A sieve for all primes of the form $x^2 + (x+1)^2$

PANAYIOTIS G. TSANGARIS

Abstract: All composite numbers of the form $x^2+(x+1)^2$ are determined in terms of suitable (non-homogeneous) linear recurrence sequences of order 2 (Theorem 4.12). As a consequence, all primes of the same form in a given interval can be determined by a sieving procedure (Theorem 4.13).

Introduction

The object of this study are the prime and composite numbers of the form $x^2 + (x + 1)^2$. Their study depends heavily on the following

Theorem 1.1. (SIERPINSKI) [3]) *The number $x^2+(x+1)^2$ is composite if and only if there exist natural numbers y, z such that:*

$$(T) \qquad\qquad T(x) = T(y) + T(z).$$

(Here $T(x), T(y), T(z)$ denote triangular numbers.)

The description of all composite numbers of the form $x^2 + (x + 1)^2$ is reduced to the study of the integral solutions of the following family of Diophantine equations of Fermat-Pell type:

$$(F_k) \qquad\qquad X^2 - 2Y^2 = 2k^2 - 1, \quad k = 0, 1, 2, \ldots.$$

Thus the study of equation (T) is reduced to the study of the family of equations (F_k) in terms of Gauss type transformations.

The detailed study of all solutions of (F_k) is carried on via Nagell's method of equivalence classes, thus avoiding any reference to fundamental units.

We will consider the Diophantine equation

$$(1.1) \qquad\qquad \xi^2 - d\eta^2 = -1 \quad (d \neq \square)$$

where $d \neq \square$ (non-square) is a natural number. The sequence of non-negative (that is $\xi_{2n+1} \geq 0$ and $\eta_{2n+1} \geq 0$) integral solutions of (1.1) is determined by the following recursive formulae:

$$(1.2) \quad \begin{aligned} \xi_{2n+3} &= 2x_1\xi_{2n+1} - \xi_{2n-1}, \text{ where } \xi_1 = \xi_1 \text{ and } \xi_3 = \xi_1^3 + 3d\xi_1\eta_1^2 \\ \eta_{2n+3} &= 2x_1\eta_{2n+1} - \eta_{2n-1}, \text{ where } \eta_1 = \eta_1 \text{ and } \eta_3 = 3\xi_1^2\eta_1 + d\eta_1^3, \end{aligned}$$

$(n = 1, 2, \ldots)$ where $\xi_1 + \eta_1\sqrt{d}$ is the fundamental solution of (1.1) and $x_1 + y_1\sqrt{d}$ is the fundamental solution of

$$(P) \qquad\qquad\qquad x^2 - dy^2 = 1 \quad (d \neq \square).$$

The following Theorems can be found in [5] (cf. also [4]).

Theorem 1.2. *Consider the Diophantine equation*

$$(F) \qquad\qquad\qquad X^2 - dY^2 = C. \quad (d \neq \square, \ C > 0).$$

Let $X_r^* + Y_r^*\sqrt{d}$ be the fundamental solution of a class A_r of integral solutions of (F) with $X_r^* > 0$ Let $x_n + y_n\sqrt{d}$, where $n = 0, 1, \ldots$, be the sequence of all non-negative integral solutions of (P). Let

$$X_n + Y_n\sqrt{d} \equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \text{ for all } n = 0, 1, \ldots,$$
$$X_n' + Y_n'\sqrt{d} \equiv (X_r^* - Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \text{ for all } n = 1, 2, \ldots$$

(for a typical r).

Then the following hold true:
(i) $Y_{n+1} > Y_n \geq 0$ for every $n = 0, 1, \ldots$.
(ii) Let $Y_r^* > 0$. Then $Y_{n+1}' \geq Y_n > Y_n' > 0$ for every $n = 1, 2, \ldots$.
(iii) Let $Y_r^* = 0$. Then $Y_n = Y_n'$ for every $n = 0, 1, \ldots$.
(iv) Let A_r be genuine ($=$ non-ambiguous). Then

$$Y_{n+1}' > Y_n > Y_n' > 0 \ \text{ for all } \ n = 1, 2, \ldots.$$

(v) Let A_r be ambiguous. Then for every m there exist n such that:

$$X_m' = X_n \ \text{ and } \ Y_m' = Y_n.$$

(vi) Let $X_r^* + Y_r^*\sqrt{d}$, where $r = 1, 2, \ldots, m$, be the only integral solutions of (F) such that

$$0 < X_r^* \leq \sqrt{(x_1 + 1)C/2} \ \text{ and } \ 0 \leq Y_r^* \leq y_1\sqrt{C}/\sqrt{2(x_1 + 1)}.$$

Then the set of all non-negative integral solutions of (F) consists of all pairs (X_n, Y_n) together with all pairs (X_n', Y_n') for all respective genuine classes A_r in addition to all pairs (X_n, Y_n) for all respective ambiguous classes

B_r. Moreover, X_n, Y_n, X_n' and Y_n' are determined by the following recursive formulae:

$$X_{n+1} = 2x_1 X_n - X_{n-1} \text{ for } n = 1, 2, \dots$$

(1.3) $\quad$ with $X_0 = X_r^*, \; X_1 = x_1 X_r^* + dy_1 Y_r^*$ and $r = 1, 2, \dots, m.$

$$Y_{n+1} = 2x_1 Y_n - Y_{n-1} \text{ for } n = 1, 2, \dots$$

$\quad$ with $Y_0 = Y_r^*, \; Y_1 = y_1 X_r^* + x_1 Y_r^*$ and $r = 1, 2, \dots, m.$

$$X_{n+1}' = 2x_1 X_n' - X_{n-1}' \text{ for } n = 1, 2, \dots$$

(1.4) $\quad$ with $X_0' = X_r^*, \; X_1' = x_1 X_r^* - dy_1 Y_r^*$ and $r = 1, 2, \dots, m.$

$$Y_{n+1}' = 2x_1 Y_n' - Y_{n-1}' \text{ for } n = 1, 2, \dots$$

$\quad$ with $Y_0' = -Y_r^*, \; Y_1' = y_1 X_r^* - x_1 Y_r^*$ and $r = 1, 2, \dots, m.$

Theorem 1.3. *Consider the Diophantine equation* (F), $C \neq 0$. *Let* $X_r^* + Y_r^* \sqrt{d}$ *be the fundamental solution of a class* A_r *of integral solutions of* (F). *Let* $x_1 + y_1 \sqrt{d}$ *be the fundamental solutions of* (P) *and*

$$X_n + Y_n \sqrt{d} \equiv (X_r^* + Y_r^* \sqrt{d})(x_1 + y_1 \sqrt{d})^n \equiv (X_r^* + Y_r^* \sqrt{d})(x_n + y_n \sqrt{d}),$$
$$X_n' + Y_n' \sqrt{d} \equiv (X_r^* - Y_r^* \sqrt{d})(x_1 + y_1 \sqrt{d})^n \quad \text{for all} \quad n = 0, 1, \dots.$$

Let $R_n \equiv Y_n^2 + k^2$ *and* $R_n' \equiv Y_n'^2 + k^2$, *where* k *is a fixed integer. Then the numbers* R_n *and* R_n' *are determined by the following recursive formulae:*

$$R_{n+1} = 2x_2 R_n - R_{n-1} - 2k^2(x_2 - 1) + 2y_1^2 C,$$

where $R_0 = Y_r^{*2} + k^2$ *and* $R_1 = (y_1 X_r^* + x_1 Y_r^*)^2 + k^2.$

$$R_{n+1}' = 2x_2 R_n' - R_{n-1}' - 2k^2(x_2 - 1) + 2y_1^2 C,$$

where $R_0' = Y_r^{*2} + k^2$ *and* $R_1' = (y_1 X_r^* - x_1 Y_r^*)^2 + k^2.$

2. Reduction of the Diophantine equation
$x(x + 1) = y(y + 1) + z(z + 1)$ to a family of Fermat equations

Theorem 2.1 below aims at reducing the problem of solving the Diophantine equation

(E) $\qquad\qquad\qquad x(x + 1) = y(y + 1) + z(z + 1)$

to that of solving each one of the Diophantine equations (F_k).

Theorem 1.3. *Consider the Diophantine equations (E) and (F_k). Then the following hold true:*

$(i)_1$ *Let (x, y, z) be an integral solution of (E) with $y \geq z$. Let*
$$X \equiv 2x + 1 \text{ and } Y \equiv 2y - (k - 1), \text{ where } k \equiv y - z.$$
Then $X + Y\sqrt{2}$ is an integral solution of (F_k).

$(i)_2$ *If $y \neq 0, -1$ and $z \neq 0, -1$ then $|Y| \neq k \pm 1$.*

$(ii)_1$ *Let $X + Y\sqrt{2}$ be an integral solution of (F_k). Let*

(2.1) $x = (X - 1)/2, \; y = (Y + k - 1)/2 \;$ and $\; z = (Y - k - 1)/2.$

Then (x, y, z) is an integral solution of (E).

$(ii)_2$ *If $|Y| \neq k \pm 1$, then $y \neq 0, -1$ and $z \neq 0, -1$.*

Proof. $(i)_1$ By direct computation.

$(i)_2$ Clear because $|Y| = k \pm 1$ implies $(y = 0, -1)$ or $(z = 0, -1)$.

$(ii)_1$ Let $X + Y\sqrt{2}$ be an integral solution of (F_k). Then it is easily proved by parity considerations that the numbers (2.1) are integers. Also

$$X = 2x + 1, \; Y = 2y - (k - 1) \text{ and } k = y - z,$$

whence (F_k) implies

$$(2x + 1)^2 - 2(2y - (y - z - 1))^2 = 2(y - z)^2 - 1,$$

that is

$$x(x + 1) = y(y + 1) + z(z + 1).$$

$(ii)_2$ Is proved in a way similar to the proof of $(i)_2$, namely $(y = 0, -1)$ or $(z = 0, -1)$ imply $|Y| = k \pm 1$.

Note. The transformation leading from (E) to (F_k) emanate from GAUSS (Art. 216 in [1])

3. Determination of all integral solutions of the equation
$$X^2 - 2Y^2 = 2k^2 - 1, \textbf{ where } k = 0, 1, \ldots$$

Proposition 3.1 is crucial for the location of the fundamental solutions of (F_k). Further, Theorem 3.4 characterizes the classes of solutions of (F_k), (as regards genuiness or ambiguity) in terms of their representing fundamental solutions. Special attention is given to the case of $2k^2 - 1$ being a square

number (cf. Theorem 3.5). The set of all non-negative solutions of (F_k) is determined recursively by Theorem 3.6 together with Corollary 3.7.

Proposition 3.1. *Consider the Diophantine equation (F_k) where k is a natural number. Let $X^* + Y^*\sqrt{2}$ be a solution of (F_k). Then $X^* + Y^*\sqrt{2}$ is the fundamental solution of a class of integral solutions of (F_k) if and only if the following (equivalent) inequalities are satisfied:*

(3.1) $$0 < |X^*| \leq 2k - 1,$$
(3.2) $$0 \leq Y^* \leq k - 1.$$

Proof. By using Theorem 109 in [2].

Note. The fundamental solution of (F_0) is $X^* + Y^*\sqrt{2} = 1 + \sqrt{2}$.

Proposition 3.2. *Let k be a natural number. Then $2k - 1 + (k-1)\sqrt{2}$ is the fundamental solution of a class of integral solutions of (F_k).*

Proof. Evident by Proposition 3.1.

Proposition 3.3. *Let A be a class of integral solutions of the Diophantine equation (F), $C \neq 0$. Let $X + Y\sqrt{d}$ be a representative of A and*

$$L = (-X^2 - dY^2)/C \quad \text{and} \quad M = -2XY/C.$$

Then the following hold true:
 (i) *A is a genuine if and only if at least one of the numbers L, M is not integral.*
 (ii) *A is ambiguous if and only if both numbers L and M are integral.*

Proof. Immediate by using Nagell's criterion (p. 205, [2]).

Theorem 3.4. *Let $X^* + Y^*\sqrt{2}$ be the fundamental solution of a class A of integral solutions of (F_k), where $k = 1, 2, \ldots$. Then the following hold true:*
 (i) *A is genuine if and only if $Y^* > 0$.*
 (ii) *A is ambiguous if and only if $Y^* = 0$.*

Proof. (i) (a) If A is genuine, then the previous Proposition 3.3 easily implies $Y^* > 0$.

(b) Let now $Y^* > 0$ and assume that A is ambiguous. Then, by the same Proposition, the numbers

$$L = (-X^{*^2} - 2Y^{*^2})/(2k^2 - 1) \quad \text{and} \quad M = -2X^*Y^*/(2k^2 - 1)$$

are integers. In particular, because L is an integer it follows that

$$(2k^2 - 1) \mid X^{*^2} + 2Y^{*^2} = 4Y^{*^2} + 2k^2 - 1.$$

Thus

$$(2k^2 - 1) \mid 4Y^{*^2}.$$

Also, $Y^* \leq \sqrt{(2k^2 - 1)/2}$, i.e.

$$4Y^{*^2} < 2(2k^2 - 1).$$

Hence

$$2k^2 - 1 < 4Y^{*^2} = h(2k^2 - 1) < 2(2k^2 - 1),$$

where h is a natural number. Hence $1 < h < 2$, which is impossible. Hence A is genuine.

(ii) Immediate by (i).

Note: (F_0) has only one class of integral solutions, which is ambiguous.

Theorem 3.5. *Let k be a natural number. Then the following are equivalent:*

(i) *$2k^2 - 1$ is a square number.*

(ii) *The totality of ambiguous classes of integral solutions of (F_k) consists of a single class.*

In consequence, if $2k^2 - 1$ is not a square number, then every class of integral solutions of (F_k) is genuine.

Proof. By using Proposition 3.1 and Theorem 3.4.

Theorem 3.6. *Consider the Diophantine equation (F_k), where k is a natural number. Let $x_n + y_n\sqrt{2}$, where $n = 0, 1, 2, \ldots$, be the sequence of all non-negative integral solutions of*

$$x^2 - 2y^2 = 1.$$

Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$), be the only integral solutions of (F_k) such that:

$$0 < X_r^* \leq 2k - 1 \quad \text{and} \quad 0 \leq Y_r^* \leq k - 1.$$

Let

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(x_n + y_n\sqrt{2}) \text{ for all } n = 0, 1, \ldots,$$
$$X_n' + Y_n'\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(x_n + y_n\sqrt{2}) \text{ for all } n = 1, 2, \ldots,$$

(for a typical r). Then the following hold true:

(i) *Let $Y_r^* > 0$ and $k \geq 2$. (Case of genuine classes of integral solutions of (F_k)). Then the pairs (X_n, Y_n) and (X'_n, Y'_n) are determined by (1.3) and (1.4) (for $x_1 = 3$, $y_1 = 2$ and $d = 2$).*

(ii) *Let $Y_r^* = 0$. (Case of ambiguous classes). Then the pairs (X_n, Y_n) are determined by (1.3).*

Moreover, in case (i) all pairs (X_n, Y_n) together with all pairs (X'_n, Y'_n) constitute the set of all non-negative integral solutions of (F_k) which belong to the class with typical fundamental solution $X_r^* + Y_r^*\sqrt{2}$. Also, in case (ii) all pairs (X_n, Y_n) constitute the set of all non-negative integral solutions of (F_k) which belong to the class with typical fundamental solution $X_r^* + 0\sqrt{2}$.

Proof. By using Theorems 3.4, 3.5, 1.2(vi) and Proposition 3.1.

Corollary 3.7. *The sequence of all positive integral solutions (X_n, Y_n) of (F_0) is determined by (1.2) (for $X_n \equiv \xi_{2n+1}, Y_n \equiv \eta_{2n+1}, \xi_1 = 1, \xi_3 = 7, \eta_1 = 1$ and $\eta_3 = 5$).*

4. Determination of all prime and composite numbers of the form $x^2 + (x + 1)^2$.

In Theorem 4.2 it is shown that every positive (integral) solution of (T) leads to a non-negative solution of a certain (F_k) and vice-versa. Theorems 4.6, 4.7 together with Corollary 4.8 determine all (F_k) whose non-negative solutions (taken together) lead to all positive solutions of (T).

In Theorem 1.1 a primality criterion is given for numbers of the form $N(x) = x^2+(x+1)^2$. Composite numbers of the form $N(x)$ are characterized (in terms of a suitable solution of (F_k)) in Theorem 4.9. The recursive determination of all composite numbers of the form $N(x)$ is given by Theorems 4.10, 4.11 and 4.12. This leads to our final Theorem 4.13, which constitutes an algorithm (sieve) for the determination of all primes of the form $N(x)$.

Lemma 4.1. *Let $X + Y\sqrt{2}$ be a non-negative integral solution of (F_k). Let*

$$x \equiv (X - 1)/2, \ y \equiv (Y + k - 1)/2 \ \text{ and } \ z \equiv (Y - k - 1)/2.$$

Then x, y, z are natural numbers if and only if $Y > k + 1$.

Proof. Easy and so omitted.

Theorem 4.2. *Consider the Diophantine equations (F_k) and (T). Then the following hold true:*

(i) Let $X + Y\sqrt{2}$ be a (non-negative) integral solution of (F_k), with $Y > k + 1$. Let

$$x \equiv (X - 1)/2, \; y \equiv (Y + k - 1)/2 \; \text{ and } \; z \equiv (Y - k - 1)/2.$$

Then (x, y, z) is a triad of positive integral solutions of (T).

(ii) Let (x, y, z) be a triad of positive integral solutions of (T) with $y \geq z$. Let

$$k \equiv y - z, \; X \equiv 2x + 1 \; \text{ and } \; Y \equiv 2y - (k - 1).$$

Then $X + Y\sqrt{2}$ is a (non-negative) integral solution of (F_k) with $Y > k+1$.

Proof. By using Theorem 2.1, Lemma 4.1 and the fact that the Diophantine equation (T) is equivalent to the equation (E).

Proposition 4.3. *Let k be a natural number. Let $X + Y\sqrt{2}$ be a non-negative integral solution of (F_k). Then the following hold true:*

(i) *Let $0 \leq Y \leq k - 1$. Then $X + Y\sqrt{2}$ is a fundamental solution of a class of integral solutions of (F_k).*

(ii) $Y \neq k$.

(iii) *Let $Y = k + 1$. Then $X = 2k + 1$. Moreover, $X + Y\sqrt{2} = (2k+1) + (k+1)\sqrt{2}$ is obtained from the fundamental solution $(X^* = 2k - 1, Y^* = k - 1)$ as follows:*

$$X + Y\sqrt{2} = (2k - 1 + (k - 1)\sqrt{2})(3 + 2\sqrt{2}) \text{ for } k = 1 \text{ and}$$
$$X + Y\sqrt{2} = (2k - 1 - (k - 1)\sqrt{2})(3 + 2\sqrt{2}) \text{ for } k > 1.$$

Proof. By direct computations.

Proposition 4.4. *Consider the Diophantine equation (F_k), where $k > 1$. Let $X^* + Y^*\sqrt{2}$ be the fundamental solution of a class A of (F_k) with $X^* > 0$. Let $3 + 2\sqrt{2}$ be the fundamental solution of the equation*

$$x^2 - 2y^2 = 1.$$

Let

$$Z_n \equiv X_n + Y_n\sqrt{2} \equiv (X^* + Y^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 0, 1, \dots, \text{ and}$$
$$Z_n' \equiv X_n' + Y_n'\sqrt{2} \equiv (X^* - Y^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 1, 2, \dots.$$

Then the following hold true:

(i) *Let A be genuine. Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k) which belong to A or to $\overline{A}$ and satisfy the inequality $Y > k + 1$ are the following:*

(a) $Z_n \in A$ and $Z'_n \in \overline{A}$ for all $n \geq 1$ if and only if $Y^* < k - 1$.

(b) $Z_n \in A$ for all $n \geq 1$ and $Z'_n \in \overline{A}$ for all $n \geq 2$ if and only if $Y^* = k - 1$.

(ii) Let A be ambiguous, (whence $Y^* = 0$, while $2k^2 - 1$ is a square number). Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k) which belong to A and satisfy the inequality $Y > k + 1$ are all Z_n for every $n \geq 1$.

Proof. (i) By Theorem 1.2 (iv) we have:

$$Y'_{n+1} > Y_n > Y'_n > 0 \ \text{ for all } n \geq 1, \text{ where } Y'_1 = 2X^* - 3Y^*.$$

(a) Hence, we have $Y'_1 = 2X^* - 3Y^* > k + 1$ if and only if $(2X^*)^2 > (3Y^* + k + 1)^2$, that is if and only if $(Y^* - (k - 1))(Y^* + 7k + 5) < 0$, and so if and only if $Y^* < k - 1$.

Consequently, by Proposition 4.3, the only (non-negative) integral solution $X + Y\sqrt{2}$ of (F_k), which belong to A or $\overline{A}$ and satisfy the inequality $Y > k + 1$ are all $Z_n \in A$ and all $Z'_n \in \overline{A}$, $n = 1, 2, \ldots$, for which $Y^* < k - 1$.

(b) Hence, $Y'_1 = 2X^* - 3Y^* = k + 1$ if and only if $Y^* = k - 1$.

Thus, the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k), which belong to A or $\overline{A}$ and satisfy the inequality $Y > k + 1$ are all $Z_n \in A$ for all $n \geq 1$ and all $Z'_n \in \overline{A}$ for all $n \geq 2$ if and only if $Y^* = k - 1$.

(ii) By Theorem 1.2 (i) the following hold true: $Y_{n+1} > Y_n \geq 0$ for all $n = 0, 1, \ldots$, while $Y^* = Y_0 = 0$ and $Y_1 = 2\sqrt{2k^2 - 1}$.

Also, (by direct computations) we show that $Y_1 > k + 1$. Consequently, the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k), which belong to A and satisfy the inequality $Y > k + 1$ are all Z_n for every $n \geq 1$.

Proposition 4.5. *Consider the Diophantine equation (F_1). Let*

$$X_n + Y_n\sqrt{2} \equiv (1 + 0\sqrt{2})(3 + 2\sqrt{2})^n \ \text{ for all } n = 0, 1, \ldots.$$

Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_1), such that $Y > 2$, are all $X_n + Y_n\sqrt{2}$ for every $n \geq 2$.

Proof. By using Theorem 1.2 (i).

Theorem 4.6. *Let k be a natural number. Consider the Diophantine equation (F_k). Let $Z_r^* \equiv X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$) be the only integral solutions of (F_k) such that:*

$$X_r^* > 0 \ \text{ and } \ 0 \leq Y_r^* \leq k - 1.$$

Let A_r be the corresponding classes of integral solutions of (F_k) with fundamental solutions Z_r^*. Let

$$Z_n \equiv X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 0, 1, \ldots,$$
$$Z_n' \equiv X_n' + Y_n'\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 1, 2, \ldots.$$

for an (arbitrary) typical r. Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k), which satisfy the inequality $Y > k + 1$, are the following:
  (i) All $Z_n \in A_r$ and all $Z_n' \in \overline{A}_r$ for every $n \geq 1$ if and only if $0 < Y_r^* < k - 1$.
 (ii) All $Z_n \in A_r$ for every $n \geq 1$ and all $Z_n' \in \overline{A}_r$ for every $n \geq 2$ if and only if $0 < Y^* = k - 1$.
(iii) All $Z_n \in A_r$ for every $n \geq 1$ if and only if $Y_r^* = 0$ for $k \geq 2$.
(iv) All $Z_n \in A_r$ for every $n \geq 2$ if and only if $Y_r^* = 0$ for $k = 1$.

Proof. By using Propositions 4.4, 4.5 and Theorem 3.6.

Theorem 4.7. *Let k be a natural number. Consider the Diophantine equation (F_k). Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$) be the only integral solutions of (F_k) such that:*

$$X_r^* > 0 \text{ and } 0 \leq Y_r^* \leq k - 1.$$

Let

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 0, 1, \ldots \text{ and } r = 1, 2, \ldots, m,$$
$$X_n' + Y_n'\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n \text{ for all } n = 1, 2, \ldots \text{ and } r = 1, 2, \ldots, m.$$

Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k) such that $Y > k + 1$ are the following:
  (i) *All $X_n + Y_n\sqrt{2}$ and all $X_n' + Y_n'\sqrt{2}$ (with $n \geq 1$) for every Y_r^* with $0 < Y_r^* < k - 1$, when $k \geq 2$.*
 (ii) *All $X_n + Y_n\sqrt{2}$ (with $n \geq 1$) and all $X_n' + Y_n'\sqrt{2}$ (with $n \geq 2$) for $0 < Y_r^* = k - 1$, when $k \geq 2$.*
(iii) *All $X_n + Y_n\sqrt{2}$ (with $n \geq 1$) for $Y_r^* = 0$, when $k \geq 2$.*
(iv) *All $X_n + Y_n\sqrt{2}$ (with $n \geq 2$) for $Y_r^* = 0$, when $k = 1$.*

Proof. By using Theorems 3.6 and 4.6.

By Corollary 3.7 it follows that

Corollary 4.8. *The only non-negative integral solutions $X + Y\sqrt{2}$ of (F_0) such that $Y > 1$ are:*

$$X_n + Y_n\sqrt{2} \text{ for every } n = 1, 2, \ldots.$$

Theorem 4.9. *Consider the Diophantine equation* (F_k), $k = 0, 1, \ldots$. *Let* $X + Y\sqrt{2}$ *be a non-negative integral solution of* (F_k). *Let* $x \equiv (X-1)/2$ *and* $N(x) \equiv x^2 + (x+1)^2$. *Then* $N(x) = Y^2 + k^2$. *Moreover, the following are equivalent:*

(i) $N(x)$ *is composite.*

(ii) $Y > k + 1$.

Proof. The equality $N(x) = Y^2 + k^2$ follows by direct computations, while the equivalence of (i) and (ii) follows from Theorems 4.2 and 1.1.

Theorem 4.10. *Let* $N(x) \equiv x^2 + (x+1)^2$. *Consider the Diophantine equation* (F_k), $k = 0, 1, \ldots$. *Let* $X_r^* + Y_r^*\sqrt{2}$, *(where* $r = 1, 2, \ldots, m$) *be the only non-negative integral solutions of* (F_k) *such that:*

$$0 \leq Y_r^* \leq k - 1 \quad \text{for } k \geq 1,$$

while, for $k = 0$ *we have:* $X_r^* = Y_r^* = 1$ *for all* $r = 1, 2, \ldots, m$. *Let*

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n,$$
$$X_n' + Y_n'\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3 + 2\sqrt{2})^n \quad \text{for all } n = 0, 1, \ldots,$$

(for a typical r). *Let* $\tilde{x}_n \equiv (X_n - 1)/2$ *and* $\tilde{x}_n' \equiv (X_n' - 1)/2$ *for every* $n = 0, 1, \ldots$. *Let* R_n, R_n', *where* $n = 0, 1, \ldots$, *be the sequences defined by the recursive formmulae:*

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0 = Y_r^{*2} + k^2$, $R_1 = (2X_r^* + 3Y_r^*)^2 + k^2$ *(for a typical* r).

$$R_{n+1}' = 34R_n' - R_{n-1}' - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0' = Y_r^{*2} + k^2$, $R_1' = (2X_r^* - 3Y_r^*)^2 + k^2$ *(for a typical* r).

Then the following hold true:

(i) *Let* $k = 0$. *The for every integer* n *there exists an integer* m *such that:*

$$R_n = R_m' = N(\tilde{x}_n) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots$, *are all composite.*

(ii) *Let* $k = 1$, *whence* $X_r^* = 1$, $Y_r^* = 0$ *for every* $r = 1, 2, \ldots, m$. *Then*

$$R_n = R_n' = N(\tilde{x}_n) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_2, R_3, \ldots,$ are all composite.

(iii) Let $k \geq 2$ and $Y_r^* = 0$ Then

$$R_n = R_n^{'} = N(\tilde{x}_n) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots,$ are all composite.

(iv) Let $k \geq 2$ and $Y_r^* = k - 1$. Then

$$R_n = N(\tilde{x}_n) \quad \text{and} \quad R_n^{'} = N(\tilde{x}_n^{'}) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots,$ and also the numbers $R_2^{'}, R_3^{'}, \ldots,$ are all composite.

(v) Let $k \geq 2$ and $0 < Y_r^* < k - 1$. Then

$$R_n = N(\tilde{x}_n) \quad \text{and} \quad R_n^{'} = N(\tilde{x}_n^{'}) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots,$ and also the numbers $R_1^{'}, R_2^{'}, \ldots,$ are all composite.

Note: For the cases (iv) and (v) we have:

$$R_m \neq R_n^{'} \quad \text{for any } m, n.$$

Proof. (i) The unique class of integral solutions of (F_0) is ambiguous. By Theorem 2.4 in [5] and Corollary 4.8 we have:

$$X_n + Y_n \sqrt{2} \equiv \xi_{2n+1} + \eta_{2n+1} \sqrt{2} = (1 + \sqrt{2})(x_n + y_n \sqrt{2}) = (1 + \sqrt{2})^{2n+1}$$

for all $n = 0, 1, \ldots.$

Hence, by the definition of ambiguous class and Theorem 1.3, for every integer n there exists an integer m such that:

$$R_n = R_m^{'} = N(\tilde{x}_n), \quad \text{where } \tilde{x}_n = (\xi_{2n+1} - 1)/2.$$

According to Corollary 4.8, the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_0) such that $Y > 1$ are all $Y_{n+1} = \eta_{2n+3}$ for every $n \geq 0$. Hence by Theorem 4.9, the numbers $R_1, R_2, \ldots$ are all composite.

(ii) Obviously $X_r^* = 1$, $Y_r^* = 0$ for every $r = 1, 2, \ldots, m$ because $k = 1$. Hence, $R_n = R_n^{'}$ for all $n = 0, 1, \ldots.$ Now, Theorem 1.3 implies

$$R_n = N(\tilde{x}_n) = Y_n^2 + k^2 = Y_n^2 + 1 \quad \text{for all } n \geq 0.$$

Also, by Theorem 4.7 (iv), we deduce that $X_{n+1}+Y_{n+1}\sqrt{2}$, where $n \geq 1$, are the only (non-negative) integral solutions of (F_1) such that $Y_{n+1} > k+1 = 2$. Hence, according to Theorem 4.9, the numbers $R_2, R_3, \ldots$ are all composite.

(iii) We have $R_n = R'_n$ for every $n = 0, 1, \ldots$ because $Y^*_r = 0$. By Theorem 4.7 (iii) the numbers $X_{n+1} + Y_{n+1}\sqrt{2}$, where $n \geq 0$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k+1$. This completes the proof by invoking Theorems 1.3 and 4.9.

(iv) By Theorem 4.7 (ii) the numbers $X_{n+1}+Y_{n+1}\sqrt{2}$ with $n \geq 0$, together with the numbers $X'_{n+1} + Y'_{n+1}\sqrt{2}$, with $n \geq 1$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k+1$ and $Y'_{n+1} > k+1$. Thus the proof is completed by Theorem 1.3 and 4.9.

(v) By Theorem 4.7 (i), the numbers $X_{n+1} + Y_{n+1}\sqrt{2}$ together with the numbers $X'_{n+1} + Y'_{n+1}\sqrt{2}$, where $n \geq 0$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k + 1$ and $Y'_{n+1} > k + 1$. This finishes the proof of the whole Theorem, again in view of Theorems 1.3 and 4.9.

Theorem 4.11. *Consider the Diophantine equation (F_k), $k = 0, 1, \ldots$. Let $X^*_r + Y^*_r\sqrt{2}$, (where $r = 1, 2, \ldots, m$) be the only non-negative integral solutions of (F_k) such that:*

$$0 \leq Y^*_r \leq k - 1 \quad \text{for } k \geq 1,$$

While, for $k = 0$ we have: $X^*_r = Y^*_r = 1$ for all $r = 1, 2, \ldots, m$. Let R_n, R'_n be the sequences, defined by the recursive formulae:

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0 = Y^{*^2}_r + k^2$, $R_1 = (2X^*_r + 3Y^*_r)^2 + k^2$ (for a typical r).

$$R'_{n+1} = 34R'_n - R'_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R'_0 = Y^{*^2}_r + k^2$, $R'_1 = (2X^*_r - 3Y^*_r)^2 + k^2$ (for a typical r).

Suppose that the number $N(x) \equiv x^2 + (x + 1)^2$ is composite. Then $N(x)$ is equal to some of the composite numbers R_n or R'_n, for a suitable index, as stated in cases (i)–(v) of Theorem 4.10 (for some value of k).

Proof. Since $N(x)$ is composite it follows from Theorem 1.1 that there exist natural numbers y, z such that

$$T(x) = T(y) + T(z).$$

Let $y \geq z$. Let also $k \equiv y - z$, $X \equiv 2x + 1$ and $Y \equiv 2y - (k - 1)$. Then, according to Theorem 4.2 (ii), $X + Y\sqrt{2}$ is a (non-negative) integral solution of (F_k), with $Y > k + 1$. Hence, $X + Y\sqrt{2}$ is a solution of type (i) or (ii) or (iii) or (iv) of Theorem 4.7 or it is a solution $X + Y\sqrt{2}$ of (F_0) with $Y > 1$ (see Corollary 4.8). Also, $N(x) = Y^2 + k^2$. Hence, by Theorem 1.3 $N(x)$ is equal to some R_n or some R_n'. Finally, the appropriate index n for which $N(x) = R_n$ or $N(x) = R_n'$ is obtained by applying Theorem 4.6 to the respective case as in (i)-(v) of Theorem 4.10. This ends the proof of the Theorem.

Theorem 4.12. (*Determination of all composites of the form* $N(x) \equiv x^2 + (x + 1)^2$) *Consider the Diophantine equations*

$$(F_k) \qquad\qquad X^2 - 2Y^2 = 2k^2 - 1, \quad where \ k = 0, 1, \dots.$$

Let $X_r^* + Y_r^*\sqrt{2}$, (*where* $r = 1, 2, \dots, m$), *be the only non-negative integral solutions of* (F_k) *such that:*

$$0 \leq Y_r^* \leq k - 1 \ \ for \ k \geq 1,$$

While, for $k = 0$ *we have:* $X_r^* = Y_r^* = 1$ *for all* $r = 1, 2, \dots, m$. *Let* R_n, R_n' *be the sequences defined by the recursive formulae:*

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \ \ for \ all \ n = 1, 2, \dots,$$

where $R_0 = Y_r^{*2} + k^2$, $R_1 = (2X_r^* + 3Y_r^*)^2 + k^2$ (*for a typical* r).

$$R_{n+1}' = 34R_n' - R_{n-1}' - 8(2k^2 + 1) \ \ for \ all \ n = 1, 2, \dots,$$

where $R_0' = Y_r^{*2} + k^2$, $R_1' = (2X_r^* - 3Y_r^*)^2 + k^2$ (*for a typical* r).
    Then, the only composite numbers of the form $N(x) \equiv x^2 + (x + 1)^2$ *are the following:*
  (i) $R_1, R_2, \dots$ (*for* $k = 0$).
 (ii) $R_2, R_3, \dots$ (*for* $k = 1$ *and* $Y_r^* = 0$).
(iii) $R_1, R_2, \dots$ (*for* $k \geq 2$ *and* $Y_r^* = 0$).
(iv) $R_1, R_2, \dots$ *together with* $R_2', R_3', \dots$ (*for* $k \geq 2$ *and* $Y_r^* = k - 1$).
 (v) $R_1, R_2, \dots$ *together with* $R_1', R_2', \dots$ (*for* $k \geq 2$ *and for all* Y_r^* *such that* $0 < Y_r^* < k - 1$).

Proof. By using Theorems 4.10 and 4.11.

Theorem 4.13. (*Sieve-algorithm for the determination of all primes of the form* $N(x) \equiv x^2 + (x + 1)^2$ *in an Interval* [5,M], *where M is a (positive) integer*)

Step 1: Determine all numbers $N(x)$ for $x = 1, 2, \ldots, \left[(-1 + \sqrt{2M-1})/2\right]$.

Step 2: Determine all R_n and R'_n, as in Theorem 4.12 obtained from the Diophantine equations

$$X^2 - 2Y^2 = 2k^2 - 1, \quad \text{where } k = 0, 1, \ldots, \left[\sqrt{M}\right].$$

Step 3: Delete from the table of the numbers in Step 1, all numbers of Step 2. The remaining numbers are the only prime numbers of the form $N(x)$ in the interval [5,M].

Proof. By using Theorem 4.12.

References

[1] C. F. GAUSS, *Disqusitiones Arithmeticae*, (English transl. A. A. Clarke, Yale University Press, New Haven, Connecticut, London, 1966.)

[2] T. NAGELL, *Introduction to Number Theory*, Chelsea, New York, 1964.

[3] W. SIERPINSKI, Sur les nombres triangularies qui sont sommes de deux nombres triangularies, *Elem. Math.*, **17** (1962), 63–65.

[4] P. G. TSANGARIS, Prime Numbers and Cyclotomy-Primes of the form $x^2+(x+1)^2$, Ph.D. Thesis, Athens University, Athens, 1984 (in Greek).

[5] P. G. TSANGARIS, Fermat–Pell Equation and the Numbers of the form $w^2+(w+1)^2$, *Publ. Math. Debrecen*, **47** (1995), 127–138.

PANAYIOTIS G. TSANGARIS

DEPARTMENT OF MATHEMATICS, ATHENS UNIVERSITY

PANEPISTIMIOPOLIS, 157 84 ATHENS, GREECE.