Acta Acad. Paed. Agriensis, Sectio Mathematicae **27** *(2000) 57–62*

INVESTIGATION OF A DISCRETE CYCLIC-WAITING PROBLEM BY SIMULATION

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Abstract. The paper investigates a discrete cyclic-waiting queueing model, introduced by L. Lakatos in [6,7] and describing the landing of airplanes, by means of simulation. The interarrival and service time distributions are geometrical, the service discipline FIFO. The simulation results show a fast convergence to the analytical ones.

1. Introduction

Queueing systems with customers arriving, after a while getting service then leaving the system often occur in real life. These phenomena constitute a special field of probability theory. Depending on the inter-arrival and service time distributions, the number of servers and service discipline lead to different mathematical problems and form an important area of applied mathematics, the theory of queues.

For the investigation of queueing systems one has two possibilities. If the system under consideration is simple enough, then it allows a mathematical description, and one can construct a model which may be examined by exact analytical methods. If the system is too complex or its features are too specific, there remains the method of simulation. In the investigation of real systems by simulation the verification and validation play an essential role. One way is to use a -possibly simpler- analytical model for which we can obtain exact results, and to compare its characteristics with the simulation one. The parallel use of analytical and simulation methods usually gives enough information about the behaviour of such systems.

In conventional queueing systems the service process runs continuously, after having completed the service of a customer, we immediately take the next one. In this paper we consider a model describing the landing of airplanes. Our system is different from the above ones, the starting moment of service is determined by the moment of the completion of previous service and the moment of the arrival of the actual customer. There appears an idle time which is necessary to get to the starting position for service. We regard this idle time as part of the service time making in such a way the service process continuous. Such systems were analytically investigated in the case of Poisson arrivals and exponentially distributed service time in [4], uniform service time in [5], and for discrete time case in [6,7]. In [1] we already compared the analytical results with data obtained by means of simulation with continuous cases, and here we do this for discrete distributions.

2. Formulation of our problem

First, let us describe the focus of investigation. There is an airport where the entering airplanes put a landing request to the control tower upon arrival in the airside. Provided there is free system, i.e. the entering entity can be serviced at the moment of the request, the airplane can start landing. However, if the server is busy, i.e. a formerly arrived plane has not accomplished landing yet or other planes are already queueing for being serviced, then the incoming plane starts to circular maneouvre. The radius of the circle is fixed in a way that it takes the airplane T time to be above the runway again, i.e. the airplane can only put further landing request to the control tower at every nT moment after arrival, where $n \in \mathbf{N}$. Naturally, the request can only be serviced if there is no airplane queueing before it. The reception and service of the incoming planes follow the FIFO rule, according to which the earlier arriving planes are given landing permission earlier. Obviously, this system only operates properly if there are not many planes cyclic queueing.

3. Theoretical results

The above described problem has been investigated by L. Lakatos in several papers. Here we shortly formulate his results to which we compare our data obtained by means of simulation.

Let us consider a queueing system in which the time between two arriving requests has geometrical distribution with parameter r, the service time with parameter $1-q$, and the service of a request can be started only at the moment of its arrival or at moments differing from it by the multiples of cycle time T according to the FIFO rule. The described system will be investigated by means of the embedded Markov chain technique (see e.g. [2]). Let us define an embedded Markov chain whose states correspond to the number of requests in the system at moments just before starting the service of a request $t_k - 0$ (where t_k is the moment of the beginning of the service of the k-th one). The matrix of transition probabilities for this chain has the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

whose elements are determined by the generating functions:

$$A(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{(1-r)(1-q)}{1-q(1-r)} + \\ + z\frac{r(1-q)}{1-q(1-r)} + z\frac{rq(1-q^T)(1-r+rz)^T}{[1-q(1-r)][1-q^T(1-r+rz)^T]},$$

$$B(z) = \sum_{i=0}^{\infty} b_i z^i = \frac{1-(1-r)^T(1-r+rz)^T}{1-(1-r)(1-r+rz)} \frac{r(1-r+rz)}{1-(1-r)^T} + \\ + \frac{1-q^T(1-r)^T(1-r+rz)^T}{1-q(1-r)(1-r+rz)} \frac{rq(1-r+rz)[(1-r+rz)^T-1]}{[1-(1-r)^T][1-q^T(1-r+rz)^T]}.$$

The generating function of ergodic distribution $P(z) = \sum_{i=0}^{\infty} p_i z^i$ for this chain has the form

$$P(z) = p_0 \frac{zA(z) - B(z) + \frac{rz}{(1-r)(1-q)}[A(z)-B(z)]}{z - B(z)},$$

where

$$p_0 = \frac{(1-r)(1-q)}{1-q(1-r)} - \frac{rq[1-(1-r)^T]}{(1-r)^{T-1}(1-q^T)[1-q(1-r)]}.$$

The condition of the existence of ergodic distribution is the fulfilment of inequality

$$\frac{rq}{1-q^T} \frac{1-q^T(1-r)^T}{1-q(1-r)} < (1-r)^T.$$

In order to prove whether the analytical results hold true in practice, we have produced a computerized model of the system which provides random data. In what follows we compare the theoretical results with the generated data.

4. Computed results

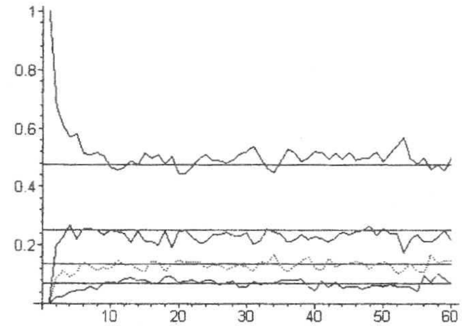
We carried out the experiments with different r, q and T values. For every fixed T, r, q, we did 500 independent experiments with different computer generated arrival and service times. On the basis of the above, we examined the probabilities in free systems of having $0, 1, 2\dots$ airplanes (marked $p_0, p_1, p_2\dots$ respectively) in the queue at the starting moments of services. In every case we recorded the results in a table where $p_0, p_1, p_2\dots$ are given in columns, and rows show the number of the incoming airplanes. All values are given in minutes.

For lack of place, we only include one table here: Figure 1 indicating those arriving and service times, where $r = 0.03,\ q = 0.9$ and $T = 6$.

	p0	p1	p2	p3	p4	p5	p6	p7	p8	p9	p10	p11	p12	p13	p14	p15	p16	p17
1.	1,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
2.	0,780	0,157	0,046	0,011	0,003	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
3.	0,746	0,183	0,049	0,017	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
4.	0,754	0,157	0,063	0,017	0,003	0,000	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
5.	0,706	0,189	0,069	0,020	0,011	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
6.	0,683	0,217	0,063	0,029	0,003	0,000	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
7.	0,689	0,189	0,094	0,014	0,009	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
8.	0,663	0,214	0,083	0,029	0,009	0,000	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
9.	0,654	0,209	0,097	0,031	0,006	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
10.	0,634	0,243	0,069	0,043	0,006	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
11.	0,671	0,186	0,091	0,026	0,023	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
12.	0,649	0,211	0,089	0,037	0,011	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
13.	0,631	0,229	0,097	0,037	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
14.	0,660	0,217	0,080	0,029	0,011	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
15.	0,694	0,197	0,063	0,034	0,011	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
16.	0,723	0,183	0,063	0,023	0,006	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
17.	0,720	0,171	0,069	0,023	0,009	0,006	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
18.	0,683	0,211	0,066	0,017	0,006	0,017	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
19.	0,689	0,180	0,086	0,023	0,020	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
20.	0,669	0,183	0,106	0,031	0,009	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
21.	0,660	0,211	0,083	0,034	0,006	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
22.	0,697	0,183	0,086	0,017	0,014	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
23.	0,671	0,217	0,074	0,029	0,009	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
24.	0,689	0,171	0,100	0,029	0,006	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
25.	0,669	0,191	0,091	0,029	0,014	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
26.	0,671	0,220	0,066	0,029	0,009	0,003	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
27.	0,663	0,209	0,089	0,031	0,000	0,003	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
28.	0,703	0,191	0,071	0,023	0,006	0,000	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
29.	0,691	0,211	0,069	0,017	0,006	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
30.	0,729	0,180	0,054	0,029	0,003	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
31.	0,697	0,186	0,089	0,023	0,003	0,000	0,000	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
32.	0,694	0,163	0,120	0,020	0,000	0,000	0,000	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
33.	0,649	0,234	0,086	0,029	0,000	0,000	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
34.	0,669	0,209	0,086	0,029	0,006	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
35.	0,709	0,194	0,054	0,023	0,011	0,006	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
36.	0,746	0,160	0,057	0,029	0,006	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
37.	0,697	0,180	0,080	0,031	0,009	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
38.	0,674	0,214	0,071	0,029	0,011	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
39.	0,703	0,189	0,066	0,029	0,009	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
40.	0,717	0,163	0,083	0,014	0,023	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
41.	0,743	0,157	0,057	0,031	0,011	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
42.	0,689	0,186	0,086	0,031	0,009	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
43.	0,691	0,191	0,080	0,026	0,011	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
44.	0,720	0,163	0,074	0,026	0,014	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
45.	0,717	0,174	0,071	0,029	0,003	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
46.	0,671	0,214	0,069	0,031	0,014	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
47.	0,686	0,191	0,091	0,023	0,009	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
48.	0,691	0,197	0,074	0,023	0,011	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
49.	0,697	0,191	0,077	0,020	0,011	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
50.	0,674	0,229	0,057	0,020	0,017	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
51.	0,737	0,166	0,057	0,034	0,000	0,003	0,000	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
52.	0,697	0,191	0,077	0,011	0,017	0,000	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
53.	0,717	0,166	0,071	0,031	0,009	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
54.	0,674	0,197	0,094	0,023	0,009	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
55.	0,671	0,203	0,077	0,031	0,011	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
56.	0,689	0,180	0,063	0,051	0,014	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
57.	0,677	0,169	0,114	0,031	0,009	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
58.	0,666	0,214	0,083	0,031	0,003	0,003	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
59.	0,657	0,220	0,083	0,034	0,006	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
60.	0,686	0,217	0,074	0,023	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000

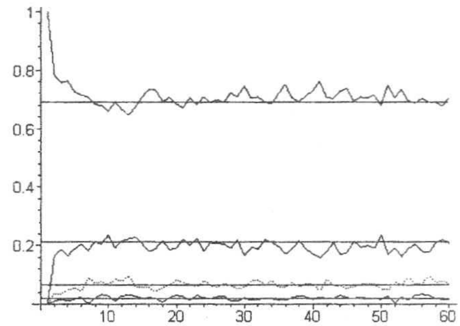
Figure 1.

T=3



r=0.05, q=0.9

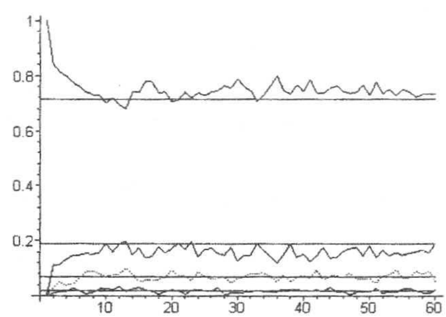
p_0	p_1	p_2	p_3
0.47419529	0.24957647	0.13712611	0.06930613
0.4923333	0.2325238	0.1335714	0.0714285
p_4	p_5	p_6	p_7
0.03476951	0.01744882	0.00875650	0.00439435
0.036	0.0163809	0.0080476	0.0055238



r=0.03, q=0.9

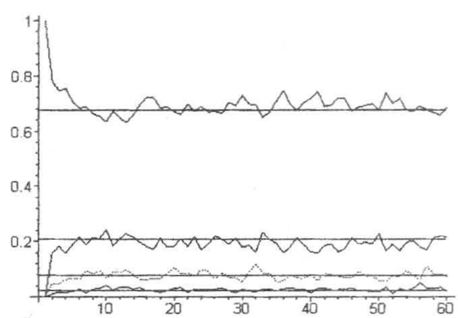
p_0	p_1	p_2	p_3
0.69096872	0.21370166	0.06771747	0.01965744
0.7056190	0.198	0.0676666	0.0210476
p_4	p_5	p_6	p_7
0.00566248	0.00163169	0.00047018	0.00013548
0.0051904	0.0019047	0.0003333	0.0002380

T=6



r=0.05, q=0.8

p_0	p_1	p_2	p_3
0.71433553	0.18798303	0.07050034	0.02018905
0.74376190	0.15961904	0.07157142	0.01876190
p_4	p_5	p_6	p_7
0.00519531	0.00133405	0.00034352	0.00008845
0.00466666	0.00119047	0.00028571	0.00014285



r=0.03, q=0.9

p_0	p_1	p_2	p_3
0.67552716	0.20892592	0.07949521	0.02502542
0.68971428	0.19366666	0.07819047	0.02676190
p_4	p_5	p_6	p_7
0.00765242	0.00234139	0.00071651	0.00021926
0.00852381	0.00223809	0.00066666	0.00014285

Figure 2.

The diagrams in Figure 2 show the calculated results where the horizontal lines from top to bottom express the probabilities p_0, p_1, p_2, p_3, whose exact values can be seen in the upper line. Below we also give p_4, p_5, p_6, p_7 values which are not included in the diagrams. In the lower lines one can see the average values obtained from numerical results shown in diagrams.

Even the examination of not more than 500 independent experiments and the cases of 60 arriving airplanes shows that the computed results clearly approximate the exact values.

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