Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae **30** *(2003) 101–108*

RECIPROCAL INVARIANT DISTRIBUTED SEQUENCES CONSTRUCTED BY SECOND ORDER LINEAR RECURRENCES

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Dedicated to the memory of Professor Péter Kiss

Abstract. In this paper we determine necessary and sufficient conditions for the sequence $(G_{n+1}/G_n)_{n=0}^{\infty}$ to become a reciprocal invariant distributed sequence modulo 1, where G_n is the n-th term of a non-degenerate second order linear recurrence of real numbers.

1. Introduction

Let $G = G(A, B, G_0, G_1) = (G_n)_{n=0}^{\infty}$ be a second order linear recursive sequence of real numbers defined by the recursion

$$G_n = AG_{n-1} + BG_{n-2} \qquad (n > 1), \tag{1}$$

where A, B and the initial terms G_0, G_1 are fixed real numbers with restrictions $AB \neq 0$, $D = A^2 + 4B \neq 0$ and $G_0^2 + G_1^2 > 0$. It is well-known that the terms of G can be written in the form

$$G_n = a\alpha^n - b\beta^n, \tag{2}$$

where α and β are the roots of the characteristic polynomial $x^2 - Ax - B$ of the sequence G and $a = \frac{G_1 - G_0\beta}{\alpha - \beta}, b = \frac{G_1 - G_0\alpha}{\alpha - \beta}$ (see e.g. I. Niven and H. S. Zuckerman [9], p. 91).

Troughout this paper we assume $|\alpha| \geq |\beta|$ and the sequence is non-degenerate, i.e. α/β is not a root of unity and $ab \neq 0$. If $G_{n_0} = 0$ we may also suppose that $G_n \neq 0$ for $n \neq n_0$, since P. Kiss [2] proved that a non-degenerate sequence G has at most one zero term.

Distribution properties of the Fibonacci sequence $G = G(1, 1, 0, 1)$ and more general integer valued and real valued recurrences were studied by several authors. Here we only mention the papers [4], [3], [5] and [7], connected with our topic.

The object of this paper is to determine necessary and sufficient conditions for the sequence $(G_{n+1}/G_n)_{n=0}^{\infty}$ to become a reciprocal invariant distributed sequence modulo 1. (The definition of reciprocal invariant will be given later.)

The sequence $\omega = (x_n)_{n=1}^{\infty}$ is said to have asymptotic distribution function modulo 1 (a.d.f. mod 1) F if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x, x_n) = F(x) \quad \text{for} \quad 0 \le x \le 1,$$

where the function χ is defined by

$$\chi(x,y) = \begin{cases} 1, & \text{if } 0 \le \{y\} < x, \\ 0, & \text{if } x \le \{y\} \end{cases}$$

and $\{y\}$ denotes the fractional part of the real number y.

In [8] the following definition was introduced.

Definition. Let $\omega = (x_n)_{n=1}^{\infty}$ and $\xi = (f(x_n))_{n=1}^{\infty}$ be sequences of real numbers, where f is a real–valued function. If the sequences ω and ξ have a.d.f. mod 1 and these functions are identical, then we say ω is f invariant distributed sequence modulo 1. (i.d. mod 1 to f.)

In special cases:

(i) if ω is i.d. mod 1 to $f(x) = \frac{1}{x}$, we say ω is reciprocal invariant distributed sequence mod 1,

(ii) if ω is i.d. mod 1 to $f(x) = \sqrt{x}$ we say ω is a square root invariant distributed sequence mod 1.

P. Kiss and R. F. Tichy in [4] investigated the asymptotic distribution function modulo 1 of the sequence $(G_{n+1}/G_n)_{n=1}^{\infty}$ when $D < 0$. Their theorem can be extended to any sequence $(G_{n+k}/G_n)_{n=1}^{\infty}$, where k is a nonzero integer. We prove:

Theorem 1. *Let $G = (G_n)_{n=0}^{\infty}$ be a linear recurring sequence defined by $G_n = AG_{n-1} + BG_{n-2}, \quad (n > 1)$ with nonzero real coefficients A and B, real initial values G_0, G_1 (not both G_0 and G_1 are zero) and with negative discriminant $D = A^2 + 4B$. Let $k \ne 0$ be an integer. If the number $\Theta = \frac{1}{\pi} \arctan \frac{\sqrt{-D}}{A}$ is irrational, then the asymptotic distribution function modulo 1 H of the sequence $(G_{n+k}/G_n)_{n=1}^{\infty}$ is given by*

$$H(x) = H_1(x - \{c\}) + H_1(\{c\}) \tag{3}$$

with

$$H_1(x) = x + \frac{1}{\pi} \arctan \frac{\sin(2\pi x)}{\exp(2\pi|d|) - \cos(2\pi x)}, \tag{4}$$

$c = r^k \cos(k\pi\Theta), \quad d = -r^k \sin(k\pi\Theta)$ *and* $r = |\alpha| = \left|\frac{A+\sqrt{A^2+4B}}{2}\right|$.

Theorem 2. *Let $G = (G_n)_{n=0}^{\infty}$ be a non-degenerate second order linear recursive sequence defined by $G_n = AG_{n-1} + BG_{n-2} \quad (n > 1)$ with nonzero real coefficients A and B, real initial values G_0, G_1 (where $G_0^2 + G_1^2 \neq 0$) and negative discriminant $D = A^2 + 4B$. The sequence $\omega = (G_{n+1}/G_n)_{n=1}^{\infty}$ is reciprocal invariant distributed modulo 1 if and only if $B = -1$.*

Theorem 3. *Let $G = (G_n)_{n=0}^{\infty}$ be a non-degenerate second order linear recursive sequence defined by the recursion $G_n = AG_{n-1} + BG_{n-2} \quad (n > 1)$ with nonzero integer coefficients A and B, integer initial values G_0, G_1 (where $G_0^2 + G_1^2 \neq 0$) and with positive discriminant $D = A^2 + 4B$. The sequence $\omega = (G_{n+1}/G_n)_{n=1}^{\infty}$ is reciprocal invariant distributed modulo 1 if and only if $B = 1$.*

2. Proofs

Proof of Theorem 1. Let G be a second order linear recursive sequence satisfying the conditions of Theorem 1. We know from [2] that the zero multiplicity of G is at most one and one element is not relevant for the asymptotic distribution function therefore without loss of generality we may assume that $G_n \neq 0$ for $n \geq 0$. In (2) α, β and a, b are complex conjugate numbers since $D = A^2 + 4B < 0$ and we can write

$$\alpha = r\exp(i\pi\Theta), \qquad \beta = r\exp(-i\pi\Theta) \tag{5}$$

and

$$a = r_1\exp(i\pi\omega), \qquad b = r_1\exp(-i\pi\omega), \tag{6}$$

where $\exp(x)$ denotes the usual exponential function and

$$0 < \Theta = \frac{1}{\pi}\arctan\frac{\sqrt{-D}}{A} < 1, \quad \omega = \frac{1}{\pi}\arctan\frac{AG_0 - 2G_1}{G_0\sqrt{-D}},$$

while r and r_1 are positive real numbers, $a \neq 0$ and $b \neq 0$. Since G is a non-degenerate sequence we have Θ is an irrational number.

By (2), (5) and (6) we obtain for all $n \geq \max\{0, -k\} = n_0$ that

$$\frac{G_{n+k}}{G_n} = \frac{r_1 r^{n+k}\exp(i\pi(\omega + (n+k)\Theta)) + r_1 r^{n+k}\exp(-i\pi(\omega + (n+k)\Theta))}{r_1 r^n\exp(i\pi(\omega + n\Theta)) + r_1 r^n\exp(-i\pi(\omega + n\Theta))}$$

$$= r^k\frac{\cos(\pi(\omega + (n+k)\Theta))}{\cos(\pi(\omega + n\Theta))} = r^k(\cos(\pi k\Theta) - \sin(\pi k\Theta)\cdot\tan(\pi(\omega + n\Theta)))$$

$$= c + d\tan(\pi(\omega + n\Theta)),$$

where $c = r^k \cos(k\pi\Theta), d = -r^k \sin(k\pi\Theta)$ are nonzero real numbers independent on n. Note that the proof of the inequality

$$\left| \frac{1}{N} \sum_{n=n_0}^{N+n_0-1} \chi\left(x, \frac{G_{n+k}}{G_n}\right) - \int_0^1 \chi(x, c + d\tan(\pi(y+\omega))dy \right|$$

$$\leq 4\sqrt{|r^k \sin(k\pi\Theta)|}\sqrt{\Delta_N} + 6\Delta_N,$$

where $\Delta_N = \Delta_N(\Theta n)$ denotes the discrepancy of the sequence $(\Theta n)_{n=1}^{\infty}$ which is analogous to described in [4] by P. Kiss and R. F. Tichy. Since we only need a.d.f. mod 1, we omit the proof.

In the following we compute the integral

$$H(x) = \int_0^1 \chi(x, c + d\tan(\pi(y+\omega))dy = \int_{-1/2}^{1/2} \chi(x, c + d\tan(\pi(y+\omega))dy \tag{7}$$

in the case $c = 0$. By the substitution $u = d\tan(\pi y)$ we get

$$H_1(x) = \frac{|d|}{\pi} \int_{-\infty}^{\infty} \frac{\chi(x,u)}{d^2+u^2} du. \tag{8}$$

We use the Fourier series expansion of the characteristic function

$$\chi(x,u) = x + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{m} \cos(2\pi mu) + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1-\cos(2\pi mx)}{m} \sin(2\pi mu)$$

and the integral formulae

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi mu)}{d^2+u^2} du = \frac{\pi}{|d|} \exp(-2\pi m|d|), \int_{-\infty}^{\infty} \frac{\sin(2\pi mu)}{d^2+u^2} du = 0 \quad \text{(see e.g. [1])}.$$

By swapping summation and integration and applying Lebesgue's theorem on dominated convergence we have

$$H_1(x) = x + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{m} \exp(-2\pi m|d|) = x + \frac{1}{\pi} \Im\left(\sum_{m=1}^{\infty} \frac{w^m}{m}\right),$$

where $w = \exp(2\pi(-|d| + ix))$. Since $-|d| < 0$, we have $|w| < 1$ and $\Re(1-w) > 0$, so $\sum_{m=1}^{\infty} \frac{w^m}{m} = -\log(1-w)$. Since

$$\Im(1-w) = \exp(-2\pi|d|)\sin(2\pi x) \text{ and } \Re(1-w) = 1 - \exp(-2\pi|d|)\cos(2\pi x)$$

it follows that

$$H_1(x) = x + \frac{1}{\pi}\arctan\frac{\exp(-2\pi|d|)\sin(2\pi x)}{1-\exp(-2\pi|d|)\cos(2\pi x)}$$

$$= x + \frac{1}{\pi}\arctan\frac{\sin(2\pi x)}{\exp(2\pi|d|)-\cos(2\pi x)}.$$

Since $H_1(-x) = -H_1(x), H(x) = H_1(x-c) - H_1(-c) = H_1(x-c) + H_1(c)$, the proof of the theorem is complete.

Proof of Theorem 2. Let G be a second order linear recursive sequence satisfying the conditions of Theorem 2. By [4] the a.d.f. mod 1 of the sequence $(G_{n+1}/G_n)_{n=1}^{\infty}$ is $F(x) = F_1(x - \{A/2\}) + F_1(\{A/2\})$, where

$$F_1(x) = x + \frac{1}{\pi}\arctan\frac{\sin(2\pi x)}{\exp(\pi\sqrt{-D})-\cos(2\pi x)}.$$

One can check that $\omega = (G_n/G_{n+1})_{n=0}^{\infty} = (G_{n-1}/G_n)_{n=1}^{\infty} = \xi$. The a. d. f. mod 1 ω and ξ are identical which is easy to derive by Theoem 1. Indeed, if $k = -1$ and $c = r^{-1}\cos(-\pi\Theta) = \frac{r\cos(\pi\Theta)}{r^2} = -\frac{A}{2B}$ and $d = -r^{-1}\sin(-\pi\Theta) = \frac{r\sin(\pi\Theta)}{r^2} = -\frac{\sqrt{-D}}{2B}$ then

$$H(x) = H_1\left(x - \left\{\frac{-A}{2B}\right\}\right) + H_1\left(\left\{\frac{-A}{2B}\right\}\right),$$

where

$$H_1(x) = x + \frac{1}{\pi}\arctan\frac{\sin(2\pi x)}{\exp\left(\frac{\pi\sqrt{-D}}{-B}\right)-\cos(2\pi x)}.$$

We have to decide some necessary and sufficient conditions for the equality

$$F(x) = H(x) \quad 0 \le x \le 1. \tag{9}$$

A straightforward calculation shows that the derivate of $F(x)$ and $H(x)$ is given by

$$F^{'}(x) = 1 + 2\frac{E_1\cos\left(2\pi\left(x - \left\{\frac{A}{2}\right\}\right)\right) - 1}{E_1^2 - 2E_1\cos\left(2\pi\left(x - \left\{\frac{A}{2}\right\}\right)\right) + 1} \tag{10}$$

and

$$H^{'}(x) = 1 + 2\frac{E_2 \cos\left(2\pi\left(x - \left\{\frac{-A}{2B}\right\}\right)\right) - 1}{E_2^2 - 2E_2\cos\left(2\pi\left(x - \left\{\frac{-A}{2B}\right\}\right)\right) + 1}, \tag{11}$$

where

$$E_1 = \exp(\pi\sqrt{-D}) \quad \text{and} \quad E_2 = \exp\left(\frac{\pi\sqrt{-D}}{B}\right).$$

This yields that the graph of $F(x)$ is steepest at $\{A/2\}$ and the graph of $H(x)$ is steepest at $\left\{\frac{-A}{2B}\right\}$. By (9) we get $x_0 = \{A/2\} = \left\{\frac{-A}{2B}\right\}$ and thus

$$F(x_0) = F_1(0) + F_1\left(\left\{\frac{A}{2}\right\}\right) = F_1\left(\left\{\frac{A}{2}\right\}\right)$$

and

$$H(x_0) = H_1(0) + H_1\left(\left\{\frac{-A}{2B}\right\}\right) = H_1\left(\left\{\frac{A}{2}\right\}\right).$$

On the other hand,

$$F_1\left(\left\{\frac{A}{2}\right\}\right) = H_1\left(\left\{\frac{A}{2}\right\}\right)$$

implies

$$\exp(\pi\sqrt{-D}) = \exp\left(\frac{\pi\sqrt{-D}}{-B}\right)$$

and $B = -1$. If $B = -1$ then $F(x) = H(x) \quad (0 \le x \le 1)$ is trivially true. Therefore $B = -1$ is a necessary and sufficient condition for $(G_{n+1}/G_n)_{n=0}^{\infty}$ to be reciprocal invariant distributed mod 1.

Proof of Theorem 3. Suppose $|\alpha| \ge |\beta|$, where α and β are the roots of the characteristic polynomial of G. By the conditions of Theorem 3, $D > 0$, therefore $|\alpha| > |\beta|$. From $\alpha\beta = -B \in \mathbf{Z}$ and $B \ne 0$ it follows that $|\alpha| > 1$. Then $(G_{n+1}/G_n)_{n=0}^{\infty}$ and $(G_n/G_{n+1})_{n=0}^{\infty}$ is convergent (c.f. [7]).

Indeed,

$$\lim_{n\longrightarrow\infty}\frac{G_{n+1}}{G_n} = \lim_{n\longrightarrow\infty}\frac{a\alpha^{n+1} - b\beta^{n+1}}{a\alpha^n - b\beta^n} = \lim_{n\longrightarrow\infty}\alpha\frac{1-(b/a)(\beta/\alpha)^{n+1}}{1-(b/a)(\beta/\alpha)^n} = \alpha$$

and

$$\lim_{n\longrightarrow\infty}\frac{G_n}{G_{n+1}} = \frac{1}{\alpha}.$$

The sequence $(G_{n+1}/G_n)_{n=0}^{\infty}$ can only be reciprocal invariant distributed mod 1 if $\alpha \equiv \frac{1}{\alpha} \pmod 1$.

If $\alpha > 1$ then $0 < \frac{1}{\alpha} < 1$, therefore there is a positive integer c, for which $\alpha - c = \frac{1}{\alpha}$. By multiplying the equality by α, we have

$$\alpha^2 - c\alpha - 1 = 0. \tag{12}$$

If $\alpha < -1$ then $-1 < \frac{1}{\alpha} < 0$, therefore

$$\alpha^2 + (c-1)\alpha - 1 = 0. \tag{13}$$

So there exists an integer A, such that α is a root of the equation

$$x^2 - Ax - 1 = 0. \tag{14}$$

The constants in (1), by the condition of Theorem 3, are integers and at the same time (14) is the characteristic equation of the sequence G, so therefore the condition $B = 1$ is necessary.

An easy calculation shows that if $|\alpha| > 1$ and $B = 1$ then the sequence $(G_{n+1}/G_n)_{n=0}^{\infty}$ and $(G_n/G_{n+1})_{n=0}^{\infty}$ are such ones that their limit points are greater and smaller, alternately. Then there exists and a.d.f. mod 1 for both sequences, which is the function

$$F(x) = \begin{cases} 0, & \text{if } 0 \le x < \{\alpha\}, \\ \frac{1}{2}, & \text{if } x = \{\alpha\}, \\ 1, & \text{if } \{\alpha\} < x \le 1. \end{cases}$$

The proof is complete.

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