

The asymptotic behavior of the real roots of Fibonacci-like polynomials

FERENC MÁTYÁS*

Abstract. The Fibonacci-like polynomials $G_n(x)$ are defined by the recursive formula $G_n(x) = xG_{n-1}(x) + G_{n-2}(x)$ for $n \geq 2$, where $G_0(x)$ and $G_1(x)$ are given seed-polynomials. In this paper the non-zero accumulation points of the set of the real roots of Fibonacci-like polynomials are determined if either both of the seed-polynomials are constants or $G_0(x) = -a$ and $G_1(x) = x \pm a$ ($a \in \mathbf{R} \setminus \{0\}$). The theorems generalize the results of G. A. Moore and H. Prodinger who investigated this problem if $G_0(x) = -1$ and $G_1(x) = x - 1$, furthermore we extend a result of Hongquan Yu, Yi Wang and Mingfeng He.

Introduction

The Fibonacci-like polynomials $G_n(x)$ are defined by the following manner. For $n \geq 2$

$$(1) \quad G_n(x) = xG_{n-1}(x) + G_{n-2}(x),$$

where $G_0(x)$ and $G_1(x)$ are fixed polynomials (so-called seed-polynomials) with real coefficients. If it is necessary to denote the seed-polynomials, then we will use the notation $G_n(x) = G_n(G_0(x), G_1(x), x)$, too. The polynomials $G_n(0, 1, x)$ are the original Fibonacci polynomials and the numbers $G_n(0, 1, 1)$ are the well-known Fibonacci numbers.

Recently, G. A. Moore [5] investigated the maximal real roots g'_n of the polynomials $G_n(-1, x - 1, x)$ and proved that g'_n exists for every $n \geq 1$ and $\lim_{n \rightarrow \infty} g'_n = 3/2$. (These numbers g'_n are called as “golden numbers”.) H. Prodinger [6] gave the asymptotic formula $g'_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n}$. Hongquan Yu, Yi Wang and Mingfeng He [3] investigated the limit of the maximal real roots g'_n of polynomials $G_n(-a, x - a, x)$ if $a \in \mathbf{R}^+$.

For brevity let us introduce the following notations. B denotes the set of the real roots of polynomials $G_n(x)$ ($n = 0, 1, 2, \dots$) and A denotes the set of the the accumulation points of set B . In [4] we investigated these sets.

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Although, the main result of [4] is formulated for seed-polynomials with integer coefficients but it is true for seed-polynomials with real coefficients, too. Since we are going to apply it, therefore we cite it as a lemma.

Lemma 1. *Let $G_0(x)$ and $G_1(x)$ be two fixed polynomials with real coefficients, $G_0(0) \cdot G_1(0) \neq 0$ and $x_0 \in \mathbf{R}$. $x_0 \in A$ if and only if one of the following conditions holds:*

$$(i) \quad -\frac{G_1(x_0)}{G_0(x_0)} = \frac{1}{\alpha(x_0)} \text{ and } x_0 > 0;$$

$$(ii) \quad -\frac{G_1(x_0)}{G_0(x_0)} = \frac{1}{\beta(x_0)} \text{ and } x_0 < 0;$$

$$(iii) \quad x_0 = 0,$$

where

$$(2) \quad \alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2}.$$

The purpose of this paper is to investigate the asymptotic behavior of the elements of the set B in the cases of simple seed-polynomials. In our discussion we are going to use the following explicit formulae for the polynomial $G_n(x) = G_n(G_0(x), G_1(x), x)$. It is known that

$$(3) \quad G_n(x) = p(x)\alpha^n(x) - q(x)\beta^n(x)$$

for $n \geq 0$, where $\alpha(x)$ and $\beta(x)$ are defined in (2), while

$$p(x) = \frac{G_1(x) - \beta(x)G_0(x)}{\alpha(x) - \beta(x)} \quad \text{and} \quad q(x) = \frac{G_1(x) - \alpha(x)G_0(x)}{\alpha(x) - \beta(x)}.$$

These formulae can be obtained by standard methods or see in [2].

Since we want to investigate the roots of the polynomials $G_n(x)$, therefore it is worth rephrasing the expression $G_n(x) = 0$ as

$$\frac{p(x)}{q(x)} = \left(\frac{\beta(x)}{\alpha(x)} \right)^n,$$

that is

$$(4) \quad \frac{G_1(x) - \beta(x)G_0(x)}{G_1(x) - \alpha(x)G_0(x)} = \left(\frac{x - \sqrt{x^2 + 4}}{x + \sqrt{x^2 + 4}} \right)^n.$$

Let us consider the polynomial $G_n(G_0(x), G_1(x), x)$. It is obvious that $G_n(0, 0, x)$ is identical to the zero polynomial for every $n \geq 0$. Using (3)

the identities $G_n(0, G_1(x), x) = G_1(x) \cdot G_n(0, 1, x)$ and $G_n(G_0(x), 0, x) = G_0(x) \cdot G_n(1, 0, x)$ yield. But it is known from [2] and can be obtained easily from (4) that neither the Fibonacci polynomials $G_n(0, 1, x)$ nor the polynomials $G_n(1, 0, x)$ have real root x' except $x' = 0$ if n is even or odd, respectively. Therefore investigating the asymptotic behavior of the roots of polynomials $G_n(G_0(x), G_1(x), x)$ we can assume that the seed-polynomials differ from the zero polynomial and at least one of them is a monic polynomial (since one can simplify the left-hand side of (4) with the leading coefficient of the polynomial $G_1(x)$ or $G_0(x)$).

Theorems and Proofs

First of all we need the following lemma, which deals with the properties of the functions $\alpha(x)$ and $\beta(x)$ defined in (2).

Lemma 2. (a) *On the interval $[0, \infty)$ the function $\frac{1}{\alpha(x)}$ is continuous and strictly monotonically decreasing, its graph is convex and $1 \geq \frac{1}{\alpha(x)} > 0$.*

(b) *On the interval $(-\infty, 0]$ the function $\frac{1}{\beta(x)}$ is continuous and strictly monotonically decreasing, its graph is concave and $0 > \frac{1}{\beta(x)} \geq -1$.*

Proof. By (2) it is obvious that the functions $\frac{1}{\alpha(x)}$ and $\frac{1}{\beta(x)}$ are continuous on the above mentioned intervals. The rest of the statement can be proved easily using the methods of differential calculus.

Further on we deal with the set A if $G_0(x) = 1$ and $G_1(x) = a$. In this case, using Lemma 1, the set A can be determined in a very simple manner.

Theorem 1. *Let $a \in \mathbf{R} \setminus \{0\}$ and $G_n(1, a, x)$ be Fibonacci-like polynomials. If $0 < |a| < 1$ then $A \setminus \{0\} = \left\{ \frac{a^2-1}{a} \right\}$, while in the case $|a| \geq 1$ $A \setminus \{0\} = \emptyset$.*

Proof. According to Lemma 1 to get the elements of the set $A \setminus \{0\}$ we have to solve the equations

$$(5) \quad -a = \frac{2}{x + \sqrt{x^2 + 4}} \quad \text{for } x > 0$$

and

$$(6) \quad -a = \frac{2}{x - \sqrt{x^2 + 4}} \quad \text{for } x < 0.$$

By Lemma 2 the functions $\frac{1}{\alpha(x)} = \frac{2}{x + \sqrt{x^2 + 4}}$ and $\frac{1}{\beta(x)} = \frac{2}{x - \sqrt{x^2 + 4}}$ are continuous, $1 > \frac{1}{\alpha(x)} > 0$ for any $x > 0$ and $0 > \frac{1}{\beta(x)} > -1$ for any $x < 0$, therefore $0 < |a| < 1$ is a necessary and sufficient condition for the solvability of (5) and (6). Solving (5) and (6) we get that the single real root x_0 is $x_0 = \frac{a^2 - 1}{a}$, where $x_0 > 0$ if $-1 < a < 0$ and $x_0 < 0$ if $0 < a < 1$. This completes the proof.

In the following theorems we prove asymptotic formulae for those real roots g_n of the polynomials $G_n(-a, x \pm a, x)$ which do not tend to 0 if n tends to infinity.

Theorem 2. *Let $G_0(x) = -a$ and $G_1(x) = x - a$, where $a \in \mathbf{R} \setminus \{0\}$. If either $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{ \frac{a(a+2)}{a+1} \right\}$, while in the case $-2 \leq a < 0$ we have $A \setminus \{0\} = \emptyset$. Furthermore for large n*

$$g_n \sim \frac{a(a+2)}{a+1} + (-1)^n \frac{a(a^2 + 2a + 2)^2}{(a+1)^2(a+2)} (a+1)^{-2n}.$$

Proof. According to Lemma 1, $x_0 \in A \setminus \{0\}$ if and only if

$$(7) \quad \frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0$$

or

$$(8) \quad \frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0$$

holds. Using the statements of Lemma 2 one can verify that (7) has a solution for x_0 if and only if $a > 0$, while (8) has a solution for x_0 if and only if $a < -2$. Solving (7) and (8) we get that

$$x_0 = \frac{a(a+2)}{a+1}.$$

To determine the asymptotic behavior of g_n we apply (4), which in our case has the following form

$$\frac{2(g_n - a) + a \left(g_n - \sqrt{g_n^2 + 4} \right)}{2(g_n - a) + a \left(g_n + \sqrt{g_n^2 + 4} \right)} = \left(\frac{g_n - \sqrt{g_n^2 + 4}}{g_n + \sqrt{g_n^2 + 4}} \right)^n.$$

This will be much nicer when we substitute

$$(9) \quad g_n = u - \frac{1}{u}.$$

Without loss of generality we can assume that $u > 0$ and we get the equality

$$(10) \quad \frac{(au + u + 1)(u - 1)}{(a + 1 - u)(u + 1)} = -(-u^2)^n.$$

Since $x_0 = u - \frac{1}{u}$ holds for $u = a + 1$ and $u = -\frac{1}{a+1}$ therefore it is plain to see that, for large n , (9) can only hold if u is either close to $a + 1$ or $-\frac{1}{a+1}$. In both cases this would mean that g_n is close to x_0 .

Let us assume that u is close to $a + 1$ and so $a > 0$ because of $u > 0$. It is clear from (10) that the cases when n is even or odd have to be distinguished.

We start with $n = 2m$ and rewrite (10) as

$$(11) \quad a + 1 - u = -\frac{(au + u + 1)(u - 1)}{u + 1} \cdot u^{-4m}.$$

We get the asymptotic behavior by a process known as “bootstrapping” which is explained in [1]. First we insert $u = a + 1 + \delta_1$ into the left-hand side of (11) and $u = a + 1$ into the right-hand side of (11). So we get an approximation for δ_1 . Then we insert $u = a + 1 + \delta_1 + \delta_2$ into the left-hand side of (11) and $u = a + 1 + \delta_1$ into the right-hand side of (11) and get an approximation for δ_2 . This procedure can be repeated to get better and better estimations for u . Now we determine only the number δ_1 . From (11) we have

$$\delta_1 \sim \frac{a(a^2 + 2a + 2)}{a + 2}(a + 1)^{-4m}$$

and so

$$u = a + 1 + \delta_1 \sim a + 1 + \frac{a(a^2 + 2a + 2)}{a + 2}(a + 1)^{-4m}.$$

Substituting u into (9) we get that

$$(12) \quad g_{2m} = a + 1 + \delta_1 - \frac{1}{a + 1 + \delta_1} \sim \frac{a(a + 2)}{a + 1} + \frac{(a(a^2 + 2a + 2))^2}{(a + 1)^2(a + 2)}(a + 1)^{-4m}.$$

If $n = 2m + 1$ then (10) can be rewrite as

$$a + 1 - u = \frac{(au + u + 1)(u - 1)}{u^2(u + 1)}u^{-4m}.$$

Using the “bootstrapping” method for $u = a + 1 + \delta'_1$ we get the estimation

$$\delta'_1 \sim \frac{a(a^2 + 2a + 2)}{(a + 2)(a + 1)^2} (a + 1)^{-4m},$$

which implies the following form:

$$(13) \quad \begin{aligned} g_{2m+1} &= a + 1 + \delta'_1 - \frac{1}{a + 1 + \delta'_1} \\ &\sim \frac{a(a + 2)}{a + 1} - \frac{a(a^2 + 2a + 2)^2}{(a + 2)(a + 1)^4} (a + 1)^{-4m}. \end{aligned}$$

Comparing (12) and (13) the desired approximation yields since $a > 0$.

One can verify in the same manner that the estimation for g_n also holds when $a < -2$. This completes the proof.

Remark. From our proof one can see that for large n $g_n = g'_n$ if $a > 0$ while g_n is the minimal real root if $a < -2$.

A similar result can be proved for the polynomials $G_n(-a, x + a, x)$.

Theorem 3. *Let $G_0(x) = -a$ and $G_1(x) = x + a$ where $a \in \mathbf{R} \setminus \{0\}$. If either $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{ -\frac{a(a+2)}{a+1} \right\}$, while $A \setminus \{0\} = \emptyset$ if $-2 \leq a < 0$. Furthermore for large n*

$$g_n \sim -\frac{a(a + 2)}{a + 1} + (-1)^n \frac{a(a^2 + 2a + 2)^2}{(a + 1)^2(a + 2)} (a + 1)^{-2n},$$

where $G_n(g_n) = 0$ and $\lim_{n \rightarrow \infty} g_n \neq 0$.

Proof. For a real number x_0 , by our Lemma 2, $x_0 \in A \setminus \{0\}$ if and only if

$$(14) \quad \frac{x_0 + a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0$$

or

$$(15) \quad \frac{x_0 + a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0$$

holds. Substituting $-x_0$ for x_0 into (14) and (15) we get that

$$(16) \quad \frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0$$

and

$$(17) \quad \frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0$$

Since (16) and (17) are identical to (8) and (7), respectively, therefore all of the statements of our theorem follows from the Theorem 2. Thus the theorem is proved.

Concluding Remarks

Using our Theorem 2 for $a = 1$ we get that $g_n = g'_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n}$, which matches perfectly with the result of H. Prodinger.

On the other hand it is quite likely that similar results can be obtained for seed-polynomials $G_0(x) = x \pm a$ and $G_0(x) = a$ or for other polynomials. This could be the subject of further research work.

References

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