case when $(p,6)=1$. By applying (14) with $Q=p$, $N=p^l$ $(l\geq 0)$, we have

$$f\left(p^{l+e(p)+2}\right)-f\left(p^{l+e(p)+1}\right)=f\left(p^{e(p)+1}\right)-f\left(p^{e(p)}\right)\quad\text{for all}\quad l\geq 0,$$

which shows that for all integers $\beta\geq e(p)$

$$f\left(p^{\beta+1}\right)-f\left(p^{\beta}\right)=f\left(p^{e(p)+1}\right)-f\left(p^{e(p)}\right). \tag{16}$$

Now we consider the case $p=2$. Applying (14) with $Q=2$ and $n=2^l$, $(l\geq 0)$ one can check as above that

$$f\left(2^{\beta+1}\right)-f\left(2^{\beta}\right)=f\left(2^{e(2)+2}\right)-f\left(2^{e(2)+1}\right). \tag{17}$$

Finally, we consider the case $p=3$. Applying (15) with $Q=3$ and $N=3^l$, $l\geq 0$ we also get

$$f\left(3^{\beta+1}\right)-f\left(3^{\beta}\right)=f\left(3^{e(3)+2}\right)-f\left(3^{e(3)+1}\right). \tag{18}$$

Now we write

$$f(n)=f^*(n)+F(n),$$

where $f^*$ is a completely additive function defined as follows:

$$f^*(p):=\begin{cases} f\left(p^{e(p)+1}\right)-f\left(p^{e(p)}\right) & \text{for } (p,6)=1\\ f\left(p^{e(p)+2}\right)-f\left(p^{e(p)+1}\right) & \text{for } p=2 \text{ or } p=3\end{cases}. \tag{19}$$

Then, from (16)-(19) it follows that

$$F\left(p^k\right)=F\left[(p^k,6C^2)\right]\quad\text{for}\quad(k=0,1,\ldots).$$

Thus, we have proved that

$$F(n)=F\left[(n,6C^2)\right] \tag{20}$$

is satisfied for all $n\in I\!N$.

We shall prove that $f^*(n)=0$ for all $n\in I\!N$ and $f_1(m)=0$ for all $m\in I\!N$ which are prime to $6AC$.

We note that, by considering $n=2m$ and taking into account (6), we have

$$f(12ACm+1)\equiv -f(54C^2m)+f(36C^2m)+f(18C^2m)+D\pmod{m}$$

Since $f = f^* + F$, from the last relation and (20) we get

$$f^*(12ACm + 1) \equiv f^*(m) + [f^*(12C^2) + F(6C^2) + D] \pmod{m},$$

which with Lemma 1 shows that $f^*(n) = 0$ for all $n \in I\!N$. This shows that $f \equiv F$, i.e.

$$f(n) = f[(n, 6C^2)]$$

holds for all $n \in I\!N$. Now, by applying (3) with $n = 6Cm$ and using the last relation and Lemma 2, we have that $f_1(n) = 0$ holds for all $n \in I\!N$ which are prime to $6AC$.

The proof of Lemma 3 is completed.

## 3. Proof of Theorem 1

It is obvious that (1) follows from (2). We shall prove that if (1) is true, then (2) holds.

Assume that the functions $f_1$ and $f_2 \in \mathcal{A}$ satisfy the congruence (1) for some integers $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$. It is obvious that (1) implies the fulfilment of

$$f_1(abn + 1) \equiv f_2(b^2cn) + d - f_1(b) \pmod{n}$$

for all $n \in I\!N$. By Lemma 3,

$$f_2(n) = f_2[(n, 6b^4c^2)] \quad \text{for all} \quad n \in I\!N \tag{21}$$

and

$$f_1(n) = 0 \tag{22}$$

for all $n \in I\!N$ which are prime to $6abc$.

We shall prove that

$$f_1(an + b) = f_2(cn) + d \tag{23}$$

is true for all $n \in I\!N$.

Let $K$ be a positive integer. By (21) and (22), we have

$$f_1(6ab^4ct + 1) = 0,$$

$$f_2[6b^4c^2(aK + b)t + cK] = f_2(cK)$$

hold for all positive integers $t$, consequently

$$f_1(aK+b)-f_2(cK)-d=f_1(aK+b)+f_1(6ab^4ct+1)-f_2(cK)-d$$
$$=f_1[a(6b^4c(aK+b)t+K)+b]-f_2[6b^4c^2(aK+b)t+cK]-d$$

holds for every positive integer $t$. Thus, by applying (1) with $n=6b^4c(aK+b)t+K$, the last relation proves that (23) holds for $n=K$.

This completes the proof of Theorem 1.

## 4. Proof of Theorem 2

As we have shown in the proof of Theorem 1, if the functions $f_1,\ f_2\in\mathcal{A}$ satisfy (2), then (21) and (22) imply

$$f_1(m)=0 \quad \text{for all} \quad m\in I\!N,\ (m,6abc)=1 \tag{24}$$

and

$$f_2(n)=0 \quad \text{for all} \quad n\in I\!N,\ (m,6bc)=1. \tag{25}$$

Let $D=(a,b),\ a_1=\frac{a}{D},\ b_1=\frac{b}{D}$. It is clear that for each positive integer $M,\ (M,a_1)=1$ there are $m_0,n_0\in I\!N$ such that

$$Mm_0=a_1n_0+b_1,\quad (m_0,\ a_1)=1 \ \text{ and } \ (M,\ n_0)=(M,\ b_1). \tag{26}$$

Let

$$u(M):=\begin{cases}1, & \text{if} \quad 2\mid a_1\frac{M}{(M,\ b_1)}\frac{b_1}{(M,\ b_1)},\\ 2, & \text{if } 2\not|\, a_1\frac{M}{(M,\ b_1)}\frac{b_1}{(M,\ b_1)}.\end{cases} \tag{27}$$

By applying the Chinese Remainder Theorem and using (26)–(27), we can choose a positive integer $t_1$ such that $m_1=a_1t_1+m_0,\ n_1=Mt_1+n_0$ satisfy the following conditions:

$$Mm_1=a_1n_1+b_1 \quad ,$$

$$\frac{n_1}{u(M)(M,\ b_1)} \quad \text{is an integer,}$$

and

$$(m_1,\ 6abc)=\left(\frac{n_1}{u(M)(M,\ b_1)},\ 6bc\right)=1.$$

Hence, we infer from (2) and (24)-(25) that

$$f_1(DM)=f_1(DMm_1)=f_1(an_1+b)=f_2(cn_1)+d=f_2\left[cu(M)(M,b_1)\right]+d,$$

consequently

$$f_1\left[DM\right] = f_2\left[cu(M)(M, b_1)\right] + d \tag{28}$$

hold for all $M \in I\!N$, $(M,\ a_1) = 1$. This implies that

$$f_1(n) = 0 \quad \text{for all} \quad n \in I\!N, \quad (n,\ \mu a b_1) = 1, \tag{29}$$

where $\mu \in \{1,\ 2\}$ such that $2|\ \mu a_1 b_1$.

Now we prove that

$$f_2(n) = 0 \quad \text{for all} \quad n \in I\!N, \quad (n,\ \mu c b_1) = 1. \tag{30}$$

For each positive integer $n$, let $M(n) := a_1 n + b_1$ and $U(n) := u(a_1 n + b_1)$. Since $(M(n),\ b_1) = (n,\ b_1)$ and

$$a_1 \frac{M(n)}{(M(n),\ b_1)} \frac{b_1}{(M(n),\ b_1)} \equiv a_1 \frac{b_1}{(n,\ b_1)} \left[\frac{n}{(n,\ b_1)} + 1\right] \pmod 2,$$

we have

$$U(n) := \begin{cases} 1, & \text{if} \quad 2 \mid a_1 \frac{b_1}{(n,\ b_1)} \left[\frac{n}{(n,\ b_1)} + 1\right], \\ 2, & \text{if } 2 \nmid a_1 \frac{b_1}{(n,\ b_1)} \left[\frac{n}{(n,\ b_1)} + 1\right]. \end{cases}$$

Hence, (2) and (28) show that

$$f_2(cn) = f_1(an + b) - d = f_1\left[DM(n)\right] - d = f_2\left[cU(n)(n, b_1)\right]$$

is satisfied for all $n \in I\!N$, which implies (29). Thus, (29) is proved.

By (29) and (30), the proof of Theorem 2 is completed.

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