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On a theorem of type Hardy–Littlewood with respect to the Vilenkin-like systems

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Abstract. In this paper we give a convergence test for generalized (by the author) Vilenkin–Fourier series. This convergence theorem is of type Hardy–Littlewood for the ordinary Vilenkin system is proved in 1954 by Yano.

Introduction and the result

First we introduce some necessary definitions and notations of the theory of the Vilenkin systems. The Vilenkin systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [7]). Let $m := (m_k, k \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, \dots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the m_k-th discrete cyclic group. Z_{m_k} can be represented by the set $\{0, \dots, m_k - 1\}$, where the group operation is the $\bmod\ m_k$ addition and every subset is open. The measure on Z_{m_k} is defined such that the measure of every singleton is $1/m_k$ $(k \in \mathbf{N})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

This gives that every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbf{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbf{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbf{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. The boundedness of the group G_m is supposed over all of this paper and denote by $\sup_{n\in\mathbf{N}} m_n < \infty$. c denotes an absolute constant (may depend only on $\sup_n m_n$) which may not be the same at different occurences.

A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

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for $x \in G_m,\ n \in \mathbf{P} := \mathbf{N} \setminus \{0\}$. Let $0 = (0, i \in \mathbf{N}) \in G_m$ denote the nullelement of G_m, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Let $\mathcal{I} := \{I_n(x) : x \in G_m,\ n \in \mathbf{N}\}$. The elements of $\mathcal{I}$ are called intervals on G_m.

Furthermore, let $L^p(G_m)$ $(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($|\,.\,|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbf{N})$ $(f \in L^1.)$

Let $M_0 := 1$, $M_{n+1} := m_n M_n$ $(n \in \mathbf{N})$ be the generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \dots, m_i - 1\},\ i \in \mathbf{N}),$$

where only a finite number of n_i's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi\imath \frac{x_n}{m_n}) \quad (x \in G_m,\ n \in \mathbf{N},\ \imath := \sqrt{-1}).$$

Then

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbf{N})$$

the nth Vilenkin function. The system $\psi := (\psi_n : n \in \mathbf{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j \pmod{m_j}) M_j \quad (k, n \in \mathbf{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n,\ \psi_n(x + y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1$ $(k, n \in \mathbf{N},\ x, y \in G_m)$.

Let functions $\alpha_n, \alpha_j^{(k)} : G_m \to \mathbf{C}$ $(n, j, k \in \mathbf{N})$ satisfy:

(i) $\alpha_j^{(k)}$ is measurable with respect to $\mathcal{A}_j$ $(j, k \in \mathbf{N})$,

(ii) $|\alpha_j^{(k)}| = \alpha_j^{(k)}(0) = \alpha_0^{(k)} = \alpha_j^{(0)} = 1$ $(j, k \in \mathbf{N})$,

(iii) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{(n^{(j)})},\ n^{(j)} := \sum_{i=j}^{\infty} n_i M_i$ $(n \in \mathbf{N})$.

Let $\chi_n := \psi_n \alpha_n$ $(n \in \mathbf{N})$. The system $\{\chi_n : n \in \mathbf{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system ([2]–[4]).

We mention some examples.

1. If $\alpha_j^{(k)} = 1$ for each $k, j \in \mathbf{N}$, then we have the "ordinary" Vilenkin systems.

2. If $m_j = 2$ for all $j \in \mathbf{N}$ and $\alpha_j^{(n^{(j)})} = (\beta_j)^{(n_j)}$, where

$$\beta_j(x) = \exp\left(2\pi\imath\left(\frac{x_{j-1}}{2^2} + \cdots + \frac{x_0}{2^{j+1}}\right)\right) \quad (n, j \in \mathbf{N},\ x \in G_m),$$

then we have the character system of the group of 2-adic integers (see e.g. [5], [4]).

3. If

$$t_n(x) := \exp\left(2\pi\imath\left(\sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}}\right)\sum_{j=0}^{\infty} x_j M_j\right) \quad (x \in G_m,\ n \in \mathbf{N}),$$

then we have a Vilenkin-like system which is usefull in the approximation theory of limit periodic, almost even arithmetical functions ([2], [4]).

In [3] we proved that a Vilenkin-like system is orthonormal and complete in $L^1(G_m)$. Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin-like system χ as follows.

$$\hat{f}^{\chi}(n) = \hat{f}(n) := \int_{G_m} f\bar{\chi}_n, \qquad S_n^{\chi} f = S_n f := \sum_{k=0}^{n-1} \hat{f}^{\chi}(k)\chi_k,$$

$$D_n^{\chi}(y, x) = D_n(y, x) := \sum_{k=0}^{n-1} \chi_n(y)\bar{\chi}_n(x),$$

It is known ([2]) that

$$D_{M_n}(y, x) = D_{M_n}(y - x) = \begin{cases} M_n, & \text{if } y - x \in I_n(0), \\ 0, & \text{if } y - x \notin I_n(0), \end{cases}$$

$$S_{M_n} f(y) = M_n \int_{I_n(y)} f\, d\mu = E_n f(y) \quad (f \in L^1(G_m),\ n \in \mathbf{N})$$

and

$$D_n(y, x) = \chi_n(y)\bar{\chi}_n(x) \sum_{j=0}^{\infty} D_{M_j}(y - x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(x)$$

$(x \in G_m,\ n \in \mathbf{N},\ f \in L^1(G_m))$. Then, $y - x \notin I_s$ gives

(1) $$|D_n(y,x)| \le cM_s \ (s \in \mathbf{N})$$

([2]). It is also known ([2]) that for $y - x \notin I_s$

(2) $$\sum_{t=0}^{M_s-1} \chi_{jM_s+t}(y)\bar{\chi}_{jM_s+t}(x) = 0 \ (j \in \mathbf{N}).$$

Moreover,

$$S_n^{\chi} f(y) = \int_{G_m} f(x) D_n(y,x)\, d\mu$$

$(n \in \mathbf{N},\ y \in G_m)$. For more details on Vilenkin-like systems see e.g. [2]–[4].

The following theorem of type Hardy–Littlewood for the ordinary Vilenkin system is proved in 1954 by Yano ([8]). We generalize this result for Vilenkin-like systems.

Theorem. *Suppose that the following two conditions hold for function $f \in L^1(G_m)$ and for a $y \in G_m$.*

(1) $M_n log M_n \int_{I_n} |f(x+y) - f(y)|\, d\mu(x) \to 0\ (n \to \infty)$,

(2) $|\hat{f}(k)| \le ck^{-\delta}$ *for some* $\delta > 0$.

Then $S_n f(y)$ converges to $f(y)$.

Proof. Denote by

(3) $$M_n \log M_n \int_{I_n} |f(x+y) - f(y)|\, d\mu(x) =: \varepsilon_n \to 0.$$

(3) implies that

(4) $$|S_{M_n} f(y) - f(y)| = M_n \left| \int_{I_n(y)} f(x) - f(y)\, d\mu(x) \right| \le \frac{\varepsilon_n}{\log M_n}$$

for $n \in \mathbf{N}$. Let $k \in \mathbf{N}$ and $n \in \mathbf{N}$ for which $M_n \le k < M_{n+1}$. Also, let $n \ge n_0 \in \mathbf{N}$ be some integer depend on n for which $r \le n/n_0$ that is the ratio of n and n_0 has a lower bound, where constant $r \in \mathbf{N}$ is discussed later.

$$\begin{aligned} S_k f(y) &= \int_{G_m} f(x) \sum_{j=0}^{k-1} \chi_j(y)\bar{\chi}_j(x)\, d\mu(x) \\ &= \int_{G_m} f(x+y) \sum_{j=0}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\, d\mu(x) \end{aligned}$$

and

$$\int_{G_m} f(y) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\,d\mu(x) = 0$$

gives

$$S_k f(y) - S_{M_n} f(y) = \int_{G_m} (f(x+y) - f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\,d\mu(x). \tag{5}$$

In (5) we integrate over G_m which is the disjoint union of I_n, $I_{n_0} \setminus I_n$ and $G_m \setminus I_{n_0}$. Since sequence m is bounded, then we have

$$\begin{aligned} &\left| \int_{I_n} (f(x+y) - f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\,d\mu(x) \right| \\ &\le (k - M_n) \int_{I_n} |f(x+y) - f(y)|\,d\mu(x) \le c\varepsilon_n / \log M_n. \end{aligned} \tag{6}$$

By (1) we have

$$\begin{aligned} &\left| \int_{I_{n_0} \setminus I_n} (f(x+y) - f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\,d\mu(x) \right| \\ &\le \sum_{s=n_0}^{n-1} cM_s \int_{I_s \setminus I_{s+1}} |f(x+y) - f(y)|\,d\mu(x) \le \sum_{s=n_0}^{n-1} \frac{c\varepsilon_s}{\log M_s}. \end{aligned} \tag{7}$$

Finally, we have $x \in G_m \setminus I_{n_0}$. This by (2) implies

$$\sum_{s=n_0}^{n} \sum_{j=0}^{k_s-1} \sum_{l=0}^{M_s-1} \bar{\chi}_{k^{(s+1)}+jM_s+l}(x+y)\chi_{k^{(s+1)}+jM_s+l}(y) = 0.$$

Denote by

$$J(x+y, y) := \sum_{s=0}^{n_0-1} \sum_{j=0}^{k_s-1} \sum_{l=0}^{M_s-1} \bar{\chi}_{k^{(s+1)}+jM_s+l}(x+y)\chi_{k^{(s+1)}+jM_s+l}(y).$$

Then,

$$
\begin{aligned}
&\left|\int_{G_m\setminus I_{n_0}} (f(x+y)-f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\, d\mu(x)\right| \\
&= \left|\int_{G_m\setminus I_{n_0}} (f(x+y)-f(y)) J(x+y,y)\, d\mu(x)\right| \\
&\le \left|\int_{I_{n_0}} (f(x+y)-f(y)) J(x+y,y)\, d\mu(x)\right| \\
&+ \left|\int_{G_m} (f(x+y)-f(y)) J(x+y,y)\, d\mu(x)\right| \\
&\le cM_{n_0} \int_{I_{n_0}} |f(x+y)-f(y)|\, d\mu(x) \\
&+ \left|\int_{G_m} f(x+y) J(x+y,y)\, d\mu(x)\right| \\
&\le c\varepsilon_{n_0}/\log M_{n_0} + \sum_{s=0}^{n_0-1}\sum_{j=0}^{k_s-1}\sum_{l=0}^{M_s-1} |\hat{f}(k^{(s+1)}+jM_s+l)| \\
&\le c\varepsilon_{n_0} + c^{n_0}2^{-\delta n} \le c\varepsilon_{n_0} + \left(\frac{c}{2^{\delta r}}\right)^{n_0}.
\end{aligned} \tag{8}
$$

At last by (4), (6), (7), (8), we get

$$
\begin{aligned}
&|S_k f(y) - S_{M_n} f(y)| \le |S_{M_n} f(y) - f(y)| \\
&+ \left|\int_{I_n} (f(x+y)-f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\, d\mu(x)\right| \\
&+ \left|\int_{I_{n_0}\setminus I_n} (f(x+y)-f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\, d\mu(x)\right| \\
&+ \left|\int_{G_m\setminus I_{n_0}} (f(x+y)-f(y)) \sum_{j=M_n}^{k-1} \chi_j(y)\bar{\chi}_j(x+y)\, d\mu(x)\right| \\
&\le c\frac{\varepsilon_n}{\log M_n} + c\varepsilon_n/\log M_n + \sum_{s=n_0}^{n-1} \frac{c\varepsilon_s}{\log M_s} + c\varepsilon_{n_0} + \left(\frac{c}{2^{\delta r}}\right)^{n_0} \\
&\le c\varepsilon_{n_0} + c\varepsilon_n + \sup_{s\ge n_0} \varepsilon_s (1/n_0 + \cdots + 1/n) + \left(\frac{\tilde{c}}{2^{\delta r}}\right)^{n_0} \to 0
\end{aligned}
$$

as $n \to \infty$, where constant $r \in \mathbf{N}$ is given as $\frac{\tilde{c}}{2^{\delta r}} < 1$ and $n_0 \to \infty$ (as $n \to \infty$) provided that $r \le n/n_0$. That is the proof of the theorem is complete.

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