ALMOST SURE FUNCTIONAL LIMIT THEOREMS IN $L^p(]0,1[)$, WHERE $1 \le p < \infty$

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Dedicated to the memory of Professor Péter Kiss

Abstract. The almost sure version of Donsker's theorem is proved in $L^p(]0,1[)$, where $1 \le p < \infty$. The almost sure functional limit theorem is obtained for the empirical process in $L^p(]0,1[)$, where $1 \le p < \infty$.

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1. Introduction

The simplest form of the central limit theorem is $\frac{1}{\sigma\sqrt{n}} S_n \Rightarrow \mathcal{N}(0,1)$, as $n \to \infty$, if S_n is the n^{th} partial sum of independent, identically distributed (i.i.d.) random variables with mean zero and variance σ^2. Here $\Rightarrow$ denotes convergence in distribution, while $\mathcal{N}(0,1)$ is the standard normal law. The functional central limit theorem, proved by Donsker, states that the broken line process connecting the points $(\frac{i}{n}, \frac{1}{\sigma\sqrt{n}} S_i)$, $i = 0, 1, \dots, n$, converges weakly to the standard Wiener process W in the space $C([0,1])$, see Billingsley [3].

A relatively new version of the CLT is the so called almost sure (a.s.) CLT, see Brosamler [4], Schatte [13], Lacey and Philipp [9]. The simplest form of the a.s. CLT is the following. Drop $\frac{1}{\log n} \frac{1}{k}$ weight to the point $\frac{1}{\sigma\sqrt{k}} S_k(\omega)$, $k = 1, \dots, n$. Then this discrete measure weakly converges to $\mathcal{N}(0,1)$ for P-almost every $\omega \in \Omega$. (Here $(\Omega, \mathcal{A}, P)$ is the underlying probability space.) The almost sure version of Donsker's theorem is also known, see e.g. Fazekas and Rychlik [7] and the references therein.

In this paper we will prove the a.s. version of Donker's theorem in $L^p(]0,1[)$, see Theorem 2.1.

In this space in contrast to the case of $C([0,1])$, we can manage without any maximal inequality. Using elementary facts of probability theory, we derive our result from the general a.s. limit theorem in Fazekas and Rychlik [7].

A well-known result of statistics is that the uniform empirical process converges to the Brownian bridge B in the space $D([0,1])$, see Billingsley [3]. The almost sure version of this theorem is also known, see e.g. Fazekas and Rychlik [7]. The proof of that theorem is based on a sophisticated inequality of Dvoretzky, Kiefer and Wolfowitz.

Here we show that the a.s. version of the limit theorem for the empirical process is valid in $L^p(]0,1[)$, see Theorem 3.1. Our proof uses only elementary facts.

We also prove the (non a.s.) functional limit theorems in $L^p(]0,1[)$. Proposition 2.1 is the Donsker theorem, Proposition 3.1 contains the convergence of the empirical process. The proof of these propositions are straightforward calculations to check the tightness conditions given in Oliveira and Suquet [12] and Marcinkiewicz and Zygmund, see e.g. in [5].

All results of this paper was proved for $p = 2$ in [14].

2. The almost sure Donsker theorem in $L^p(]0,1[)$

In this part we consider the process

(1) $$Y_n(t) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}, \quad \text{if} \quad t \in [0,1],$$

where $S_0 = 0$, $S_k = X_1 + X_2 + \cdots + X_k$, $k \geq 1$, and $X_1, X_2, \ldots$ are i.i.d. real random variables with $EX_1 = 0$ and $D^2 X_1 = \sigma^2$ and $E|X_1|^p < \infty$. Here $[\cdot]$ denotes the integer part. We shall prove a.s. limit theorem for $Y_n(t)$ in $L^p(]0,1[)$. For the sake of completeness first we prove the usual limit theorem.

We will use the next result due to Marcinkiewicz and Zygmund (see [5]) and its consequence (Remark 2.1).

Remark 2.1. If $\{X_n, n \geq 1\}$ are independent random variables with $EX_n = 0$, then for every $p \geq 1$ there exists a positive constant C_p depending only upon p for which
$$\left\| \sum_{i=1}^n X_i \right\|_p \leq C_p \left\| \left(\sum_{i=1}^n X_i^2 \right)^{1/2} \right\|_p .$$

Remark 2.2. If $\{X_n, n \geq 1\}$ are i.i.d. with $EX_1 = 0$, $E|X_1|^p < \infty$ if $2 \leq p < \infty$ and $E|X_1|^2 < \infty$ if $1 \leq p \leq 2$ and $S_n = \sum_{i=1}^n X_i$, then $E|S_n|^p = O\left(n^{p/2}\right)$.

We also need the result below due to Oliveira and Suquet [12].

Remark 2.3. Let $(Y_n(t), n \geq 1)$ be a sequence of random elements in $L^p(]0,1[)$, $p \geq 1$. Assume that

(i) for some $\gamma > 1$, $\sup_{n \geq 1} E\|Y_n\|_1^\gamma < \infty$,

(ii) $\lim_{h \to 0} \sup_{n \geq 1} E\|Y_n(\cdot + h) - Y_n(\cdot)\|_p^p = 0$.

Then $(Y_n(t), n \geq 1)$ is tight in $L^p(]0,1[)$.

Proposition 2.1. *The sequence of processes $(Y_n(t), n \geq 1)$ defined in (1) converges weakly to the standard Wiener process W in $L^p(]0,1[)$, where $1 \leq p < \infty$.*

Proof. According to Theorem 6.2.2. of [8] we have to prove that the family $(Y_n(t), n \geq 1)$ is tight and $\langle f, Y_n(t)\rangle \Rightarrow \langle f, W\rangle$ for each f from the dual space of $L^p(]0,1[)$. First consider the convergence in distribution of $\int_0^1 Y_n(t)f(t)dt$ to $\int_0^1 W(t)f(t)dt$ for f in $L^q(]0,1[)$, the dual space of $L^p(]0,1[)$.

But $\int_0^1 Y_n(t)f(t)dt$ converges weakly to normal distribution with mean zero and variance $\int_0^1\int_0^1 min\{s,t\}f(s)f(t)dsdt$. However it is the distribution of the random variable $\int_0^1 W(t)f(t)dt$.

Now, we prove that the conditions (i) and (ii) of Remark 2.3 are satisfied.

First we show that (i) is fulfilled with $\gamma = 2$, i.e., $\sup_{n\geq 1} E\|Y_n\|_1^2 < \infty$ is satisfied. This is implied by the following calculation:

$$\begin{aligned}
\sup_{n\geq 1} E\|Y_n\|_1^2 &= \sup_{n\geq 1} E\left\|\frac{1}{\sigma\sqrt{n}}S_{[nt]}\right\|_1^2 = \sup_{n\geq 1} E\left(\int_0^1\left|\frac{1}{\sigma\sqrt{n}}S_{[nt]}\right|dt\right)^2\\
&= \sup_{n\geq 1} E\left(\sum_{i=0}^{n-1}\int_{i/n}^{(i+1)/n}\left|\frac{S_i}{\sigma\sqrt{n}}\right|dt\right)^2 = \sup_{n\geq 1} E\left(\frac{1}{\sigma\sqrt{n}}\frac{1}{n}\sum_{i=0}^{n-1}|S_i|\right)^2\\
&\leq \sup_{n\geq 1}\frac{1}{\sigma^2 n}E\left(\frac{1}{n}\sum_{i=0}^{n-1}|S_i|^2\right) = \sup_{n\geq 1}\frac{1}{\sigma^2 n^2}\sum_{i=0}^{n-1}E|S_i|^2\\
&= \sup_{n\geq 1}\frac{1}{\sigma^2 n^2}\sigma^2\sum_{i=0}^{n-1} i = \sup_{n\geq 1}\left(\frac{1}{n^2}\frac{n(n-1)}{2}\right) = \sup_{n\geq 1}\left(\frac{n-1}{2n}\right) < \infty.
\end{aligned}$$

Now we prove condition (ii). This follows from the argument below, where $\{\cdot\}$ denotes the fractional part.

$$\begin{aligned}
E\|Y_n(t+h) - Y_n(t)\|_p^p &= E\int_0^1 |Y_n(t+h) - Y_n(t)|^p\,dt\\
&= E\int_0^{1-h}\left|\frac{1}{\sigma\sqrt{n}}S_{[n(t+h)]} - \frac{1}{\sigma\sqrt{n}}S_{[nt]}\right|^p dt\\
&\quad + E\int_{1-h}^1\left|\frac{1}{\sigma\sqrt{n}}S_{[nt]}\right|^p dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{1-h} E\left|\frac{1}{\sigma\sqrt{n}}\left(X_{[n(t+h)]}+\cdots+X_{[nt]+1}\right)\right|^p dt \\
&\quad + \int_{1-h}^1 E\left|\frac{1}{\sigma\sqrt{n}} S_{[nt]}\right|^p dt \\
&= \int_0^{1-h} \frac{1}{n^{p/2}\sigma^p} E\left|X_{[n(t+h)]}+\cdots+X_{[nt]+1}\right|^p dt \\
&\quad + \int_{1-h}^1 \frac{1}{n^{p/2}\sigma^p} E|S_{[nt]}|^p\, dt \\
&\le \int_0^{1-h} \frac{1}{n^{p/2}\sigma^p} C([n(t+h)]-[nt])^{p/2}\, dt \\
&\quad + \int_{1-h}^1 \frac{1}{n^{p/2}\sigma^p} C[nt]^{p/2}\, dt \\
&\le \frac{C}{n^{p/2}\sigma^p}\int_0^1 ([\{nt\}+\{nh\}]+[nh])^{p/2}\, dt + \frac{C}{n^{p/2}} h n^{p/2} \\
&\le C^* h \to 0, \quad \text{as } h \to 0.
\end{aligned}
$$

The proof of Proposition 2.1 is complete.

To prove a.s. Donsker's theorem we shall need the next result due to Fazekas and Rychlik [7] (see also Chuprunov and Fazekas [6]). Let μ_X denote the distribution of X. Let δ_x be the unit mass at x.

Remark 2.4. Let (M,ρ) be a complete separable metric space and X_n, $n \in N$, be a sequence of random elements in M. Assume that there exist $C > 0$, $\varepsilon > 0$ and an increasing sequence of positive numbers C_n with $\lim_{n\to\infty} C_n = \infty$, $C_{n+1}/C_n = O(1)$, and M-valued random elements $X_{k,l}$, $k,l \in N$, $k < l$, such that the random elements X_k and $X_{k,l}$ are independent for $k < l$ and

$$
E\rho(X_{k,l}, X_l) \le C\left(\frac{C_k}{C_l}\right)^{\beta} \tag{2}
$$

for $k < l$, where $\beta > 0$. Let $0 \le d_k \le \log(C_{k+1}/C_k)$, assume that $\sum_{k=1}^{\infty} d_k = \infty$. Let $D_n = \sum_{k=1}^n d_k$. Then, for any probability distribution μ on the Borel σ-algebra of M, the following two statements are equivalent

$$
\begin{aligned}
\frac{1}{D_n}\sum_{k=1}^n d_k \delta_{X_k(\omega)} \Rightarrow \mu, &\qquad \text{as } n\to\infty \text{ for almost every } \omega\in\Omega; \\
\frac{1}{D_n}\sum_{k=1}^n d_k \mu_{X_k} \Rightarrow \mu, &\qquad \text{as } n\to\infty.
\end{aligned}
$$

The following result is the a.s. Donsker's theorem in $L^p(]0,1[)$, where $1 \le p < \infty$.

Theorem 2.1. *Let* $1 \le p < \infty$.

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{Y_{k(\cdot,\omega)}} \Rightarrow \mu_W,$$

in $L^p(]0,1[)$, *as* $n \to \infty$, *for almost every* $\omega \in \Omega$, *where* W *is the standard Wiener process and* $Y_k(t,\omega) = Y_k(t)$ *is defined in (1).*

Proof. We shall prove that the conditions of Remark 2.4 are fulfilled. The separability and completeness of space $L^p(]0,1[)$ $(1 \le p < \infty)$ are well-known facts.

Let us define the process

$$Y_{k,n}(t) = \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}}\right) I_{]k/n,1]}(t), \qquad k = 1,2,\ldots,n-1,\ t \in [0,1],$$

where I_A denotes the indicator function of the set A. Then $Y_{k,n}$ and Y_k are independent for $k < n$.

$$\begin{aligned}
E\rho(Y_n, Y_{k,n}) &= E\left(\int_0^1 \left|Y_n(t) - \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}}\right) I_{]k/n,1]}(t)\right|^p dt\right)^{1/p} \\
&\le \left(E\int_0^1 \left|Y_n(t) - \left(Y_n(t) - \frac{S_k}{\sigma\sqrt{n}}\right) I_{]k/n,1]}(t)\right|^p dt\right)^{1/p} \\
&= \left(E\left(\left|\frac{S_1}{\sigma\sqrt{n}}\right|^p \frac{1}{n} + \left|\frac{S_2}{\sigma\sqrt{n}}\right|^p \frac{1}{n} + \cdots + \left|\frac{S_{k-1}}{\sigma\sqrt{n}}\right|^p \frac{1}{n} + \left|\frac{S_k}{\sigma\sqrt{n}}\right|^p \frac{n-k}{n}\right)\right)^{1/p} \\
&\le \left(\frac{1}{\sigma^p n^{p/2} n} C\left(1^{p/2} + 2^{p/2} + \cdots + (k-1)^{p/2} + (k)^{p/2}(n-k)\right)\right)^{1/p} \\
&\le \left(\frac{C}{\sigma^p n^{p/2} n} k^{p/2}[(k-1) + (n-k)]\right)^{1/p} \\
&\le C^* \left(\frac{k^{p/2}}{n^{p/2}}\right)^{1/p} = C^*\sqrt{\frac{k}{n}}.
\end{aligned}$$

So condition (2) of Remark 2.4 holds and the proof of Theorem 2.1 is complete.

3. The empirical process in $L^p(]0,1[)$

In this section, we consider the empirical process

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (I_{[0,t]}(U_i) - t), \qquad t \in [0,1],$$

where U_i $(i = 1, 2, \ldots)$ are independent random variables with uniform distribution on the interval $[0,1]$.

For the sake of completeness we prove the weak convergence of Z_n.

Proposition 3.1. *The process $(Z_n(t),\ n \geq 1)$ weakly converges to the Brownian bridge B in space $L^p(]0,1[)$, where $1 \leq p < \infty$.*

Proof. First we prove the convergence in distribution of $\int_0^1 Z_n(t)f(t)dt$ to $\int_0^1 B(t)f(t)dt$ for each f in $L^q(]0,1[)$ the dual space of $L^p(]0,1[)$. $\int_0^1 Z_n(t)f(t)dt$ converges weakly to the normal distribution with mean zero and variance $\int_0^1 \int_0^1 (min\{s,t\} - st)f(s)f(t)dsdt$. But it is the distribution of $\int_0^1 B(t)f(t)dt$.

Now we prove that the condition (i) of Remark 2.3 is fulfilled with $\gamma = 2$. Since $\|\cdot\|_1 \leq \|\cdot\|_2$ this will be done if we show $\sup_{n\geq 1} E\|Z_n\|_2^2 < \infty$.

$$\begin{aligned} E\|Z_n\|_2^2 &= E\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(I_{[0,t]}(U_i) - t)\right\|_2^2 = E\int_0^1 \left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(I_{[0,t]}(U_i) - t)\right|^2 dt \\ &= \frac{1}{n}E\int_0^1 \left|\sum_{i=1}^{n}(I_{[0,t]}(U_i) - t)\right|^2 dt \\ &= \frac{1}{n}\int_0^1 E(\xi - nt)^2\,dt = \frac{1}{n}\int_0^1 nt(1-t)\,dt = \frac{1}{6}, \end{aligned}$$

where ξ is a binomial random variable with parameters t and n.

Now, we will show that condition (ii) of Remark 2.3 is fulfilled.

$$\begin{aligned} E\|Z_n(\cdot + h) - Z_n(\cdot)\|_p^p &= E\int_0^1 |Z_n(t+h) - Z_n(t)|^p\,dt \\ &= E\int_0^{1-h} |Z_n(t+h) - Z_n(t)|^p\,dt + E\int_{1-h}^{1} |Z_n(t)|^p\,dt \end{aligned}$$

$$
\begin{aligned}
&= E\int_0^{1-h}\Bigg|\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(I_{[0,t+h]}(U_i)-(t+h)\big)\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(I_{[0,t]}(U_i)-t\big)\Bigg|^p dt\\
&\quad + E\int_{1-h}^{1}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(I_{[0,t]}(U_i)-t\big)\right|^p dt\\
&= E\frac{1}{n^{p/2}}\int_0^{1-h}\left|\sum_{i=1}^n \big(I_{]t,t+h]}(U_i)-h\big)\right|^p dt\\
&\quad + E\frac{1}{n^{p/2}}\int_{1-h}^{1}\left|\sum_{i=1}^n \big(I_{[0,t]}(U_i)-t\big)\right|^p dt\\
&= \frac{1}{n^{p/2}}\int_0^{1-h} E\left|\sum_{i=1}^n (I_{]t,t+h]}(U_i)-h)\right|^p dt\\
&\quad + \frac{1}{n^{p/2}}\int_{1-h}^{1} E\left|\sum_{i=1}^n (I_{[0,t]}(U_i)-t)\right|^p dt\\
&\le \frac{1}{n^{p/2}}\int_0^{1-h} A_p^p E\left(\left(\sum_{i=1}^n (I_{]t,t+h]}(U_i)-h)^2\right)^{1/2}\right)^p dt\\
&\quad + \frac{1}{n^{p/2}}\int_{1-h}^{1} B_p^p E\left(\left(\sum_{i=1}^n (I_{[0,t]}(U_i)-t)^2\right)^{1/2}\right)^p dt\\
&= \frac{1}{n^{p/2}}\int_0^{1-h} A_p^p E\left(\sum_{i=1}^n (I_{]t,t+h]}(U_i)-h)^2\right)^{p/2} dt\\
&\quad + \frac{1}{n^{p/2}}\int_{1-h}^{1} B_p^p E\left(\sum_{i=1}^n (I_{[0,t]}(U_i)-t)^2\right)^{p/2} dt,
\end{aligned}
$$

where the used the Marcinkiewicz–Zygmund inequality, see Remark 2.1. We will distinguish two cases. In the first case $1 \le p \le 2$.

$$
\begin{aligned}
&\frac{1}{n^{p/2}}\int_0^{1-h} A_p^p E\left(\sum_{i=1}^n (I_{]t,t+h]}(U_i)-h)^2\right)^{p/2} dt\\
&\qquad + \frac{1}{n^{p/2}}\int_{1-h}^{1} B_p^p E\left(\sum_{i=1}^n (I_{[0,t]}(U_i)-t)^2\right)^{p/2} dt
\end{aligned}
$$

$$\leq \frac{1}{n^{p/2}} \int_0^{1-h} A_p^p \left(E \sum_{i=1}^n (I_{]t,t+h]}(U_i) - h)^2 \right)^{p/2} dt$$

$$+ \frac{1}{n^{p/2}} \int_{1-h}^1 B_p^p \left(E \sum_{i=1}^n (I_{[0,t]}(U_i) - t)^2 \right)^{p/2} dt$$

$$= \frac{A_p^p}{n^{p/2}} \int_0^{1-h} (E(\xi - nh)^2)^{p/2} \, dt + \frac{B_p^p}{n^{p/2}} \int_{1-h}^1 (E(\eta - nt)^2)^{p/2} \, dt$$

$$= \frac{A_p^p}{n^{p/2}} \int_0^{1-h} (nh(1-h))^{p/2} \, dt + \frac{B_p^p}{n^{p/2}} \int_{1-h}^1 (nt(1-t))^{p/2} \, dt$$

$$\leq A_p^p h^{p/2} (1-h)^{\frac{p+2}{2}} + B_p^p \int_{1-h}^1 (t(1-t))^{p/2} \, dt$$

$$\leq A_p^p h^{p/2} (1-h)^{\frac{p+2}{2}} + B_p^p h^{\frac{p+2}{2}} \to 0, \qquad \text{as } h \to 0,$$

where ξ is a binomial random variable with parameters h and n, and η is binomial with parameters t and n.

In the second case $2 < p < \infty$.

$$\frac{1}{n^{p/2}} \int_0^{1-h} A_p^p E \left(\sum_{i=1}^n (I_{]t,t+h]}(U_i) - h)^2 \right)^{p/2} dt$$

$$+ \frac{1}{n^{p/2}} \int_{1-h}^1 B_p^p E \left(\sum_{i=1}^n (I_{[0,t]}(U_i) - t)^2 \right)^{p/2} dt$$

$$\leq \frac{A_p^p}{n^{p/2}} \int_0^{1-h} E \left(n^{\frac{p-2}{p}} \left(\sum_{i=1}^n |I_{]t,t+h]}(U_i) - h|^p \right)^{2/p} \right)^{p/2} dt$$

$$+ \frac{B_p^p}{n^{p/2}} \int_{1-h}^1 E \left(n^{\frac{p-2}{p}} \left(\sum_{i=1}^n |I_{[0,t]}(U_i) - t|^p \right)^{2/p} \right)^{p/2} dt$$

$$= \frac{A_p^p}{n^{p/2}} n^{\frac{p-2}{2}} \int_0^{1-h} E \left(\sum_{i=1}^n |I_{]t,t+h]}(U_i) - h|^p \right) dt$$

$$+ \frac{B_p^p}{n^{p/2}} n^{\frac{p-2}{2}} \int_{1-h}^1 E \left(\sum_{i=1}^n |I_{[0,t]}(U_i) - t|^p \right) dt$$

$$= \frac{A_p^p}{n} \int_0^{1-h} \sum_{i=1}^n E|I_{]t,t+h]}(U_i) - h|^p \, dt$$

$$+ \frac{B_p^p}{n} \int_{1-h}^1 \sum_{i=1}^n E|I_{[0,t]}(U_i) - t|^p \, dt$$

$$= A_p^p \int_0^{1-h} E|I_{]t,t+h]}(U_i) - h|^p\,dt + B_p^p \int_{1-h}^1 E|I_{[0,t]}(U_i) - t|^p\,dt$$
$$= A_p^p \int_0^{1-h} E|\xi - h|^p + B_p^p \int_{1-h}^1 E|\eta - t|^p$$
$$= A_p^p \left[(1-h)^{p+1}h + h^p(1-h)^2\right] + B_p^p \int_{1-h}^1 [(1-t)^p t + t^p(1-t)]\,dt$$
$$\leq A_p^p[(1-h)^{p+1}h + h^p(1-h)^2] + B_p^p \int_{1-h}^1 2dt$$
$$= A_p^p[(1-h)^{p+1}h + h^p(1-h)^2] + 2B_p^p h \to 0, \qquad \text{as } h \to 0,$$

where ξ is a Bernoulli random variable with parameter h and η is Bernoulli with parameter t. This completes the proof of the Proposition 3.1.

Theorem 3.1.

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{Z_k(\cdot,\omega)} \Rightarrow \mu_B,$$

in $L^p(]0,1[)$, as $n \to \infty$, for almost every $\omega \in \Omega$, where B is the Brownian bridge.

Proof. We shall prove that the conditions of Remark 2.4 are fulfilled.

The separability and completeness of $L^p(]0,1[)$ are well-known facts. Let us define the process

$$Z_{k,n}(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(I_{[0,t]}(U_i) - t) - \frac{1}{\sqrt{n}}\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t).$$

Then $Z_{k,n}$ and Z_k are independent for $k < n$.

We show that the condition (2) is valid.

$$E\rho(Z_n, Z_{k,n}) = E\left(\int_0^1 \left|\frac{1}{\sqrt{n}}\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t)\right|^p dt\right)^{1/p}$$
$$= \frac{1}{\sqrt{n}}E\left(\int_0^1 \left|\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t)\right|^p dt\right)^{1/p}$$
$$\leq \frac{1}{\sqrt{n}}\left(\int_0^1 E\left|\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t)\right|^p dt\right)^{1/p}$$
$$\leq \frac{1}{\sqrt{n}}\left(\int_0^1 C_p^p E\left(\left(\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t)^2\right)^{1/2}\right)^p dt\right)^{1/p}$$
$$\leq \frac{1}{\sqrt{n}}\left(\int_0^1 C_p^p E\left(\sum_{i=1}^{k}(I_{[0,t]}(U_i) - t)^2\right)^{p/2} dt\right)^{1/p},$$

where we used the Marcinkiewicz–Zygmund inequality, see Remark 2.1. We will distinguish two cases. In the first case $1 \le p \le 2$.

$$\frac{1}{\sqrt{n}}\left(\int_0^1 C_p^p E\left(\sum_{i=1}^k (I_{[0,t]}(U_i)-t)^2\right)^{p/2} dt\right)^{1/p}$$

$$\le \frac{C_p}{\sqrt{n}}\left(\int_0^1 \left(E\sum_{i=1}^k (I_{[0,t]}(U_i)-t)^2\right)^{p/2} dt\right)^{1/p}$$

$$= \frac{C_p}{\sqrt{n}}\left(\int_0^1 (E(\xi-kt)^2)^{p/2}\,dt\right)^{1/p}$$

$$= \frac{C_p}{\sqrt{n}}\left(\int_0^1 (kt(1-t))^{p/2}\,dt\right)^{1/p} = C^*\frac{\sqrt{k}}{\sqrt{n}},$$

where ξ has binomial distribution with parameters t and k.

In the second case $2 < p < \infty$.

$$\frac{1}{\sqrt{n}}\left(\int_0^1 C_p^p E\left(\sum_{i=1}^k (I_{[0,t]}(U_i)-t)^2\right)^{p/2} dt\right)^{1/p}$$

$$\le \frac{C_p}{\sqrt{n}}\left(\int_0^1 E\left(k^{\frac{p-2}{p}}\left(\sum_{i=1}^k |I_{[0,t]}(U_i)-t|^p\right)^{2/p}\right)^{p/2} dt\right)^{1/p}$$

$$= \frac{C_p}{\sqrt{n}}\left(\int_0^1 E\left(k^{\frac{p-2}{2}}\left(\sum_{i=1}^k |I_{[0,t]}(U_i)-t|^p\right)\right) dt\right)^{1/p}$$

$$= \frac{C_p}{\sqrt{n}}\left(\int_0^1 k^{\frac{p-2}{2}}\left(\sum_{i=1}^k E|I_{[0,t]}(U_i)-t|^p\right) dt\right)^{1/p}$$

$$= \frac{C_p}{\sqrt{n}}\left(\int_0^1 k^{p/2}E|\xi-t|^p\,dt\right)^{1/p}$$

$$= C_p\frac{\sqrt{k}}{\sqrt{n}}\left(\int_0^1 E|\xi-t|^p\,dt\right)^{1/p}$$

$$= C_p\frac{\sqrt{k}}{\sqrt{n}}\left(\int_0^1 [(1-t)^p t + (1-t)t^p]\,dt\right)^{1/p}$$

$$\le C_p\frac{\sqrt{k}}{\sqrt{n}}2^{1/p} = C^*\frac{\sqrt{k}}{\sqrt{n}},$$

where ξ is a Bernoulli random variable with parameter t. This completes the proof of the Theorem 3.1.

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