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EXPONENTIAL STABILITY OF REGULAR LINEAR SYSTEMS ON BANACH SPACES

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Abstract: The article deals with the vd-transformation in Banach space and its application in studying the stability of trivial solution of differential equations. A sufficient condition of exponential stability of regular linear systems with burfication on Banach space will be proved.

vd-transformation and it's properties

In this section we shall give the definition, examples and some properties of a vd-transformation on Banach spaces. It is an expansion of a vd-transformation on finite dimension spaces given by Yu. S. Bogdanov ([2]–[6]). From that, we shall give the definition of regular linear equations which are applied to study the stability of regular linear equations with burfication on Banach spaces.

Let E be a Banach space and G be an open simple connected domain containing the origin O of E.

We define H as follows $H = G \times \mathbf{R} = \{\eta = (x, t) \colon\ x \in G, \quad t \in \mathbf{R}\}$.

Let $v_0 \colon \mathbf{R}^+ \to \mathbf{R}^+$ be a function which is continuous, monotone strictly increasing and satisfies the following conditions:

$$v_0(0) = 0; \quad v_0(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Let $d \colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ be a given real function of two variables: and d satisfies the following conditions for all $\gamma > 0, \gamma_3 > \gamma_2 > \gamma_1 > 0$;

(d_1) $d(\gamma_2, \gamma_1) = -d(\gamma_1, \gamma_2)$,

(d_2) $d(\gamma_2, \gamma) > d(\gamma_1, \gamma)$,

(d_3) $d(\gamma_3, \gamma_2) + d(\gamma_2, \gamma_1) \geq d(\gamma_3, \gamma_1)$,

(d_4) $\cup_{\gamma \in \mathbf{R}_+} \{d(\gamma, \gamma_1)\} = \mathbf{R}$.

Suppose that, $l \colon H \to H$ is a diffeomorphism,

$$\eta = (x, t) \mapsto \eta' = (x', t')$$

satisfying the following equalities:

$$l(0, t) = (0, t), \qquad l(x, t) = (x', t)$$

for all $t \in \mathbf{R}$. It is easy to prove that the set L of all those transformations $L = \{l\}$ is a group with the composition of maps.

Let v be a real function

$$v\colon H^\star \to \mathbf{R}_+, \quad \eta = (x,t) \to v(\eta) = v_0(\|x\|)$$

where $H^\star = G^\star \times \mathbf{R} = (G \setminus \{0\}) \times \mathbf{R}$.

Since the function $v\colon H^\star \to \mathbf{R}_+$ is independent of t, that is, $v(x,t) = v(x,t')$ for all $t, t' \in \mathbf{R}$, we can denote by $v(x)$ the value of $v(x,t)$ for any $x \in G^\star$ and $t \in \mathbf{R}$.

Definition. The transformation $l \in L$ is called vd-transformation iff

$$\sup_{\eta \in H^\star} |d\{v(\eta), v[l(\eta)]\}| < +\infty \tag{1}$$

From the definition of function d, we also have

$$\sup_{\eta' \in H^\star} \left| d\left\{v(\eta'), v(l^{-1}(\eta'))\right\}\right| < +\infty.$$

Consequently, if we denote by L_{vd} the set of vd-transformation then it is a subgroup of L.

Examples

1. Suppose $v_0(x,t) = \|x\|, d_0(\gamma_1,\gamma_2) = ln\frac{\gamma_1}{\gamma_2}$, and $l(x,t)$ (with a fixed t) is a linear transformation having bounded partial derivation with respect to t. Then, l is v_0d_0-transformation if and only if it's a Lyapunov transformation ([1]).
2. If $v(x,t) = |x|^2; E = \mathbf{R}$

$$d(\gamma_1,\gamma_2) = \begin{cases} \sqrt{\gamma_1} - \sqrt{\gamma_2} & \text{if } \gamma_1 \cdot \gamma_2 \geq 1 \\ \frac{1}{\sqrt{\gamma_2}} - \frac{1}{\sqrt{\gamma_1}} & \text{if } \gamma_1 \cdot \gamma_2 < 1, \end{cases}$$

then all conditions from d_1 are satisfied. So

$$l(x,t) = (x + \frac{1}{2}\sin t \sin^2 x, t)$$

is a vd-transformation.

From example 1, we can see that a vd-transformation is an expansion of Lyapunov transformation, but it still keeps an important property, stability of the trivial solution of a following differential equation on the Banach space E:

$$\begin{cases} \frac{dx}{dt} = f(x,t) \\ f(0,t) \equiv 0 \end{cases} \tag{2}$$

We denote by $x(t;\xi)$ the solution of equation (2) which satisfies the initial condition $x(t_0,\xi)=\xi$ and suppose that

$$\lambda = \lim_{\varepsilon\to 0^+} \sup_{\substack{\|\xi\|\le\varepsilon \\ t\ge t_0}} \|(x(t;\xi)\|;\lambda_1 = \lim_{\varepsilon\to 0^+} \sup_{\varepsilon\to 0^+} \sup_{\substack{v(\xi)\le\varepsilon \\ t\ge t_0}} v(x(t;\xi)).$$

Definition. ([7]). The solution $x=0$ of differential equation (2) is said to be Lyapunov stable if for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for each solution $x(t)$ of (2), with its initiala value $x(t_0)=\xi$ satisfying the condition $\|\xi\|<\delta(\varepsilon)$

then the inequality $\|x(t,\xi)\|<\varepsilon$ holds for all $t\ge t_0$.

From the definition we can see that the solution $x=0$ of differential equation (2) is stable iff $\lambda=0$.

Proposition 1. $\lambda=0$ *if and only if* $\lambda_1=0$.

Proof. By the continuity of the function v we immediately have $\lim\limits_{\xi\to 0} v(\xi)=0$. Since $v(\|x\|)$ is monotone strictly increasing we can deduce $\lim\limits_{v(\xi)\to 0}\xi=0$. Therefore

$$\lim_{k\to\infty}\xi_k=0 \Leftrightarrow \lim_{k\to\infty} v(\xi_k)=0. \tag{3}$$

We assume that $\lambda=0$, then:

$$\lim_{k\to\infty}\|x(t_k,\xi_k)\|=0$$

for all sequences $\{\varepsilon_k\}\subset\mathbf{R}_+:\ \varepsilon_k\to 0;\{\xi_k\}\subset E:\ \|\xi_k\|<\varepsilon_k$ and $\{t_k\}\subset\mathbf{R}, t-k\ge t_0$. Because of (3), we have

$$\lim_{k\to\infty}\|x(t_k,\xi_k)\|=0 \Leftrightarrow \lim_{k\to\infty} v\left(x\left(t_k,\xi_k\right)\right)=0.$$

It follows that $\lambda=0\Leftrightarrow\lambda_1=0$.

Proposition 2. *A vd-transformation conserves the stability of trivial solution* $x=0$ *of differential equation (2).*

Proof. By the vd-transformation

$$(x,t)\to l(x,t)=(y,t),$$

the equation (2) is transformed to the following one:

$$\frac{dy}{dt}=g(y,t) \tag{4}$$

By assumption, the solution $x = 0$ of equation (2) is stable, that means:

$$\lim_{\varepsilon\to 0^+} \sup_{\substack{\|x_0\|\le\varepsilon \\ t\ge t_0}} \|x(t;x_0)\| = 0 \Leftrightarrow \lim_{\varepsilon\to 0^+} \sup_{\substack{v(x_0)\le\varepsilon \\ t\ge t_0}} v[x(t;x_0)] = 0.$$

If the solution $y = 0$ of (4) is unstable, then

$$\lim_{\varepsilon\to 0^+} \sup_{\substack{v(y_0)\le\varepsilon \\ t\ge t_0}} v\left[y\left(t, y_0\right)\right] > 0.$$

It means that there exists a positive number δ such that

$$\exists\{\eta_n\} \subset E:\ \eta_n \to 0;\ \exists\{t_n\} \subset \mathbf{R}_+:\ t_n \ge t_0;\ \forall nN:\ v\left[y\left(t_n;\eta_n\right)\right] \ge \delta. \tag{5}$$

Since $v\left[x\left(t_n, \xi_n\right)\right] \to 0$ as $n \to \infty$, where $\xi_n, t_n) = l^{-1}(\eta_n, t_n)$ one could say

$$v\left[x\left(t_n;\xi_n\right)\right] < \delta, \forall n \in N. \tag{6}$$

From (5), (6) and d_4) we deduce:

$$|d\{v\left[x\left(t_n;\xi_n\right)\right], v\left[y\left(t_n;\eta_n\right)\right]\}| = d\{v\left[y(t_n;\eta_n)\right], v\left[x\left(t_n;\xi_n\right)\right]\}$$

$$> d\{\delta, v\left[x\left(t_n;\xi_n\right)\right]\} \to +\infty \quad \text{as} \quad n \to \infty.$$

Consequently

$$\sup_{n\in N} |d\{v\left[x\left(t_n;\xi_n\right)\right], v\left[l\left(x\left(t_n;\xi_n\right)\right)\right]\}| = +\infty$$

that contradicts the definition of 1.

Regular system

Definition. A transformation $l \in L$, satisfying the following condition for all $\eta \in H^\star$:

$$d\{v\left(\eta\right), v\left[l\left(\eta\right)\right]\} = o(t) \quad \text{as} \quad t \to \pm\infty,$$

is called a generalized vd-transformation.

Definition. A transformation $y = L(t)x$ is a generalized Lyapunov one if:

$$\chi\left[L\left(t\right)\right] = \chi\left[L^{-1}\left(t\right)\right] = 0 \tag{7}$$

where $\chi\left[L\left(t\right)\right] := \overline{\lim_{t\to\infty}} \frac{1}{t} ln\|L(t)\|$ is called characteristic exponent of L(t).

By definition we immediately have following remarks:

Remark 1. Generalized Lyapunov transformations conserve Lyapunov exponents [1].

Remark 2. A generalized Lyapunov transformation is generalized vd-transformation when

$$v(x) = \|x\|, d\,(\gamma_1, \gamma_2) = ln\frac{\gamma_1}{\gamma_2},$$

and l is homogeneously linear for x (where $l(x,t) = (L(t)x, t)$).

Now we shall prove a necessary and sufficient condition for which a differential system on finite dimension spaces are regular. Since this condition plays an important role for the conception of a regular differential equations on Banach spaces and we could not find it in literature, we shall formulate it as a lemma.

We consider the following linear differential system:

$$\frac{dx}{dt} = A(t)x \tag{8}$$

where $x \in \mathbf{R}^n, A(t) \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and is real continuous for all $t \in \mathbf{R}$ and $\sup \|A(t)\| < \infty$.

Let $X(t)$ be a normal fundamental matrix of (8) and $\sigma_x = \sum\limits_{k=1}^{m} n_k \alpha_k$ be the sum of all its exponent numbers ([1]).

Definition. ([1]) The linear system (8) is said to be regular iff

$$\sigma_x = \varliminf_{t\to\infty} \frac{1}{t} \int\limits_{t_0}^{t} \mathrm{Sp}\, A(\tau) d\tau.$$

Lemma. *A necessary and sufficient condition that the system (8) to be regular one is there exists a generalized Lyapunov transformation carrying the system (8) to the system with constant matrix $B \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$:*

$$\frac{dy}{dt} = By \tag{9}$$

Proof. Let $y = L(t)x$ be a generalized Lyapunov transformation, $X(t)$ be a normal fundamental matrix of system (8). It follows that $Y(t) = L(t)X(t)$ is a fundamental matrix of system (9). Since

$$\det Y(t) = \det L(t) \det X(t),$$

we have

$$\det Y(t_0)\exp(t-t_0)\,\mathrm{Sp}\,B = \det L(t)\det X(t_0)\exp\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1$$

$$\exp\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = |c(t_0)|\left|\det L^{-1}(t)\right|\exp\left[(t-t_0)\,\mathrm{Sp}\,B\right],$$

$$\text{where}\quad c(t_0)=\det\left[Y(t_0)X^{-1}(t_0)\right],$$

$$\Rightarrow\quad \frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = \frac{1}{t}ln\,|c(t_0)| + \frac{1}{t}ln\left|\det L^{-1}(t)\right| + \left(1-\frac{t_0}{t}\right)\mathrm{Sp}\,B$$

$$\Rightarrow\quad \overline{\lim_{t\to\infty}}\,\frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = \mathrm{Sp}\,B + \chi\left[\det L^{-1}(t)\right].$$

Because of $\chi\left[L^{-1}(t)\right]=0$ we have

$$\chi\left[\det L^{-1}(t)\right]\le n\chi\left[L^{-1}(t)\right]=0$$

Analogously, from $\chi\left[L(t)\right]=0$ it follows that

$$\chi\left[\det L(t)\right]\le 0$$

On the other hand, since

$$\det L(t).\det L^{-1}(t)=0,$$

the following is held: $\chi\left[\det L(t)\right]+\chi\left[\det L^{-1}(t)\right]\ge 0$.

Therefore $\chi\left[\det L(t)\right]=\chi\left[\det L^{-1}(t)\right]=0$

It follows from these equalities that

$$\lim_{t\to\infty}\frac{1}{t}ln\left|\det L^{-1}(t)\right|=0$$

and finally

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1=\mathrm{Sp}\,B.$$

Since the Lyapunov transformation conserves Lyapunov exponents and the X is normal, Y is normal too and

$$\sigma_x = \sigma_y = \operatorname{Sp} B,$$

we have

$$\sigma_x = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Sp} A(t_1) dt_1,$$

i.e. the system (8) is regular.

Conversely, let the system (8) be regular. We will denote by $X(t)$ the fundamental normal matrix of (8), which has the exponent numbers: $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. Consider the Jordan matrix B, in which $\lambda_1, \ldots, \lambda_n$ are elements on the diagonal.

Denoting $Y(t)$ the fundamental normal matrix of the system (9), we constate that the column of which has the same exponent numbers (with the same order): $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Putting $L(t) = Y(t)X^{-1}(t)$ we will prove that $y = L(t)x$ is a generalized Lyapunov transformation.

Suppose that

$$Y(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \ldots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix}$$

$$X^{-1}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t) \end{pmatrix}.$$

Tdilehen $\chi\left[y^{(k)}\right] = \lambda_k$, where $y^{(k)}(t) = \operatorname{colon}(y_{1k}(t) \ldots y_{nk}(t))$.

Because of the regularity of (8), we have $\chi\left[x^{(k)}(t)\right] = -\lambda_k$, where $x^{(k)}(t) = (x_{k1}(t) \ldots x_{kn}(t))$.

We consider now the diagonal matrix

$$\Delta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We find then

$$L(t) = Y(t)e^{-t\Delta}e^{t\Delta}X^{-1}(t) = \phi(t)\Psi(t)$$

in which $\phi(t) = Y(t)e^{-t\Delta}, \Psi(t) = e^{t\Delta}X^{-1}(t)$. It follows that

$$\chi[\phi(t)] = \max_{j,k} \chi\left[y_{jk}(t)e^{-\lambda_k t}\right] = 0$$

$$\chi[\Psi(t)] = \max_{j,k} \chi\left[x_{j,k}(t)e^{\lambda_j t}\right] = 0.$$

Consequently,

$$\chi[L(t)] \leq \chi[\phi(t)] + \chi[\Psi(t)] = 0$$

Analogously we can prove that $\chi\left[L^{-1}(t)\right] \leq 0$.

However, from $L(t) \cdot L^{-1}(t) = E$, we immediately find that $\chi[L(t)] + \chi\left[L^{-1}(t)\right] \geq 0$, i.e. $\chi[L(t)] = \chi\left[L^{-1}(t)\right] = 0$. The lemma is proved.

Definition. A linear differential equation:

$$\frac{dx}{dt} = A(t)x, \tag{10}$$

where $A(t) \in \mathcal{L}(E, E)$ and is continuous for all $t \in \mathbf{R}$ and $\sup_t \|A(t)\| < \infty$, is said to be regular one iff there is a generalized Lyapunov transformation $y = L(t)x$ carrying which to the linear differential equation with constant operator:

$$\frac{dy}{dt} = By. \tag{11}$$

Now we shall give a main theorem to regular differential equations on Banach spaces.

Let consider differential equation

$$\frac{dx}{dt} = A(t)x + f(x,t), \tag{12}$$

where $A(t) \in \mathcal{L}(E, E)$ and $\sup_{t\in\mathbf{R}} \|A(t)\| < \infty, f \in C^{(1,0)}(E \times \mathbf{R}), f(0,t) \equiv 0,$

$$\|f(x,t)\| \leq \Psi(t)\|x\|^m \quad (m > 1); \quad \chi[\Psi(t)] = 0.$$

Under these conditions, we show the following theorem:

Theorem. *If the equation (10) is regular and all its characteristic exponents are not larger than $-\lambda < 0$, the trivial solution $x = 0$ of the equation (10) is exponential stability* ([7]). *I.e there exist $N > 0, A > 0$ such that*

$$\|x(t)\| \leq Ae^{-N(t-t_0)}\|x(t_0)\|$$

for all solutions $x(t)$ *of* (12).

Proof. We denote by $X(t)(X(t_0) = Id_E)$ its Cauchy operator of equation (10) ([7], p. 147).

1. First we will estimate the resolvant operator $K(t,\tau) = X(t)X^{-1}(\tau)$ $(t_0 \leq \tau \leq t)$.

Because of the regularity of the equation (10), there is a generalized Lyapunov transformation $y = L(t)x$ carrying equation (10) to equation (11).

We have $Y(t) = L(t)X(t)$ is resolvant operator of the equation (11).

If we put $H(t,\tau) = Y(t)Y^{-1}(\tau)$ then $K(t,\tau) = L(t)H(t,\tau)L^{-1}(\tau)$.

Suppose that all characteristic exponents of the equation (10) are not larger than α.

Hence all those of the equation (11) are not too than α, that is for every solution $y(t) = Y(t)y_0$ and $\varepsilon > 0$ there exists $c > 0$ we have

$$\|y(t)\| \leq ce^{(\alpha+\varepsilon/2)t}, \quad \forall t \geq t_0.$$

Then, the operator's family $\left\{e^{-(\alpha+\varepsilon/2)t}Y(t), t \geq t_0\right\}$ is point-bounded.

By virtue of the Banach–Steinhauss there exists $c_1, > 0$ such that:

$$\|e^{(-\alpha+\varepsilon/2)t}Y(t)\| \leq c_1 \Leftrightarrow \|Y(t)\| \leq c_1e^{(\alpha+\varepsilon/2)t}.$$

Therefore $\|H(t,\tau)\| = \|Y(t-\tau\| \leq c_1e^{(\alpha_\varepsilon/2)(t-\tau)}$ for the equation with constant operator (11)

On the other hand

$$\chi\left[L(t)\right] = \chi\left[L^{-1}(t)\right] = 0 \Leftrightarrow \begin{cases} \|L(t)\| \leq c_2e^{\frac{\varepsilon}{2}t} \\ \|L^{-1}(\tau)\| \leq c_3e^{\frac{\varepsilon}{2}\tau}. \end{cases}$$

It follows that

$$\|K(t,\tau)\| \leq \|L(t)\|\|H(t,\tau)\|\|L^{-1}(\tau)\|$$

(13)
$$\leq c_1c_2c_3e^{(\alpha+\varepsilon)(t-\tau)}e^{\varepsilon\tau} = c(\varepsilon,t_0)e^{(\alpha+\varepsilon)(t-\tau)}$$

where $c = c_1c_2c_3e^{-(\alpha+\varepsilon)\tau}$.

Since $K(t,t_0) = X(t)$ we have

$$\|X(t)\| \leq ce^{(\alpha+\varepsilon)t} \tag{14}$$

In the case, when $\alpha < 0$, there exists a positive number ε such that $\alpha + \varepsilon \leq 0$, whence

$$\|K(t,\tau)\| \leq ce^{\varepsilon t}, \quad \|X(t)\| \leq c. \tag{15}$$

2. We will now prove the theorem. Denoting

$$y = \chi e^{\gamma(t-t_0)} \tag{16}$$

where γ is a positive number such that $0 < \gamma < \lambda$, the equation (12) will be transformed to:

$$\frac{dy}{dt} = B(t)y + g(t,y) \tag{17}$$

with $B(t) = A(t) + \gamma Id_E$

$$g(t,y) = \exp(\gamma(t-t_0))f\left(t, ye^{-\gamma(t-t_0)}\right). \tag{18}$$

Now we show that the equation

$$\frac{d\eta}{dt} = B(t)\eta \tag{19}$$

is regular. Indeed, by the regularity of (10) there is a generalized Lyapunov transformation $z = L(t)x$ carrying (10) to the equation with constant operator:

$$\frac{dz}{dt} = Cz$$

where

$$C = L'(t)L^{-1}(t) + L(t)A(t)L^{-1}(t).$$

The transformation $\xi = L(t)\eta$ implies the following:

$$\frac{d\xi}{dt} = \left[L'(t)L^{-1}(t) + L(t)B(t)L^{-1}(t)\right]\xi = (C + \gamma Id_E)\,\xi.$$

The regularity of (19) is proved.

We denote by $\eta(t)$ the solution of (19) and then $e^{-\gamma(t-t_0)}\eta(t)$ is the solution of (10).

This implies:

$$\chi\left[\eta(t)e^{-\gamma(t-t_0)}\right] \leq -\lambda$$

$$\Rightarrow \chi\left[\eta(t)\right] \leq \chi\left[e^{\gamma(t-t_0)}\right] + \chi\left[\eta(t)e^{-\gamma(t-t_0)}\right] \leq -\lambda + \gamma < 0.$$

By virtue of the estimation of the resolvant operator the following inequality is true:

$$\|K(t,\tau)\| \leq Ne^{\varepsilon\tau}; \quad t_0 \leq \tau < \infty,$$

where $K(t,\tau)$ is the resolvant operator of (10).

Now considerint the solution of (17)

$$y(t) = K(t,t_0)y(t_0) + \int\limits_{t_0}^{t} K(t,\tau)g(\tau,y(\tau))d\tau,$$

we have

$$\|y(t)\| \leq \|K(t,t_0)\| \cdot \|y(t_0)\| + \int\limits_{t_0}^{t} \|K(t,\tau)\| \cdot \|g(\tau,y(\tau))\|d\tau$$

$$\leq Ne^{\varepsilon(t_0)}\|y(t_0)\| + \int\limits_{t_0}^{t} Ne^{\varepsilon\tau}e^{\gamma(\tau-t_0)}\Psi(\tau)\|y(\tau)\|^m e^{-m\gamma(\tau-t_0)}d\tau$$

$$\leq Ne^{\varepsilon t_0}\|y(t_0)\| + \int\limits_{t_0}^{t} Ne^{\varepsilon\tau}e^{(1-m)\gamma(\tau-t_0)}ce^{\varepsilon\tau}\|y(\tau)\|^m d\tau$$

$$= c_1\|y(t_0)\| + \int\limits_{t_0}^{t} c_2 e^{[2\varepsilon-(m-1)\gamma](\tau-t_0)}\|y(\tau)\|^m d\tau$$

where $c_1 = Ne^{\varepsilon t_0}, c_2 = cNe^{-2\varepsilon t_0}$.

Hence

$$\|y(t)\| \leq c_1\|y(t_0)\| + \int\limits_{t_0}^{t} c_2 e^{-\delta(\tau-t_0)}\|y(\tau)\|^m d\tau, \tag{20}$$

where $\delta = (m-1)\gamma - 2\varepsilon$.

We will find the positive number ε such that $\delta > 0$.

Since

$$\int\limits_{t_0}^{t} e^{-\delta(\tau-t_0)}d\tau = \frac{1}{\delta} - \frac{1}{\delta}e^{-\delta(t-t_0)} < \frac{1}{\delta}$$

there is $\Delta > 0$ such that

$$N = (m-1)c_1^{m-1}\|y(t_0)\|^{m-1}\int\limits_{t_0}^{t} c_2 e^{-\delta(\tau - t_0)}d\tau < 1$$

provided that

$$\|y(t_0)\| < \Delta.$$

We apply here the lemma of Bihari [8] and find

$$\|y(t)\| \leq \frac{c_1\|y(t_0)\|}{[1-N]^{\frac{1}{m-1}}} = A\|y(t_0)\|, A = \frac{c_1}{[1-N]^{\frac{1}{m-1}}}$$

$$\Rightarrow \|x(t)\| \leq Ae^{-\gamma(t-t_0)}\|x(t_0)\|, (x(t_0) = y(t_0))$$

that is the exponential stability of the solution $x = 0$ of (12), and the proof of the theorem is finished.

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