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General solution of the differential equation $y''(x) - \big(y'(x)\big)^2 + x^2 e^y(x) = 0$

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Abstract. In this note we prove that the general solution of the differential equation $y''(x)-\big(y'(x)\big)^2+x^2e^y(x)=0$, $x>0$ is the function $y(x)=-\ln W(x)$, where $W(x)=\frac{1}{12}x^4+Ax+B$ and A,B are arbitrary constants.

1. Introduction

In this note we prove that the general solution of the differential equation

$$y''(x) - \big(y'(x)\big)^2 + x^2 e^{y(x)} = 0, \quad x > 0 \tag{1}$$

is the function

$$y(x) = -\ln W(x), \quad \text{where} \quad W(x) = \frac{1}{12}x^4 + Ax + B \tag{2}$$

and A, B are arbitrary constants. First, we note that such type of differential equations as (1) are difficult to solve. For example, E. Y. Rodin (see [1], p. 474, Unsolved problems, SIAM 81-17) posed the following problem. Find the general solution of the differential equation:

$$y''(x) + x^2 e^{y(x)} = 0, \quad x > 0. \tag{3}$$

We prove that (1) has general solution given by (2), however we can't find the general solution of (3).

2. The Result

We prove the following theorem:

Theorem. *The general solution of the differential equation (1) is the function*

$$y(x) = \ln\left(\frac{1}{12}x^2 + Ax + B\right)$$

where A, B are arbitrary constants.

Proof. Putting $y(x) = \ln z(x)$ we obtain

$$y'(x) = \frac{z'(x)}{z(x)} \tag{4}$$

and consequently we have

$$y''(x) = \frac{z''(x)}{z(x)} - \left(\frac{z'(x)}{z(x)}\right)^2. \tag{5}$$

Since $y(x) = \ln z(x)$, then $e^{y(x)} = z(x)$ and by (4) and (5) it follows that (1) can be reduced to the following form:

$$\frac{z''(x)}{z^2(x)} - 2\frac{\big(z'(x)\big)^2}{z^3(x)} = -x^2. \tag{6}$$

Integrating (6) with respect to x we obtain:

$$\int \left(\frac{z''(x)}{z^2(x)} - 2\frac{\big(z'(x)\big)^2}{z^3(x)}\right) dx = -\frac{1}{3}x^3 + C_1.$$

Denote by

$$f\big(z(x)\big) = \frac{z''(x)}{z^2(x)} - 2\frac{\big(z'(x)\big)^2}{z^3(x)}. \tag{7}$$

Then we see that the function

$$\frac{z'(x)}{z^2(x)} = F\big(z(x)\big) \tag{8}$$

satisfies the following condition

$$F'\big(z(x)\big) = \big(\frac{z'(x)}{z^2(x)}\big)' = \frac{z''(x)}{z^2(x)} - 2\frac{\big(z'(x)\big)^2}{z^3(x)} = f\big((z(x)\big)$$

and therefore by (7) and (8) it follows that

$$\frac{z'(x)}{z^2(x)} = -\frac{1}{3}x^3 + C_1. \tag{9}$$

Integrating the last equality with respect to x we obtain

$$\int \frac{z'(x)}{z^2(x)}dx = -\frac{1}{12}x^4 + C_1x + C_2. \tag{10}$$

On the other hand it easy to see that

$$\left(-\frac{1}{z(x)}\right)' = \frac{z'(x)}{z^2(x)} \tag{11}$$

and consequently by (10) and (11) it follows that

$$-\frac{1}{z(x)} = -\frac{1}{12}x^4 + C_1x + C_2$$

and we have

$$y(x) = \ln z(x) = -\ln\left(\frac{1}{12}x^4 + Ax + B\right) = -\ln W(x)$$

where $A = -C_1$, $B = -C_2$. The proof of the theorem is complete.

References

[1] S. Rabinowitz, Index to Mathematical Problems, Westford, Massachusetts, USA, 1992.

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