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ON A PROBLEM CONCERNING PERFECT POWERS IN LINEAR RECURRENCES

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Abstract: For a linear recurrence sequence $\{G_n\}_{n=0}^{\infty}$ of rational integers of order $k \geq 2$ satisfying some conditions, we show that the equation $sG_x^r = w^q$, where $w > 1$ and r are positive integers and s contains only given primes as its prime factors, implies the inequality $q < q_0$, where q_0 is an effective computable constant depending on the sequence, the prime factors of s and r.

Let $G = \{G_n\}_{n=0}^{\infty}$ be a linear recurrence sequence of order $k \geq 2$ defined by

$$G_n = A_1G_{n-1} + A_2G_{n-2} + \cdots + A_kG_{n-k} \quad (n \geq k),$$

where $A_1, \ldots, A_k$ are given rational integers with $A_k \neq 0$ and the initial values $G_0, G_1, \ldots, G_{k-1}$ are not all zero integers. We denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ the distinct roots of the polynomial

$$g(x) = x^k - A_1x^{k-1} - A_2x^{k-2} - \cdots - A_k,$$

furthermore we suppose that $|\alpha| > |\alpha_i|$ for $2 \leq i \leq s$, and the roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ have multiplicity $m_1 = 1, m_2, \ldots, m_s$. In this case $|\alpha| > 1$ and, as it is well known, the terms of G can be writen in the form

$$G_n = a\alpha^n + g_2(n)\alpha_2^n + \cdots + g_s(n)\alpha_s^n \quad (n \geq 0), \tag{1}$$

where g_i $(2 \leq i \leq s)$ is a polynomial of degree $m_i - 1$, furthermore a and the coefficients of g_i are elements of the algebraic number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_s)$.

Several authors investigated the perfect powers in the recurrences G. Among others T. N. Shorey and C. L. Stewart [6] proved that for a given integer $d(\neq 0)$ the equation

$$G_x = dw^q$$

with positive integers $x, w(> 1)$ and q implies the inequality $q < N$, where N is an effectively computable constant depending only on d and G. In [4] we gave an improvement of this result substituting d by integers containing only fixed prime factors. For second order recurrences ($k = 2$) A. Pethő obtained more strict

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results (e.g. see [5]). In [2] B. Brindza, K. Liptai and L. Szalay proved, under some conditions, that for recurrences G and H the equation

$$G_x H_y = w^q$$

can be satisfied only if q is bounded above. We proved [3] that for a sequence G and fixed positive integer n from

$$G_n^r G_x^{q-r} = w^q,$$

with $0 < r \le q/2$, it follows that q is bounded above.

In this note we prove the following theorems.

Theorem 1. *For given primes $p_1, \dots, p_t$ let S be a set of integers defined by*

$$S = \{n : n \in \mathbb{N},\ n = \prod_{i=1}^{t} p_i^{\beta_i},\ \beta_i \ge 0\}.$$

Let $r \ge 1$ be an integer and let G be a linear recurrence defined in (1) satisfying the conditions $a \ne 0$ and $G_n \ne a\alpha^n$ for $n \ge n_0$. Then the equation

$$sG_x^r = w^q \tag{2}$$

with positive integers $s \in S$, $w > 1$ and $x > x_0$ (x_0 depends on G and r) implies that $q < q_0$, where q_0 is an effectively computable constant depending on n_0, r, the sequence G and the primes $p_1, \dots, p_t$.

Theorem 2. *Let G be a linear recurrence defined by (1) satisfying the conditions $a \ne 0$ and $G_n \ne a\alpha^n$ for $n \ge n_0$. If*

$$G_y^{q-r} G_x^r = w^q$$

for positive integers x, y, q and r with the conditions $(q, r) = 1$ and $y < n_1$, then $q < q_1$, where q_1 is a constant depending on G, n_0 and n_1, but does not depend on r.

In the proofs we need some lemmas.

Lemma 1. *Let $\omega_1, \omega_2, \dots, \omega_v$ ($\omega_i \ne 0$ or 1) be algebraic numbers and let $\gamma_1, \gamma_2, \dots, \gamma_v$ be not all zero rational integers. Suppose that $\omega_1, \dots, \omega_v$ have heights $M_1, \dots, M_v (\ge 4)$, furthermore $|\gamma_i| \le B$ ($B \ge 4$) for $i = 1, 2, \dots v-1$ and $|\gamma_v| \le B'$. Further let*

$$\Lambda = \gamma_1 \log \omega_1 + \gamma_2 \log \omega_2 + \dots + \gamma_v \log \omega_v,$$

where the logarithms mean their principal values. If $\Lambda \neq 0$, then there exists an effectively computable constant $c > 0$, depending only on $v, M_1, \ldots, M_{v-1}$ and the degree of the field $\mathbb{Q}(\omega_1, \ldots, \omega_v)$, such that for any δ with $0 < \delta < 1/2$ we have

$$|\Lambda| > (\delta/B')^{c \log M_v} e^{-\delta B}.$$

(see A. Baker [1]).

Lemma 2. *Let a, b, c, q and r be positive integers with $0 < r < q$ and $(q, r) = 1$. If*

$$a^{q-r} b^r = c^q, \tag{3}$$

then for any integer r_1 with $0 < r_1 < q$ there is a positive integer d, such that

$$a^{q-r_1} b^{r_1} = d^q.$$

Proof of Lemma 2. From (3)

$$\left(\frac{b}{a}\right)^r = \left(\frac{c}{a}\right)^q$$

follows. Let x and y be integers for which $rx + qy = r_1$. Then

$$\left(\frac{b}{a}\right)^{rx} = \left(\frac{c}{a}\right)^{qx}$$

and

$$\left(\frac{b}{a}\right)^{rx} \left(\frac{b}{a}\right)^{qy} = \left(\frac{b}{a}\right)^{r_1} = \left(\frac{c}{a}\right)^{qx} \left(\frac{b}{a}\right)^{qy}$$

from which

$$\left(\frac{b}{a}\right)^{r_1} a^q = a^{q-r_1} b^{r_1} = \left(\frac{c^x b^y a}{a^{x+y}}\right)^q = d^q$$

follows, where d is an integer since $a^{q-r_1} b^{r_1}$ is integer.

Proof of Theorem 1. In the proof we denote by $c_1, c_2, \ldots$ effectively computable positive constants, which depend only on n_0, r, the sequence G and the primes $p_1, \ldots, p_t$. Suppose that equation (2) holds with the conditions given in the theorem. We can suppose that

$$s = p_1^{u_1} \cdots p_t^{u_t}, \text{ where } 0 \leq u_i < q \text{ for } 1 \leq i \leq t. \tag{4}$$

Namely if

$$s=\prod_{i=1}^{t} p_i^{u_i+qv_i}=\prod_{i=1}^{t} p_i^{u_i}\left(\prod_{i=1}^{t} p_i^{v_i}\right)^q,$$

then $w/\prod_{i=1}^{t} p_i^{v_i}$ is also an integer and greater than 1.

By (1), equation (2) can be written in the form

$$\lambda=\frac{w^q}{sa^r\alpha^{xr}}=\left(1+\frac{g_2(x)}{a}\left(\frac{\alpha_2}{\alpha}\right)^x+\cdots\right)^r, \tag{5}$$

where $|\lambda|\neq 1$ if $x>n_0$. Using the properties of the exponential and logarithm functions, by (5) and $|\alpha|>|\alpha_i|$ $(i=2,\dots,s)$

$$|\lambda|<\left|1+e^{-c_1x}\right|^r$$

and

$$|\log|\lambda||<re^{-c_2x}=e^{\log r-c_2x} \tag{6}$$

follows. On the other hand, by (5)

$$|\log|\lambda||=\left|q\log w-r\log|a|-xr\log|\alpha|-\sum_{i=1}^{t}u_i\log p_i\right|. \tag{7}$$

By (2) and (4) it follows that

$$G_x^r>\left(\frac{w}{\prod_{i=1}^{t} p_i}\right)^q,$$

where $w/\prod_{i=1}^{t} p_i>1$ is an integer since any prime factor of s divides w. From this inequality, using (1),

$$\log\left(|a|^r|\alpha|^{rx}\right)>c_3q\log w\left(1-\frac{\log\left(\prod_{i=1}^{t} p_i\right)}{q\log w}\right)$$

follows and so, if q is large enough,

$$q<c_4rx \text{ and } x>c_5q\log w. \tag{8}$$

Since $u_i < q < c_4 rx$, using Lemma 1 with $v \leq t+3$, $\omega_v = w$, $M_v = 2w$ (≥ 4), $B' = q$ and $B = c_4 xr$, from (7) we obtain the inequality

$$|\log|\lambda|| > (\delta/q)^{c_6 \log 2w} e^{-\delta c_4 rx} = e^{-(\log q - \log\delta) c_6 \log 2w - \delta c_4 rx} \tag{9}$$

for any $0 < \delta < 1/2$. By (6) and (9) we obtain that

$$(\log q - \log\delta) c_6 \log 2w + \delta c_4 rx > -\log r + c_2 x$$

and

$$c_7 \log q \log w > c_8 x, \tag{10}$$

if we choose x_0 and δ such that

$$c_2 - \delta c_4 r - \frac{\log r}{x} > 0,$$

i.e.

$$\delta < \frac{c_2 - \frac{\log r}{x}}{c_4 r}.$$

But by (10) and (8)

$$\log q \log w > c_9 q \log w$$

which implies that q is bounded above.

Proof of Theorem 2. Using Lemma 2 with $a = G_y$, $b = G_x$ and $r_1 = 1$, the equation of the theorem can be transformed into the form

$$G_y^{q-1} G_x = d^q,$$

where d is an integer. From this, by Theorem 1, our assertion follows if we choose the set S such that $G_i \in S$ for any $0 < i < n_1$.

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