REAL NUMBERS THAT HAVE GOOD DIOPHANTINE APPROXIMATIONS OF THE FORM r_{n+1}/r_n

Andreas Dress & Florian Luca (Bielefeld & Morelia)

Abstract. In this note, we show that if α is a real number such that there exist a constant c and a sequence of non-zero integers $(r_n)_{n\geq 0}$ with $\lim_{n\to\infty}|r_n| = \infty$ for which $\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{c}{|r_n|^2}$ holds for all $n \geq 0$, then either $\alpha \in \mathbf{Z}\backslash\{0, \pm 1\}$ or α is a quadratic unit. Our result complements results obtained by P. Kiss who established the converse in Period. Math. Hungar. 11 (1980), 281-187.

AMS Classification Number: 11J04, 11J70

1. Introduction

Let α be a real number. In this paper, we deal with the topic of approximating α by rationals. It is well known that there exist a constant c and two sequences of integers $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ with $v_n > 0$ for all $n \geq 0$ and v_n diverging to infinity (with n) such that

$$\left|\alpha - \frac{u_n}{v_n}\right| \leq \frac{c}{v_n^2} \tag{1}$$

holds for all $n \geq 0$. By work of Hurwitz (see [5]), one can take $c := 1/\sqrt{5}$ and the above constant is well known to be best-possible for $\alpha := \frac{1+\sqrt{5}}{2}$.
Several papers in the literature deal with the question of approximating α by rationals u_n/v_n requiring u_n and v_n to satisfy (1) as well as some additional conditions. For example, if α is irrational and a, b and k are integers with $k > 1$, then there exist a constant c and two sequences of integers $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ with $v_n > 0$ and v_n diverging to infinity such that

$$\left|\alpha - \frac{u_n}{v_n}\right| < \frac{c}{v_n^2} \quad \text{and} \quad u_n \equiv a \pmod{k},\ \ v_n \equiv b \pmod{k} \tag{2}$$

The second author's research was partially sponsored by the Alexander von Humboldt Foundation.

holds for all $n \geq 0$. The best-possible constant c in (2) is $k^2/4$ in case a and b are not both divisible by k (see [3] and [4]).

If α is algebraic and $\mathcal{P}$ is a fixed finite set of prime numbers, then Ridout [10] inferred from Roth's work [11] that one cannot approximate α too well by rational numbers u/v where either u or v is divisible only by primes from $\mathcal{P}$. More precisely, for every given $\epsilon > 0$, the inequality

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{v^{1+\epsilon}} \tag{3}$$

has only finitely many integer solutions $(u,\ v)$ with $v > 0$ and either u or v divisible by primes from $\mathcal{P}$, only.

A different type of question was considered by P. Kiss in [6] and [7] (see also [8] and [9]). In [6], it was shown that if α is a quadratic unit with $|\alpha| > 1$, then there exist a constant c and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity such that

$$\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{c}{|r_n|^2} \tag{4}$$

holds for all $n \geq 0$. In [7] it was shown that, in fact, a statement similar to (4) holds for both α and α^s where $s \geq 2$ is some positive integer: There exist a constant c and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity such that both

$$\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{c}{|r_n|^2} \quad \text{and} \quad \left|\alpha^s - \frac{r_{n+s}}{r_n}\right| < \frac{c}{|r_n|^2} \tag{5}$$

hold for all $n \geq 0$.

An explicit description of a sequence $(r_n)_{n\geq 0}$ satisfying inequalities (5) above was also given in [7]: Let

$$f = X^2 + AX + B \quad (A, B \in \mathbf{Z})$$

be the minimal polynomial of α over $\mathbf{Q}$. Let β be the other root of f. Since α is a unit, $|B| = |\alpha\beta| = 1$ must hold which implies that the sequence

$$r_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \geq 1 \tag{6}$$

fulfills the inequalities (5) for all n with $c := 2\sum_{i=0}^{s-1} |\alpha|^i|\beta|^{s-1-i}$.

One may ask if one can characterize all real numbers α for which there exist a constant c and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity such that inequality (4) or, respectively, inequalities (5) hold for all $n \geq 0$. From the above remarks, we saw that quadratic units α with $|\alpha| > 1$ have these properties. Moreover, the sequence $r_n := \alpha^n$ $(n \geq 1)$ shows that integers α with $|\alpha| > 1$ also belong to this class. It seems natural therefore to inquire if there are any other candidates α satisfying the above conditions. The perhaps not too surprising, answer is no. Our exact result is the following.

Theorem 1. *Let α be a real number.*

(i) Assume that there exist $\epsilon > 0$ and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity such that

$$\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{1}{|r_n|^{\frac{3}{2}+\epsilon}} \tag{7}$$

holds for all $n \geq 0$. Then, α is a real algebraic integer of absolute value larger than 1 and of degree at most 2. Moreover, if α is irrational, then the absolute value of its norm is smaller than $\sqrt{|\alpha|}$.

(ii) Assume, moreover, that there exist a constant c and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity such that

$$\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{c}{|r_n|^2} \tag{8}$$

holds for all $n \geq 0$. Then α is a quadratic unit or a rational integer different from 0 or ± 1.

The following result characterizes real numbers α for which - as in (5) - two different powers can be well approximated by rationals.

Theorem 2. *Let α be a real number. Assume that there exist two coprime positive integers s_1 and s_2, two positive integers t_1 and t_2, a real number $\epsilon > 0$, and a sequence of integers $(r_n)_{n\geq 0}$ with $|r_n|$ diverging to infinity with n such that*

$$\left|\alpha^{s_i} - \frac{r_{n+t_i}}{r_n}\right| < \frac{1}{|r_n|^{\frac{3}{2}+\epsilon}} \tag{9}$$

hold for all $n \geq 0$ and for both $i = 1$ and 2. Then, either $\alpha \in \mathbf{Z}\backslash\{0, \pm 1\}$ or α is quadratic irrational with norm smaller than $\sqrt{|\alpha|}$ in absolute value. If moreover α is irrational and there exists a constant c with

$$\left|\alpha^{s_1} - \frac{r_{n+t_1}}{r_n}\right| < \frac{c}{r_n^2}, \tag{10}$$

then α is a quadratic unit.

The proofs of both Theorems 1 and 2 are based on the following result which follows right away from our recent work [1] and [2].

Theorem DL. *Let $(r_n)_{n\geq 0}$ be a sequence of integers with $|r_n|$ diverging to infinity.*

(i) Assume that

$$\overline{\lim}_{n\to\infty} \frac{|r_n^2 - r_{n+1}r_{n-1}|}{\sqrt{|r_n|}} < \frac{1}{\sqrt{2}}. \tag{11}$$

Then, the sequence $\left(\frac{r_{n+1}}{r_n}\right)_{n\geq 0}$ *is convergent to a limit* α *that is a non-zero algebraic integer of degree at most 2. If* α *is irrational, then its norm is smaller than* $\sqrt{|\alpha|}$*. Moreover, there exists* $n_0 \in \mathbf{N}$ *such that* $(r_n)_{n\geq n_0}$ *is binary recurrent.*

(ii) If

$$|r_n^2 - r_{n+1}r_{n-1}| < c \tag{12}$$

holds for some constant c *and all* n*, then* α *is a quadratic unit or a non-zero integer.*

We point out that in our work [1] and [2], we gave more precise descriptions for both the sequences $(r_n)_{n\geq 0}$ satisfying (11) or (12), respectively, and the limit $\alpha = \lim_{n\to\infty} \frac{r_{n+1}}{r_n}$, but the above Theorem DL suffices for our present purposes. We now proceed to the proofs of Theorems 1 and 2.

2. The Proofs

Proof of Theorem 1. We will prove (i) in detail and we will only sketch the proof of (ii).

(i) By replacing the sequence $(r_n)_n$ by the sequence $\left((-1)^n r_n\right)_n$ and α by $-\alpha$ if $\alpha < 0$, we may assume $\alpha \geq 0$ and $r_n > 0$ for all $n \geq 0$. By letting n tend to infinity in (7), we get $\alpha = \lim_{n\to\infty} \frac{r_{n+1}}{r_n}$. Since r_n diverges to infinity, we must have $\alpha \geq 1$. We now show that $\alpha > 1$. Indeed, if $\alpha = 1$, then inequality (7) becomes

$$\left|1 - \frac{r_{n+1}}{r_n}\right| < \frac{1}{r_n^{\frac{3}{2}+\epsilon}},$$

or

$$|r_{n+1} - r_n| < \frac{1}{r_n^{\frac{1}{2}+\epsilon}} \leq 1,$$

therefore $r_{n+1} = r_n$ for all $n \geq 0$. This contradicts the fact that r_n diverges to infinity. Hence, $\alpha > 1$.

Now let δ be a real number with $1 < \delta < \alpha$, note that $\gamma := 2\alpha - \delta$ exceeds α, and choose n_0 such that

$$r_n^{\frac{3}{2}+\epsilon} > \frac{1}{\alpha - \delta}$$

holds for all $n \geq n_0$. From inequality (7), we get that

$$\delta r_n < r_{n+1} < \gamma r_n \tag{13}$$

holds for all $n \geq n_0$. From inequalities (7) for n and $n+1$ and the triangular inequality, we get

$$\frac{|r_{n+1}^2 - r_n r_{n+2}|}{r_n r_{n+1}} = \left|\frac{r_{n+1}}{r_n} - \frac{r_{n+2}}{r_{n+1}}\right| < \left|\alpha - \frac{r_{n+1}}{r_n}\right| + \left|\alpha - \frac{r_{n+2}}{r_{n+1}}\right| < \Big(\frac{1}{r_n^{\frac{3}{2}+\epsilon}} + \frac{1}{r_{n+1}^{\frac{3}{2}+\epsilon}}\Big),$$

or

$$\frac{|r_{n+1}^2 - r_{n+2} r_n|}{\sqrt{r_{n+1}}} < \frac{1}{r_n^\epsilon} \cdot \sqrt{\frac{r_{n+1}}{r_n}} + \frac{1}{r_{n+1}^\epsilon} \cdot \Big(\frac{r_n}{r_{n+1}}\Big). \tag{14}$$

Using inequality (13) in (14), we get

$$\frac{|r_{n+1}^2 - r_{n+2} r_n|}{\sqrt{r_{n+1}}} < \frac{c_1}{r_n^\epsilon} + \frac{c_2}{r_{n+1}^\epsilon} \tag{15}$$

for all $n \geq n_0$, where $c_1 = \sqrt{\gamma}$ and $c_2 = 1/\delta$. We now let n tend to infinity in (15) and get

$$\lim_{n\to\infty} \frac{|r_n^2 - r_{n+1} r_{n-1}|}{\sqrt{r_n}} = 0 < \frac{1}{\sqrt{2}}. \tag{16}$$

Consequently, the conclusion of part (i) of Theorem 1 follows from part (i) of Theorem DL.

The remaining assertions of part (ii) now follow from putting $\epsilon := 1/2$ in (15) and invoking $r_{n+1}/r_n < \gamma$ as well as part (ii) of Theorem DL.

Theorem 1 is therefore established.

Remark 1. The occurence of $\epsilon > 0$ in the exponent in inequality (7) is unnecessary. A closer investigation of the arguments used in the proof of Theorem 1 shows that the conclusion of part (i) of Theorem 1 remains valid if inequality (7) is replaced by the weaker inequality

$$\left|\alpha - \frac{r_{n+1}}{r_n}\right| < \frac{1-\epsilon}{\sqrt{2}(\sqrt{|\alpha|} + 1/|\alpha|)} \cdot \frac{1}{r_n^{\frac{3}{2}}}. \tag{7'}$$

Remark 2. Assume that α is a real number such that the hypotheses of either part (i9 or part (ii) of Theorem 1 are fulfilled. Using the full strength of our results from [1] and [2], we can infer that if α is an integer, then $(r_n)_{n\geq 0}$ is a geometrical progression of ratio α from some n on. However, if α is quadratic and the hypotheses of part (ii) of Theorem 1 are fulfilled, we can only infer that $(r_n)_{n\geq 0}$ is binary recurrent from some n on, and that its charateristic equation is precisely the minimal polynomial of α over $\mathbf{Q}$. However, we cannot infer that $(r_n)_{n\geq 0}$ is the

Lucas sequence of the first kind for α given by formula (6), mostly because the constant c appearing in inequality (8) is arbitrary. Of course, if one imposes that the constant c appearing in inequality (8) is small enough (for example, $c = 1/2$), then the rational numbers r_{n+1}/r_n are exactly the convergents of α, therefore r_n is indeed given by formula (6) for all n (up to some linear shift in the index n).

Proof of Theorem 2. If one replaces the sequence $(r_n)_{n\geq 0}$ by the sequence $(R_n)_{n\geq 0} = (r_{nt_1})_{n\geq 0}$, then the first inequality (9) together with part (i) of Theorem 1 show that α^{s_1} is an algebraic integer, different than 0 or ± 1, of degree at most 2. Similarly, if one replaces the sequence $(r_n)_{n\geq 0}$ by the sequence $(R_n)_{n\geq 0} = (r_{nt_2})_{n\geq 0}$, then the second part of inequality (9) together with part (ii) of Theorem 1 show that α^{s_2} is an algebraic integer, different that 0 or ± 1, of degree at most 2.

From here on, all we need to establish is that α is itself algebraic of degree at most 2. Assume that this is not so and let $K := \mathbf{Q}[\alpha]$ and $K_i := \mathbf{Q}[\alpha^{s_i}]$ for $i = 1,\ 2$. Since s_1 and s_2 are coprime, we get that $K = \mathbf{Q}[\alpha^{s_1},\ \alpha^{s_2}]$. Moreover, we must have $[K_i : \mathbf{Q}] = 2$ for both $i = 1$ and 2, i.e. K is a biquadratic real extension of $\mathbf{Q}$ and $\mathrm{Gal}(K/\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Hence, there exist two non-trivial elements σ_1 and σ_2 in $\mathrm{Gal}(K/\mathbf{Q})$ with $\sigma_i(\alpha^{s_i}) = \alpha^{s_i}$, i.e.

$$1 = \frac{\sigma_i(\alpha^{s_i})}{\alpha^{s_i}} = \left(\frac{\sigma_i(\alpha)}{\alpha}\right)^{s_i} \tag{17}$$

for $i = 1,\ 2$. Since K is a real field and σ_i is non-trivial, formula (17) implies that $\sigma_i(\alpha) = -\alpha$ for $i = 1,\ 2$. Hence, $\sigma_1(\alpha) = \sigma_2(\alpha)$, which implies $\sigma_1 = \sigma_2$. But this is a contradiction. The remaining of the assertions of Theorem 2 follow from Theorem 1.

Theorem 2 is therefore established.

Acknowledgements

Work by the second author was done while he visited Bielefeld. He would like to thank the Graduate College Strukturbildungsprozesse and the Forschungsschwerpunkt Mathematisierug there for their hospitality and the Alexander von Humboldt Foundation for support.

References

[1] Dress, A., Luca, F., Unbounded Integer Sequences $(A_n)_{n\geq 0}$ with $A_{n+1}A_{n-1} - A_n^2$ Bounded are of Fibonacci Type, to appear in the *Proceedings of ALCOMA99.*

[2] Dress, A., Luca, F., A Characterization of Certain Binary Recurrence Sequences, to appear in the *Proceedings of ALCOMA99.*

[3] Elsner, C., On the Approximation of Irrationals by Rationals, *Math. Nachr.* **189** (1998), 243–256.

[4] Elsner, C., On Diophantine Approximations with Rationals restricted by Arithmetical Conditions, *Fibonacci Quart.* **38** (2000), 25–34.

[5] Hurwitz, A., Über die angenäherte Darstellung der Irrationalzahlen durch rationalle Brüche, *Math. Ann.* **39** (1891), 279–284.

[6] Kiss, P., A diophantine approximative property of the second order linear recurrences, *Period. Math. Hungar.* **11** (1980), 281–287.

[7] Kiss, P., On a simultaneous approximation problem concerning binary recurrence sequences, preprint, 2000.

[8] Kiss, P., Tichy, R.F., A discrepancy problem with applications to linear recurrences I, *Proc. Japan Acad. (ser. A)* **65** (1989), 135–138.

[9] Kiss, P., Tichy, R.T., A discrepancy problem with applications to linear recurrences II, *Proc. Japan Acad. (ser. A)* **65** (1989), 191–194.

[10] Ridout, D., Rational approximations to algebraic numbers, *Mathematika* **4** (1957), 125–131.

[11] Roth, K.F., Rational approximations to algebraic numbers, *Mathematika* **2** (1955), 1–20, corrigendum 168.

Andreas Dress
Mathematics Department
Bielefeld University
Postfach 10 01 31
33 501 Bielefeld, Germany
e-mail: dress@mathematik.uni-bielefeld.de

Florian Luca
Instituto de Matemáticas de la UNAM
Campus Morelia
Apartado Postal 61-3 (Xangari), CP 58089
Morelia, Michoácan, Mexico
e-mail: fluca@matmor.unam.mx