Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae, 24. (1997) pp. 41–45

A generalization of an approximation problem concerning linear recurrences

BÉLA ZAY*

Abstract. Let $\{G_n\}$ be a linear recursive sequence of order $t(\geq 2)$ defined by $G_n = A_1G_{n-1}+\dots+A_tG_{n-t}$ for $n\geq t$, where $A_1,\dots,A_t$ and $G_0,\dots,G_{t-1}$ are given rational integers. Denote by $\alpha_1,\alpha_2,\dots,\alpha_t$ the roots of the polynomial $x^t-A_1x^{t-1}-\dots-A_t$ and suppose that $|\alpha_1|>|\alpha_i|$ for $2\leq i<t$. It is known that $\lim\limits_{n\to\infty}\frac{G_{n+s}}{G_n}=\alpha_1^s$, where s is a positive integer. The quality of the approximation of α_1 by rational numbers $\frac{G_{n+s}}{G_n}$ in the case $s=1$ was investigated in several papers. Extending the earlier results we show that the inequality

$$\left|\alpha_1^s-\frac{G_{n+s}}{G_n}\right|<\frac{1}{cG_n^r}$$

holds for infinitely many positve integers n with some constant c if and only if

$$r\leq 1-\frac{\log|\alpha_2|}{\log|\alpha_1|}.$$

Let $\{G_n\}_{n=1}^{\infty}$ be a k^{th} order $(k\geq 2)$ linear recursive sequence defined by

$$G_n = A_1G_{n-1}+A_2G_{n-2}+\dots+A_kG_{n-k} \quad \text{for} \quad n\geq k,$$

where $A_1,\dots,A_k$, and $G_1,\dots,G_k$ are given rational integers with $A_k\neq 0$ and $G_0^2+\dots+G_{k-1}^2\neq 0$. Denote by $\alpha_1,\dots,\alpha_t$ the distinct roots of the characteristic polinomial

$$f(x)=x^k-A_1x^{k-1}-\dots-A_k=(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_t)^{m_t}.$$

Using the well known explicite form of the terms of linear recursive sequences, G_n can be expressed by

$$G_n=\sum_{i=1}^{t}\left(\sum_{j=1}^{m_i}a_{ij}n^{j-1}\right)\alpha_i^n=\sum_{i=1}^{t}P_i(n)\alpha_i^n \quad (n\geq 0) \tag{1}$$

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.

where the coefficients a_{ij} of polinomials $P_i(n)$ are elements of the algebraic number field $Q(\alpha_1,\dots,\alpha_t)$. We assume that the sequence G is a non degenerate one, i.e. $a_{11}, a_{21},\dots,a_{t_1}$ are non zero algebraic numbers and α_i/α_j is not a root of unity for any $1\le i<j\le t$. We can also assume that $G_n\ne 0$ for $n>0$ since the sequence have only finitely many zero terms and after a movement of indices this condition will be fulfilled. If $|\alpha_i|<\alpha_1$ for $i=2,3,\dots,t$ than from (1) it follows that $\lim\limits_{n\to\infty}\frac{G_{n+1}}{G_n}=\alpha_1$. In the case $k=2$ the quality of the approximation of α_1 by rational numbers G_{n+1}/G_n was investigated some earlier papers (e.g. set [2], [3], [4] and [5]). In the general case P. Kiss ([1]) proved the following result. Let G be a t^{th} order linear recurrence with conditions $|\alpha_1|>|\alpha_2|\ge|\alpha_3|>\dots>|\alpha_t|$, where $m_1=\dots=m_t=1$). Then

$$\left|\alpha_1-\frac{G_{n+1}}{G_n}\right|<\frac{1}{cG_n^k}$$

holds for infinitely many positive integers n with some constant c if and only if $k\le k_0$, where

$$k_0=1-\frac{\log|\alpha_2|}{\log|\alpha_1|}\le 1+\frac{1}{t-1}$$

and the equation $k_0=1+\frac{1}{t-1}$ can be held only if $|A_t|=1$ and $|\alpha_1|>|\alpha_2|=\dots=|\alpha_t|$.

In [1] the following lemma was also proved.

Lemma. *Let β and γ be complex algebraic numbers for which $|\beta|=|\gamma|=1$ and γ is not a root of unity. Then there are positive numbers δ and n_0 depending only on β and γ such that*

$$|1+\beta\gamma^n|>e^{\delta\log n}$$

for any $n>n_0$.

In the case $|\alpha_1|>\alpha_i$ $(2\le i\le t)$ it is clear that $\lim\limits_{n\to\infty}\frac{G_{n+s}}{G_n}=\alpha_1^s$ for any fixed positive integer s.

The purpose of this paper is the investigation of the quality of the approximation of α_1^s by rational numbers $\frac{G_{n+s}}{G_n}$ and to prove an extension of P. Kiss's theorem.

Theorem. *Let G be a non degenerate k^{th} order linear recurrence sequence with conditions:*

$$|\alpha_1|>|\alpha_2|\ge|\alpha_3|>|\alpha_4|\ge\dots\ge|\alpha_t|,\quad m_1=m_2=1,\quad \sum_{i=1}^t m_i=k$$

(where m_i is the multiplicity of α_i in the characteristic polinomial of G) and $G_n > 0$ for $n > 0$. Then

$$\left|\alpha_1^s - \frac{G_{n+s}}{G_n}\right| < \frac{1}{cG_n^r} \tag{2}$$

holds for infinitely many positive integers n with some positive constant c if and only if

$$r \le r_0 = 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|}. \tag{3}$$

We remark that in the case of $s = 1$, $m_1 = \cdots = m_t = 1$ we get the result of P. Kiss ([1]). In the next proof we shall use similar arguments wich was used by P. Kiss.

Proof of the Theorem. Since $m_1 = m_2 = 1$ the polinomials $P_1(n)$ and $P_2(n)$ are non zero constants (denoted by a_{11} and a_{21} respectively) and so by (1) we have

$$\begin{aligned}\left|\alpha_1^s - \frac{G_{n+s}}{G_n}\right| &= \left|\alpha_1^s - \frac{P_1(n+s)\alpha_1^{n+s} + \cdots + P_t(n+s)\alpha_t^{n+s}}{P_1(n)\alpha_1^n + \cdots + P_t(n)\alpha_t^n}\right| \\ &= \left|G_n^{-1}\right| \left|a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n + \sum_{i=3}^t (\alpha_1^s p_i(n) - \alpha_i^s P_i(n+s))\alpha_i^n\right| \\ &= \left|G_n^{-1} a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n\right| H_3(n)\end{aligned}$$

where

$$H_3(n) = \left|1 + \sum_{i=3}^t \frac{\left(\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s)\right)\alpha_i^n}{a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n}\right|.$$

Since $G_n = a_{11}\alpha_1^n(1 + d_n)$, where $\lim\limits_{n\to\infty} d_n = 0$, (2) holds if and only if

$$\begin{aligned}&c\left|a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n G_n^{r-1}\right| H_3(n) \\ &= c\left|a_{11}^{r-1} a_{21}(\alpha_1^s - \alpha_2^s)(1 + d_n)^{r-1}\right| \left|\alpha_2 \alpha_1^{r-1}\right|^n H_3(n) < 1.\end{aligned}$$

Denoting the second and the third factors of the last product by $H_1(n)$ and $H_2(n)$ respectively, (2) holds if and only if

$$cH_1(n)H_2(n)H_3(n) < 1. \tag{4}$$

It is easy to see that

$$e^{c_1} < H_1(n) < e^{c_2}$$

holds with suitable real numbers c_1, c_2.

From this it follows that

$$ce^{hn+c_1} < cH_1(n)H_2(n) < ce^{hn+c_2} \tag{5}$$

where $h = \log \alpha_2 + (r-1)\log \alpha_1$.

If we assume that $|\alpha_2| > |\alpha_3|$ then $\lim\limits_{n\to\infty} cH_1(n)H_3(n) = cc_0$, where $c_0 = \left|a_{11}^{r-1}a_{21}(\alpha_1^s - \alpha_2^s)\right|$.

Using the well known fact

$$\lim_{n\to\infty} H_2(n) = \lim_{n\to\infty} \left|\alpha_1\alpha_2^{r-1}\right|^n = \begin{cases} 0, & \text{if } r < r_0 = 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|} \\ 1, & \text{if } r = r_0 \\ \infty, & \text{if } r > r_0 \end{cases}$$

it is clear that (4) (and so (2), too) holds for infinitely many positive integers n with some positive constant c $(0 < c \le c_0^{-1})$ if and only if $r \le r_0$. Now we assume that

$$|\alpha_1| > |\alpha_2| = |\alpha_3| > |\alpha_4| \ge \cdots \ge |\alpha_t|.$$

Since α_1 is real and α_3/α_2 is not a root of unity α_3 and α_2 are (not real) conjugate complex numbers and $m_2 = m_3$ (i.e. $m_1 = m_2 = 1 = m_3$ and $P_3(n) = P_3(n+s) = a_{31}$). Furthermore a_{21} and a_{31} also are conjugate numbers since they are solutions of the system of linear equations

$$G_n = \sum_{i=1}^{t}\left(\sum_{j=1}^{m_i} a_{ij}n^{j-1}\right)\alpha_i^n, \quad 0 \le n \le k-1.$$

Hence $\frac{a_{3,1}(\alpha_1^s-\alpha_3^s)}{a_{2,1}(\alpha_1-\alpha_2^s)}$ and $\frac{\alpha_3}{\alpha_2}$ are algebraic numbers with absolute value 1 and so using the Lemma (proved by P. Kiss in [1]), we obtain the estimation

$$\left|1 + \frac{a_{31}(\alpha_1^s - \alpha_3^s)}{a_{21}(\alpha_1^s - \alpha_2^s)}\left(\frac{\alpha_3}{\alpha_2}\right)^n\right| > e^{-\delta\log n}$$

with some positive real δ.

But $|\alpha_i| < |\alpha_2|$ for $i \geq 4$, so by the last inequality

$$
\begin{aligned}
(6) \qquad e^{-c_3 \log n} < \Bigg| 1 + \frac{a_3(\alpha_1^s - \alpha_3^s)}{(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_3}{\alpha_2}\right)^n \\
+ \sum_{i=4}^{t} \frac{\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s)}{a_{21}(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_i}{\alpha_2}\right)^n \Bigg| = H_3(n) < 3
\end{aligned}
$$

with some $c_3 > 0$ if n is large enough.

By (5) and (6) we have

$$
(7) \qquad ce^{hn - c_3 \log n + c_1} < cH_1(n)H_2(n)H_3(n) < ce^{hn + c_2 + \log 3}.
$$

(7) holds for infinitely many positive integers if and only if $h \leq 0$, which is equivalent to $r \leq r_0$.

This completes the proof of the theorem.

References

[1] P. Kiss, An approximation problem concerning linear recurrences, to appear.

[2] P. Kiss, A Diophantine approximative property of the second order linear recurrences, *Period. Math. Hungar.* **11** (1980), 281–287.

[3] P. Kiss and Zs. Sinka, On the ratios of the terms of second order linear recurrences, *Period. Math. Hungar.* **23** (1991), 139–143.

[4] P. Kiss and R. F. Tichy, A discrepancy problem with applications to linear recurrences I., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 135–138.

[5] P. Kiss and R. F. Tichy, A discrepancy problem with applications to linear recurrences II., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 131–194.

Béla Zay
Eszterházy Károly Teachers' Training College
Department of Mathematics
Leányka u. 4.
3301 Eger, Pf. 43.
Hungary