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ON THE DERIVATIVES OF A SPECIAL FAMILY OF B-SPLINE CURVES

Miklós Hoffmann (Eger, Hungary)

Abstract. This paper is devoted to the geometrical examination of a family of B-spline curves resulted by the modifiaction of one of their knot values. These curves form a surface, the other parameter lines of which are the paths of the points of the original curve at a fixed parameter value. The first and second derivatives of these curves are examined yielding geometrical results concerning their tangent lines and osculating planes.

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1.Introduction

B-spline and NURBS curves are well-known and widely used description methods in computer aided geometric design today. The data structure of these curves are very simple, containing control points, knot values and - in terms of NURBS curves - weights. The modification of the control points and the weights has well-known effects on the curves (see e.g.[9]), while more sophisticated possibilities of curve modification by these data can be found in [1], [3], [4], [8], [10].

The modification of the knot values also affects the shape of the curves, but this effect has been examined only numerically. Some geometrical aspects of the behavior of a B-spline or NURBS curve modifying one of its knot values have been described recently in [5], [6], [7]. The purpose of this paper is to extend these geometrical representations by examining the curves around the parameter value of the modified knot.

Definition. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\mathbf{d}_l \qquad u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order k (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are

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necessary. The points $\mathbf{d}_i$ are called control points or de Boor-points, while the polygon formed by these points is called control polygon.

Definition. The j^{th} span of the B-spline curve can be written as

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \quad u \in [u_j, u_{j+1}).$$

Modifying the knot u_i, the point of this span associated with the fixed parameter value $\tilde{u} \in [u_j, u_{j+1})$ will move along the curve

$$\mathbf{s}_j(\tilde{u}, u_i) = \sum_{l=j-k+1}^{j} N_l^k(\tilde{u}, u_i)\, \mathbf{d}_l, \quad u_i \in [u_{i-1}, u_{i+1}].$$

Hereafter, we refer to this curve as the *path* of the point $\mathbf{s}_j(\tilde{u})$. In [5] and [6] Juhász and Hoffmann proved important properties of these paths, among which the most important is the following

Theorem 1. *Modifying the knot value $u_i \in [u_{i-1}, u_{i+1}]$ of the k^{th} order B-spline curve, the points of the spans $\mathbf{s}_{i-k+1}(u), ..., \mathbf{s}_{i+k-2}(u)$ moves along rational curves. The degree of these paths decreases symmetrically from $k-1$ to 1 as the indices of the spans getting farther from i, i.e. the paths $\mathbf{s}_{i-m}(\tilde{u}, u_i)$ and $\mathbf{s}_{i+m-1}(\tilde{u}, u_i)$ are rational curves of degree $k-m$ with respect to u_i, $(m = 1, \ldots, k-1)$.*

Beside these paths we can also consider the one-parameter family of B-spline curves

$$\mathbf{s}(u, \tilde{u}_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, \tilde{u}_i), \quad u \in [u_{k-1}, u_{n+1}]$$

yielded by the modification of the knot value u_i. In trems of these curves another property has been proved by Juhász and Hoffmann (see [6]), namely the family of these curves has an envelope, which is also a B-spline curve.

Theorem 2. *The family of the k^{th} order B-spline curves $\mathbf{s}(u, u_i)$, $(k > 2)$ has an envelope. The envelope is also a B-spline curve of order $(k-1)$ and can be written in the form*

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v),\ v \in [v_{i-1}, v_i],$$

where $v_j = u_j$ if $j < i$ and $v_j = u_{j+1}$ otherwise, that is the i^{th} knot value is removed from the knot vector u_j of the original curves.

Hence two families of curves have been received, the paths of the points and the B-spline curves themselves. These two families of curves can be considered as parameterlines of the surface patch

$$\mathbf{s}\left(u,u_i\right)=\sum_{l=0}^{n}\mathbf{d}_l N_l^k\left(u,u_i\right),\quad u\in\left[u_{k-1},u_{n+1}\right],\quad u_i\in\left[u_{i-1},u_{i+1}\right).$$

The envelope mentioned above in Theorem 2. is a curve on this surface, but the parameter lines behave in a singular way at the points of that curve. We have seen that it is an envelope of the family of B-spline curves. In the next sections, where we will restrict our consideration to the cubic case ($k=4$) the derivatives of the two families of curves will be computed in the points of the quadratic envelope by the help of which we will prove, that this curve is also the envelope of the paths and both families have the same osculating plane at every point of this envelope, which plane is also the plane of the envelope itself.

2. The derivatives of the curves

Let the knot value u_i of a cubic B-spline curve defined above be modified. At first the family of B-spline curves will be considered, the derivatives of which can be calculated by a well-known iterative formula, which can be found e.g. in [9]:

$$\text{(1)}\qquad \frac{\partial \mathbf{s}_i}{\partial u}=\sum_{l=i-3}^{i}\mathbf{d}_l 3\left(\frac{1}{u_{l+3}-u_l}N_l^3(u,u_i)-\frac{1}{u_{l+4}-u_{l+1}}N_{l+1}^3(u,u_i)\right)$$

Using this rule the first derivatives of the coefficients are

$$\begin{aligned}
\frac{\partial N_{i-3}^4}{\partial u}=&-3\frac{1}{u_{i+1}-u_{i-2}}\frac{u_{i+1}-u}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i},\\
\frac{\partial N_{i-2}^4}{\partial u}=&3\left(\frac{1}{u_{i+1}-u_{i-2}}\frac{u_{i+1}-u}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i}\right.\\
&\left.-\frac{1}{u_{i+2}-u_{i-1}}\left(\frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i}+\frac{u_{i+2}-u}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i}\right)\right),\\
\frac{\partial N_{i-1}^4}{\partial u}=&3\left(\frac{1}{u_{i+2}-u_{i-1}}\left(\frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i}+\frac{u_{i+2}-u}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i}\right)\right.\\
&\left.-\frac{1}{u_{i+3}-u_i}\frac{u-u_i}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i}\right),\\
\frac{\partial N_i^4}{\partial u}=&3\frac{1}{u_{i+3}-u_i}\frac{u-u_i}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i}.
\end{aligned}$$

The second derivatives of these curves can also be calculated applying the equation (1) iteratively for the basis functions of degree 3. The second derivatives are

$$\frac{\partial^2 \mathbf{s}_i}{\partial u^2} = \sum_{l=i-3}^{i} \mathbf{d}_l \frac{\partial^2 N_l^4}{\partial u^2}$$

where the coefficients are

$$\begin{aligned}
\frac{\partial^2 N_{i-3}^4}{\partial u^2} =& -3\frac{1}{u_{i+1}-u_{i-2}}\left(-\frac{1}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i} - \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}}\frac{1}{u_{i+1}-u_i}\right),\\
\frac{\partial^2 N_{i-2}^4}{\partial u^2} =& 3\frac{1}{u_{i+1}-u_{i-2}}\left(-\frac{1}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i} - \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}}\frac{1}{u_{i+1}-u_i}\right)\\
&-3\frac{1}{u_{i+2}-u_{i-1}}\left(\frac{1}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i} - \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\frac{1}{u_{i+1}-u_i}\right.\\
&\left.-\frac{1}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i} + \frac{u_{i+2}-u}{u_{i+2}-u_i}\frac{1}{u_{i+1}-u_i}\right),\\
\frac{\partial^2 N_{i-1}^4}{\partial u^2} =& 3\frac{1}{u_{i+2}-u_{i-1}}\left(\frac{1}{u_{i+1}-u_{i-1}}\frac{u_{i+1}-u}{u_{i+1}-u_i} - \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\frac{1}{u_{i+1}-u_i}\right.\\
&\left.-\frac{1}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i} + \frac{u_{i+2}-u}{u_{i+2}-u_i}\frac{1}{u_{i+1}-u_i}\right)\\
&-3\frac{1}{u_{i+3}-u_i}\frac{1}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i} - 3\frac{1}{u_{i+3}-u_i}\frac{u-u_i}{u_{i+2}-u_i}\frac{1}{u_{i+1}-u_i},\\
\frac{\partial^2 N_i^4}{\partial u^2} =& 3\left(\frac{1}{u_{i+3}-u_i}\frac{1}{u_{i+2}-u_i}\frac{u-u_i}{u_{i+1}-u_i} + \frac{1}{u_{i+3}-u_i}\frac{u-u_i}{u_{i+2}-u_i}\frac{1}{u_{i+1}-u_i}\right).
\end{aligned}$$

Now the other family of curves, namely the paths will be considered. The first derivative of this family is

$$\frac{\partial \mathbf{s}_i}{\partial u_i} = \sum_{l=i-3}^{i} \mathbf{d}_l \frac{\partial N_l^4}{\partial u_i},$$

where the coefficients are

$$
\begin{aligned}
\frac{\partial N_{i-3}^4}{\partial u_i} =& \frac{u_{i+1}-u}{u_{i+1}-u_{i-2}} \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}} \frac{u_{i+1}-u}{(u_{i+1}-u_i)^2}, \\
\frac{\partial N_{i-2}^4}{\partial u_i} =& \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}} \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}} \frac{u_{i+1}-u}{(u_{i+1}-u_i)^2} \\
&+ \frac{u_{i+2}-u}{u_{i+2}-u_{i-1}} \left(\frac{u-u_{i-1}}{u_{i+1}-u_{i-1}} \frac{u_{i+1}-u}{(u_{i+1}-u_i)^2} \right. \\
&\left. + \frac{u_{i+2}-u}{(u_{i+2}-u_i)^2} \frac{u-u_i}{u_{i+1}-u_i} + \frac{u_{i+2}-u}{u_{i+2}-u_i} \frac{u-u_{i+1}}{(u_{i+1}-u_i)^2} \right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial N_{i-1}^4}{\partial u_i} =& \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}} \left(\frac{u-u_{i-1}}{u_{i+1}-u_{i-1}} \frac{u_{i+1}-u}{(u_{i+1}-u_i)^2} \right. \\
&\left. + \frac{u_{i+2}-u}{(u_{i+2}-u_i)^2} \frac{u-u_i}{u_{i+1}-u_i} + \frac{u_{i+2}-u}{u_{i+2}-u_i} \frac{u-u_{i+1}}{(u_{i+1}-u_i)^2} \right) \\
&+ \frac{u_{i+3}-u}{(u_{i+3}-u_i)^2} \frac{u-u_i}{u_{i+2}-u_i} \frac{u-u_i}{u_{i+1}-u_i} \\
&+ \frac{u_{i+3}-u}{u_{i+3}-u_i} \left(\frac{u-u_{i+2}}{(u_{i+2}-u_i)^2} \frac{u-u_i}{u_{i+1}-u_i} + \frac{u-u_i}{u_{i+2}-u_i} \frac{u-u_{i+1}}{(u_{i+1}-u_i)^2} \right), \\
\frac{\partial N_i^4}{\partial u_i} =& \frac{u-u_{i+3}}{(u_{i+3}-u_i)^2} \frac{u-u_i}{u_{i+2}-u_i} \frac{u-u_i}{u_{i+1}-u_i} \\
&+ \frac{u-u_i}{u_{i+3}-u_i} \left(\frac{u-u_{i+2}}{(u_{i+2}-u_i)^2} \frac{u-u_i}{u_{i+1}-u_i} + \frac{u-u_i}{u_{i+2}-u_i} \frac{u-u_i}{(u_{i+1}-u_i)^2} \right).
\end{aligned}
$$

The second dertivatives of these paths are the following

$$
\frac{\partial^2 \mathbf{s}_i}{\partial u_i^2} = \sum_{l=i-3}^{i} \mathbf{d}_l \frac{\partial^2 N_l^4}{\partial u_i^2},
$$

where the coefficient functions are large polynomials thus, for the sake of brevity they are not presented here.

2. New results

Using the derivatives of the preceeding section the following theorems can be proved (in these proofs the Maple software was applied for the evaluation and simplification of polynomials):

Theorem 3. *If we consider the surface* $\mathbf{s}_i(u, u_i), u \in [u_{k-1}, u_{n+1}], \quad u_i \in [u_{i-1}, u_{i+1})$ *then the envelope of the family of B-spline curves* $\mathbf{s}_i(u, \tilde{u_i})$ *is also the envelope of the family of paths* $\mathbf{s}_i(\tilde{u}, u_i)$ *at the points corresponding to* $u = u_i$.

Proof. It is sufficient to prove, that the two families of curves have points and tangent lines on common at the points corresponding to the parameter value $u = u_i$. If we fix the parameters $u = \tilde{u}$ and $u_i = \tilde{u_i}$ then a member of both families of curves has been selected. Substituting these parameters to both of the curves the existence of the common point $\mathbf{s}_i(\tilde{u}, \tilde{u_i}) = \mathbf{s}_i(\tilde{u}, \tilde{u_i})$ immediately follows. For the proof of the common tangent lines the first derivatives of these curves will be used. Substituting the parameter $u = u_i$ to the coefficients after some calculations one can receive, that

$$\left.\frac{\partial N_{i-3}^4}{\partial u_i}\right|_{u=u_i} = -\frac{1}{3}\left.\frac{\partial N_{i-3}^4}{\partial u}\right|_{u=u_i} = \frac{1}{u_{i+1} - u_{i-2}}\frac{u_{i+1} - u_i}{u_{i+1} - u_{i-1}},$$

$$\left.\frac{\partial N_{i-2}^4}{\partial u_i}\right|_{u=u_i} = -\frac{1}{3}\left.\frac{\partial N_{i-2}^4}{\partial u}\right|_{u=u_i} = \frac{1}{u_{i+1} - u_{i-1}}\left(\frac{u_i - u_{i-2}}{u_{i+1} - u_{i-2}} - \frac{u_{i+2} - u_i}{u_{i+2} - u_{i-1}}\right),$$

$$\left.\frac{\partial N_{i-1}^4}{\partial u_i}\right|_{u=u_i} = -\frac{1}{3}\left.\frac{\partial N_{i-1}^4}{\partial u}\right|_{u=u_i} = -\frac{1}{u_{i+1} - u_{i-1}}\frac{u_i - u_{i-1}}{u_{i+2} - u_{i-1}},$$

$$\left.\frac{\partial N_{i}^4}{\partial u_i}\right|_{u=u_i} = \left.\frac{\partial N_{i}^4}{\partial u}\right|_{u=u_i} = 0,$$

which yield, that

$$\left.\frac{\partial \mathbf{s}_i(u, u_i)}{\partial u}\right|_{u=u_i} = -\frac{1}{3}\left.\frac{\partial \mathbf{s}_i(u, u_i)}{\partial u_i}\right|_{u=u_i}$$

i.e. the curves have also tangent lines on common at the points of the envelope.

With the help of the second derivatives of the coefficient functions the osculating plane of these curves can also be examined.

Theorem 4. *The osculating planes of the two families of curves* $\mathbf{s}_i(u, \tilde{u_i})$ *and* $\mathbf{s}_i(\tilde{u}, u_i)$ *coincide at every point of the envelope and this plane is that of the three control points* $\mathbf{d}_{i-3}, \mathbf{d}_{i-2}, \mathbf{d}_{i-1}$ *for every* u_i.

Proof. The osculating plane is uniquely defined by the first and second derivatives of the curve. Since Theorem 3 holds for the first derivatives it is sufficient to prove that the second derivatives of these curves are also parallel to each other. Using the second derivatives of the coefficient functions and substituting the parameter value $u = u_i$ the following result can be obtained:

$$\begin{aligned}
\left.\frac{\partial^2 N_{i-3}^4}{\partial {u_i}^2}\right|_{u=u_i} &= \frac{1}{3}\left.\frac{\partial^2 N_{i-3}^4}{\partial u^2}\right|_{u=u_i} = 2\frac{1}{u_{i+1}-u_{i-2}}\frac{1}{u_{i+1}-u_{i-1}},\\
\left.\frac{\partial^2 N_{i-2}^4}{\partial {u_i}^2}\right|_{u=u_i} &= \frac{1}{3}\left.\frac{\partial^2 N_{i-2}^4}{\partial u^2}\right|_{u=u_i} = 2\frac{-u_{i+1}+u_{i-2}-u_{i+2}+u_{i-1}}{\left(-u_{i+2}+u_{i-1}\right)\left(u_{i+1}-u_{i-1}\right)\left(-u_{i+1}+u_{i-2}\right)},\\
\left.\frac{\partial^2 N_{i-1}^4}{\partial {u_i}^2}\right|_{u=u_i} &= \frac{1}{3}\left.\frac{\partial^2 N_{i-1}^4}{\partial u^2}\right|_{u=u_i} = 2\frac{1}{\left(u_{i+1}-u_{i-1}\right)\left(u_{i+2}-u_{i-1}\right)},\\
\left.\frac{\partial^2 N_{i}^4}{\partial {u_i}^2}\right|_{u=u_i} &= \left.\frac{\partial^2 N_{i}^4}{\partial u^2}\right|_{u=u_i} = 0,
\end{aligned}$$

which immediately yield, that

$$\left.\frac{\partial^2 \mathbf{s}_i\left(u,u_i\right)}{\partial u^2}\right|_{u=u_i} = \frac{1}{3}\left.\frac{\partial^2 \mathbf{s}_i\left(u,u_i\right)}{\partial {u_i}^2}\right|_{u=u_i}.$$

Hence the osculating planes of the two families of curves coincide at the parameter values $u = u_i$. Moreover, the second derivatives do no depend on u_i, and using the notations

$$A := \frac{\partial^2 N_{i-3}^4}{\partial {u_i}^2}, \qquad B := \frac{\partial^2 N_{i-1}^4}{\partial {u_i}^2}$$

they can be written in the form

$$\begin{aligned}
\left.\frac{\partial^2 \mathbf{s}_i\left(u,u_i\right)}{\partial {u_i}^2}\right|_{u=u_i} &= A\left(\mathbf{d}_{i-3}-\mathbf{d}_{i-2}\right)+B\left(\mathbf{d}_{i-2}-\mathbf{d}_{i-1}\right),\\
\left.\frac{\partial^2 \mathbf{s}_i\left(u,u_i\right)}{\partial u^2}\right|_{u=u_i} &= \frac{1}{3}A\left(\mathbf{d}_{i-3}-\mathbf{d}_{i-2}\right)+\frac{1}{3}B\left(\mathbf{d}_{i-2}-\mathbf{d}_{i-1}\right).
\end{aligned}$$

This means that these derivative vectors are in the plane of the control points $\mathbf{d}_{i-3}, \mathbf{d}_{i-2}, \mathbf{d}_{i-1}$ for every u_i. The same holds for the first derivative vectors since the envelope is a quadratic B-spline curve (a parabola) defined by these control points and it has common tangent lines with both of the families of the curves at $u = u_i$. This yields, that the osculating planes of the curves coincide with the plane of the three control points mentioned above for every u_i.

4. Further Research

Some geometrical aspects of the modification of a knot value of a cubic B-spline curve have been discussed. Defining a special surface with two families of curves it turned out that these two families have the same envelope at a certain parameter value and even the osculating planes coincide. This plane is a constant plane and defined by three control points of the original B-spline curve. Natural extensions of these results would be desired for B-spline curves of arbitrary degree, but the derivatives of these curves in the direction u_i should be calculated by recursive formulae of the derivatives of the basis functions and these formulae have not been found yet.

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Miklós Hoffmann
Károly Eszterházy College
Department of Mathematics
H-3300 Eger, Hungary
Leányka str. 4.
e-mail: hofi@ektf.hu