ON A PROBLEM CONCERNING
PERFECT POWERS IN LINEAR RECURRENCES

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Abstract: For a linear recurrence sequence \( \{G_n\}_{n=0}^{\infty} \) of rational integers of order \( k \geq 2 \) satisfying some conditions, we show that the equation \( sG_x = w^q \), where \( w > 1 \) and \( r \) are positive integers and \( s \) contains only given primes as its prime factors, implies the inequality \( q < q_0 \), where \( q_0 \) is an effective computable constant depending on the sequence, the prime factors of \( s \) and \( r \).

Let \( G = \{G_n\}_{n=0}^{\infty} \) be a linear recurrence sequence of order \( k \geq 2 \) defined by

\[
G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad (n \geq k),
\]

where \( A_1, \ldots, A_k \) are given rational integers with \( A_k \neq 0 \) and the initial values \( G_0, G_1, \ldots, G_{k-1} \) are not all zero integers. We denote by \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_s \) the distinct roots of the polynomial

\[
g(x) = x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k,
\]

furthermore we suppose that \( |\alpha| > |\alpha_i| \) for \( 2 \leq i \leq s \), and the roots \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_s \) have multiplicity \( m_1 = 1, m_2, \ldots, m_s \). In this case \( |\alpha| > 1 \) and, as it is well known, the terms of \( G \) can be written in the form

\[
G_n = a \alpha^n + g_2(n) \alpha_2^n + \cdots + g_s(n) \alpha_s^n \quad (n \geq 0),
\]

where \( g_i \) \( (2 \leq i \leq s) \) is a polynomial of degree \( m_i - 1 \), furthermore \( a \) and the coefficients of \( g_i \) are elements of the algebraic number field \( \mathbb{Q}(\alpha_1, \ldots, \alpha_s) \).

Several authors investigated the perfect powers in the recurrences \( G \). Among others T. N. Shorey and C. L. Stewart [6] proved that for a given integer \( d(\neq 0) \) the equation

\[
G_x = dw^q
\]

with positive integers \( x, w(> 1) \) and \( q \) implies the inequality \( q < N \), where \( N \) is an effectively computable constant depending only on \( d \) and \( G \). In [4] we gave an improvement of this result substituting \( d \) by integers containing only fixed prime factors. For second order recurrences \( (k = 2) \) A. Pethő obtained more strict

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results (e.g. see [5]). In [2] B. Brindza, K. Liptai and L. Szalay proved, under some conditions, that for recurrences $G$ and $H$ the equation

$$G_x H_y = w^q$$

can be satisfied only if $q$ is bounded above. We proved [3] that for a sequence $G$ and fixed positive integer $n$ from

$$G_n^r G_x^{q-r} = w^q,$$

with $0 < r \leq q/2$, it follows that $q$ is bounded above.

In this note we prove the following theorems.

**Theorem 1.** For given primes $p_1, \ldots, p_t$ let $S$ be a set of integers defined by

$$S = \{ n : n \in \mathbb{N}, \ n = \prod_{i=1}^t p_i^{\beta_i}, \ \beta_i \geq 0 \}.$$

Let $r \geq 1$ be an integer and let $G$ be a linear recurrence defined in (1) satisfying the conditions $a \neq 0$ and $G_n \neq a \alpha^n$ for $n \geq n_0$. Then the equation

$$(2) \quad sG_x^r = w^q$$

with positive integers $s \in S$, $w > 1$ and $x > x_0$ ($x_0$ depends on $G$ and $r$) implies that $q < q_0$, where $q_0$ is an effectively computable constant depending on $n_0, r$, the sequence $G$ and the primes $p_1, \ldots, p_t$.

**Theorem 2.** Let $G$ be a linear recurrence defined by (1) satisfying the conditions $a \neq 0$ and $G_n \neq a \alpha^n$ for $n \geq n_0$. If

$$G_y^q G_x^r = w^q$$

for positive integers $x, y, q$ and $r$ with the conditions $(q, r) = 1$ and $y < n_1$, then $q < q_1$, where $q_1$ is a constant depending on $G, n_0$ and $n_1$, but does not depend on $r$.

In the proofs we need some lemmas.

**Lemma 1.** Let $\omega_1, \omega_2, \ldots, \omega_v$ ($\omega_i \neq 0$ or 1) be algebraic numbers and let $\gamma_1, \gamma_2, \ldots, \gamma_v$ be not all zero rational integers. Suppose that $\omega_1, \ldots, \omega_v$ have heights $M_1, \ldots, M_v(\geq 4)$, furthermore $|\gamma_i| \leq B$ ($B \geq 4$) for $i = 1, 2, \ldots, v-1$ and $|\gamma_v| \leq B'$. Further let

$$\Lambda = \gamma_1 \log \omega_1 + \gamma_2 \log \omega_2 + \cdots + \gamma_v \log \omega_v,$$
where the logarithms mean their principal values. If \( \Lambda \neq 0 \), then there exists an effectively computable constant \( c > 0 \), depending only on \( v \), \( M_1, \ldots, M_{v-1} \) and the degree of the field \( \mathbb{Q}(\omega_1, \ldots, \omega_v) \), such that for any \( \delta \) with \( 0 < \delta < 1/2 \) we have

\[
|\Lambda| > \left( \frac{\delta}{B'} \right)^c \log M_v e^{-\delta B}.
\]

(see A. Baker [1]).

**Lemma 2.** Let \( a, b, c, q \) and \( r \) be positive integers with \( 0 < r < q \) and \( (q, r) = 1 \). If

\[
(3) \quad a^{q-r} b^r = c^q,
\]

then for any integer \( r_1 \) with \( 0 < r_1 < q \) there is a positive integer \( d \), such that

\[
(4) \quad a^{q-r_1} b^{r_1} = d^q.
\]

**Proof of Lemma 2.** From (3)

\[
\left( \frac{b}{a} \right)^r = \left( \frac{c}{a} \right)^q
\]

follows. Let \( x \) and \( y \) be integers for which \( rx + qy = r_1 \). Then

\[
\left( \frac{b}{a} \right)^{rx} = \left( \frac{c}{a} \right)^{qx}
\]

and

\[
\left( \frac{b}{a} \right)^{rx} \left( \frac{b}{a} \right)^{qy} = \left( \frac{b}{a} \right)^{r_1} = \left( \frac{b}{a} \right)^{qy} \left( \frac{b}{a} \right)^{rx}
\]

from which

\[
\left( \frac{b}{a} \right)^{r_1} a^q = a^{q-r_1} b^{r_1} = \left( \frac{c^x b^y a}{a^{x+y}} \right)^q = d^q
\]

follows, where \( d \) is an integer since \( a^{q-r_1} b^{r_1} \) is integer.

**Proof of Theorem 1.** In the proof we denote by \( c_1, c_2, \ldots \) effectively computable positive constants, which depend only on \( n_0, r \), the sequence \( G \) and the primes \( p_1, \ldots, p_t \). Suppose that equation (2) holds with the conditions given in the theorem. We can suppose that

\[
(4) \quad s = p_1^{u_1} \cdots p_t^{u_t}, \text{ where } 0 \leq u_i < q \text{ for } 1 \leq i \leq t.
\]
Namely if
\[ s = \prod_{i=1}^{t} p_i^{u_i+qv_i} = \prod_{i=1}^{t} p_i^{u_i} \left( \prod_{i=1}^{t} p_i^{v_i} \right)^q, \]
then \( w/ \prod_{i=1}^{t} p_i^v \) is also an integer and greater than 1.

By (1), equation (2) can be written in the form
\[
\lambda = \frac{w^q}{sa^r \alpha^x} = \left( 1 + \frac{g_2(x)}{a} \left( \frac{\alpha_2}{\alpha} \right)^x + \cdots \right)^r,
\]
where \(|\lambda| \neq 1\) if \( x > n_0 \). Using the properties of the exponential and logarithm functions, by (5) and \(|\alpha| > |\alpha_i|\) \( (i=2,\ldots,s)\)
\[ |\lambda| < |1 + e^{-c_1 x}|^r \]
and
\[
|\log |\lambda|| < r e^{-c_2 x} = e^{\log r - c_2 x}
\]
follows. On the other hand, by (5)
\[
|\log |\lambda|| = \left| q \log w - r \log |a| - x r \log |\alpha| - \sum_{i=1}^{t} u_i \log p_i \right|.\]

By (2) and (4) it follows that
\[
C_x^r > \left( \frac{w}{t \prod_{i=1}^{t} p_i} \right)^q,
\]
where \( w/ \prod_{i=1}^{t} p_i > 1 \) is an integer since any prime factor of \( s \) divides \( w \). From this inequality, using (1),
\[
\log (|a|^r |\alpha|^r x) > c_3 q \log w \left( 1 - \frac{\log (\prod_{i=1}^{t} p_i)}{q \log w} \right)
\]
follows and so, if \( q \) is large enough,
\[
q < c_4 r x \quad \text{and} \quad x > c_5 q \log w.
\]
Since $u_i < q < c_4 rx$, using Lemma 1 with $v \leq t + 3$, $\omega_v = w$, $M_v = 2w$ ($\geq 4$), $B' = q$ and $B = c_4 rx$, from (7) we obtain the inequality

$$|\log |\lambda|| > \left(\frac{\delta}{q}\right)^c_6 \log 2w e^{-\delta c_4 rx} = e^{-(\log q - \log \delta)c_6 \log 2w - \delta c_4 rx}$$

for any $0 < \delta < 1/2$. By (6) and (9) we obtain that

$$(\log q - \log \delta)c_6 \log 2w + \delta c_4 rx > -\log r + c_2 x$$

and

$$c_7 \log q \log w > c_8 x,$$

if we choose $x_0$ and $\delta$ such that

$$c_2 - \delta c_4 r - \frac{\log r}{x} > 0,$$

i.e.

$$\delta < \frac{c_2 - \log r}{c_4 r}.$$

But by (10) and (8)

$$\log q \log w > c_9 q \log w$$

which implies that $q$ is bounded above.

**Proof of Theorem 2.** Using Lemma 2 with $a = G_y$, $b = G_x$ and $r_1 = 1$, the equation of the theorem can be transformed into the form

$$G_y q^{-1} G_x = d',$$

where $d$ is an integer. From this, by Theorem 1, our assertion follows if we choose the set $S$ such that $G_i \in S$ for any $0 < i < n_1$.

**References**


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