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On the Fejér kernel functions with respect to the Walsh–Paley system

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Abstract. In this paper we prove some lemmas with respect to the Fejér kernels of the Walsh–Paley system. This lemmas give a new proof for the known a.e. convergence $\sigma_n f \to f$ $(n\to\infty,\ f\in L^1)$.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $I := [0,1)$ the unit interval. Denote the Lebesgue measure of any set $E \subset I$ by $|E|$. Denote the $L^p(I)$ norm of any function f by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbf{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbf{N}$ choose the expansion which terminates in zeros (these numbers are the dyadic rationals)). n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} (x_j + y_j \bmod 2) 2^{-j-1}.$$

The sets

$$I_n(x) := \{y \in I : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbf{P}$ and $I_0(x) := I$ are the dyadic intervalls of I. Set $e_n := (0, \dots, 0, 1, 0, \dots)$ where the n-th coordinate of e_n is 1 the rest are zeros for all $n \in \mathbf{N}$. The dyadic rationals are the finite $0, 1$ combinations of the elements of the set $\{e_n : n \in \mathbf{N}\}$ (which dense in I).

Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh–Paley system ([2], [8]) that is,

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}, \quad n \in \mathbf{N},\ x \in I.$$

Denote by $D_n := \sum_{k=0}^{n-1} \omega_k$, the Walsh–Dirichlet kernels.

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It is well-known that ([2], [8])

$$S_nf(y) = \int_I f(x)D_n(y+x)dx = f * D_n(y)$$

($y \in I,\ n \in \mathbf{P}$) the n-th partial sum of the Walsh–Fourier series. Moreover, ([8], p. 28.)

$$D_{2^n}(x) := \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$D_n(x) = \omega_n(x)\sum_{k=0}^{\infty} n_k(D_{2^{k+1}}(x) - D_{2^k}(x)) = \omega_n(x)\sum_{k=0}^{\infty} n_k(-1)^{x_k}D_{2^k}(x), \tag{2}$$

$n \in \mathbf{N}$, $x \in I$.

Define the n-th Fejér means [8] of function $f \in L^1(I)$ as

$$\sigma_nf(y) := \frac{1}{n}\sum_{k=0}^{n-1} S_kf(y)$$

for $y \in I$ and $n \in \mathbf{P}$ and define n-th Fejér kernel [8]

$$K_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x)$$

for $x \in I$ and $n \in \mathbf{P}$. This gives

$$\sigma_nf(y) = \int_I f(x)K_n(x+y)dx = f * K_n(y) \quad (y \in I,\ n \in \mathbf{P}).$$

Set

$$K_{a,b} := \sum_{j=a}^{b-1} D_j \quad a, b \in \mathbf{N} \ \text{ and } \ n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i \quad (n, s \in \mathbf{N}).$$

Also set for $n \in \mathbf{N}$ $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. That is, $2^{|n|} \leq n < 2^{|n|+1}$. In this paper c denotes an absolute constant which may not be the same at different occurences. Then we have by an easy calculation that

Lemma 1. $nK_n = \sum_{s=0}^{|n|} n_s K_{n^{(s+1)},2^s}$ *for all* $n \in \mathbf{P}$. ∎

Lemma 2. *Suppose that $s,t,n \in \mathbf{N}$, $x \in I_t \setminus I_{t+1}$. If $s \le t \le |n|$, then $\left|K_{n^{(s+1)},2^s}(x)\right| \le c2^{s+t}$. On the other hand, if $t < s \le |n|$, we have*

$$K_{n^{(s+1)},2^s}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x)2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases}$$

Proof. If $s \le t$, then for all $k \in \mathbf{N}$ by (1) and (2) we have $|D_k(x)| \le c\sum_{j=0}^{t} 2^j \le c2^t$, thus in this case $\left|K_{n^{(s+1)},2^s}(x)\right| \le c2^{s+t}$. On the other hand, let $|n| \ge s > t$. Then

$$\begin{aligned} D_{n^{(s+1)}+j}(x) &= \omega_{n^{(s+1)}+j}(x) \sum_{k=0}^{t} (n^{(s+1)}+j)_k r_k(x) \\ &= \omega_{n^{(s+1)}+j}(x) \left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t \right). \end{aligned}$$

This implies that

$$\begin{aligned} K_{n^{(s+1)},2^s}(x) &= \sum_{j=0}^{2^s-1} D_{n^{(s+1)}+j}(x) \\ &= \omega_{n^{(s+1)}}(x) \sum_{j=0}^{2^s-1} \omega_j(x) \left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t \right) =: \sum\nolimits_1 - \sum\nolimits_2. \end{aligned}$$

$$\begin{aligned} \sum\nolimits_1 &= \omega_{n^{(s+1)}}(x) \sum_{j_0,\dots,j_{s-1}} \omega_j(x) \sum_{k=0}^{t-1} j_k 2^k \\ &= \sum_{j_i=0, i\ne t, i=0,\dots,s-1}^{1} \sum_{k=0}^{t-1} j_k 2^k \sum_{j_t=0}^{1} \omega_j(x) = 0, \end{aligned}$$

since

$$\sum_{j_t=0}^{1} \omega_j(x) = \sum_{j_t=0}^{1} (-1)^{j_0x_0+\dots+j_{t-1}x_{t-1}+j_{t+1}x_{t+1}+\dots+j_{s-1}x_{s-1}} = 0.$$

That is,

$$\begin{aligned} K_{n^{(s+1)},2^s}(x) &= -\omega_{n^{(s+1)}}(x) \sum_{j=0}^{2^s-1} \omega_j(x) j_t 2^t \\ &= \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x)2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases} \quad \blacksquare \end{aligned}$$

As a straightforward consequence of Lemma 2 we get

Lemma 3. $\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \le c\sqrt{2^{s+t}}$, *where* $m \ge s, t \in \mathbf{N}$ *are fixed.*

Proof. If $s > t$, then by Lemma 2 it follows that

$$\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx = \int_{I_s(e_t)} 2^{s+t-1} dx = 2^{t-1}.$$

On the other hand, if $s \le t$, then also by Lemma 2 we have

$$\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \le c \int_{I_t\setminus I_{t+1}} c2^{s+t} \le c2^s. \ \blacksquare$$

Lemma 4. $\int_{I\setminus I_k} \sup_{|n|\ge A} |K_n(x)|\, dx \le c\sqrt{2^{k-A}}$, *for all* $A \ge k \in \mathbf{N}$.

Proof. By Lemma 1 we have

$$n\,|K_n| \le \sum_{s=0}^{|n|} \left|K_{n^{(s+1)},2^s}\right|,$$

consequently,

$$\begin{aligned}
\int_{I\setminus I_k} \sup_{|n|\ge A} |K_n(x)|\, dx &\le \sum_{t=0}^{k-1} \int_{I_t\setminus I_{t+1}} \sum_{m=A}^{\infty} \sup_{|n|=m} |K_n(x)|\, dx \\
&\le \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} n\,|K_n(x)|\, dx \\
&\le \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \left(\sum_{s=0}^{t} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \right. \\
&\quad \left. + \sum_{s=t+1}^{m} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \right) \\
&\le c \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \sum_{s=0}^{m} 2^{\frac{s+t}{2}} \le c \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} 2^{\frac{t-m}{2}} \le c2^{\frac{k-A}{2}}. \ \blacksquare
\end{aligned}$$

The following Theorem shows that the maximal operator

$$Tf := \sup_{n\in\mathbf{P}} |\sigma_n f|$$

is quasi-local. The conception of quasi-locality is introduced by F. Schipp [8]. Let $f \in L^1(I)$, $\operatorname{supp} f \subset I_k(x^0)$ for some $k \in \mathbf{N}, x^0 \in I$ and suppose that the integral of Tf on the set $I \setminus I_k(x^0)$ is bounded by $c\,|f|_1$. Then we call T quasi-local. That is, we prove

Theorem 5. $\int_{I\setminus I_k(x^0)} Tf \le c\,|f|_1$.

Proof. If $n < 2^k$, then $\hat{f}(n) = \int_I f\omega_n = \int_{I_k(x^0)} f\omega_n = \omega_n(x^0)\int_{I_k(x^0)} f = 0$, thus $S_n f = 0, \sigma_n f = 0$. That is, we have $Tf = \sup_{n\ge 2^k} |\sigma_n f|$. By Lemma 4 it follows

$$\begin{aligned}\int_{I\setminus I_k(x^0)} \sup_{n\ge 2^k} &\left|\int_{I_k(x^0)} f(x)K_n(x+y)dx\right| dy \\ &\le \int_{I_k(x^0)} |f(x)| \int_{I\setminus I_k(x^0)} \sup_{n\ge 2^k} |K_n(x+y)dy|\, dx \\ &= \int_{I_k(x^0)} |f(x)| \int_{I\setminus I_k} \sup_{n\ge 2^k} |K_n(y)dy|\, dx \le c\,|f|_1 . \ \blacksquare\end{aligned}$$

Define the Hardy space H as follows. Let $f^* := \sup_{n\in\mathbf{N}} |S_{2^n} f|$ be the maximal function of the integrable function $f \in L^1(I)$. Then,

$$H(I) := \{f \in L^1(I) :\, f^* \in L^1(I)\},$$

moreover H is a Banach space endowed with the norm $|f|_H := |f^*|_1$. By standard argument (see e.g. [8]) and by the help of Theorem 5 one can prove that the operator T is of type (H, L^1) which means that $|Tf|_1 \le c\,|f|_H$ for all $f \in H$. This result with respect to the Walsh system is due to Schipp [7] and Fujii [2]. With respect to bounded Vilenkin system it is proved by Simon [6]. The noncommutative case is discussed by the author ([4]).

Also by standard argument (see e.g. [8]) and by the help of Theorem 5 we have that for all $f \in L^1(I)$ the almost everywhere convergence $\sigma_n f \to f$ $(n \to \infty,\ f \in L^1(I))$ holds. This result with respect to the Walsh system is due to Fine [1]. With respect to bounded Vilenkin systems it is proved by Pál and Simon [5]. The so-called 2-adic integers and the noncommutative case are discussed by the author ([3], [4]).

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